Acceleration lies in the plane of T and N

Math 311

Recall that for arbitrary parameter t,

$$s' = \frac{ds}{dt} = \frac{d}{dt}\int_a^t \|\mathbf{r}'(u)\|\,du = \|\mathbf{r}'\|$$

$$\kappa = \frac{\|\mathbf{T}'\|}{\|\mathbf{r}'\|} = \frac{\|\mathbf{T}'\|}{s'} \quad \text{and} \quad \mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|}$$

so that

$$\kappa s' = \|\mathbf{T}'\| \quad \text{and} \quad \mathbf{T}' = \|\mathbf{T}'\|\,\mathbf{N} = \kappa s' \mathbf{N}.$$

Furthermore,

$$\mathbf{v} = \mathbf{r}' = \|\mathbf{r}'\|\frac{\mathbf{r}'}{\|\mathbf{r}'\|} = s'\mathbf{T}$$

and by the product rule we have

$$\mathbf{a} = \mathbf{s}''\mathbf{T} + \mathbf{s}'\mathbf{T}'.$$

Therefore

$$\boxed{\mathbf{a} = \mathbf{s}''\mathbf{T} + \kappa\,(s')^2\,\mathbf{N}.}$$

Since **T** and **N** are unit vectors,

$$a_{\mathbf{T}} = s'' = \text{comp}_{\mathbf{T}}\,\mathbf{a} = \mathbf{a} \bullet \mathbf{T} \quad \text{and} \quad a_{\mathbf{N}} = \kappa\,(s')^2 = \text{comp}_{\mathbf{N}}\,\mathbf{a} = \mathbf{a} \bullet \mathbf{N}$$

so that

$$\boxed{\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}.}$$

Since **T** and **N** are orthogonal, the Pythagorean Theorem gives

$$\boxed{\|\mathbf{a}\|^2 = a_{\mathbf{T}}^2 + a_{\mathbf{N}}^2.}$$