CURVATURE OF SMOOTH CURVES WITH ARBITRARY PARAMETER

Theorem 1 *Every plane curve C with smooth parametrization may be reparametrized with unit speed.*

Proof: Let C be a plane curve with smooth parametrization $\mathbf{r}(t) = \langle f(t), g(t)\rangle$, where $t \in (a,b)$. Choose any reference point $t_0 \in (a,b)$; then as the parameter varies from t_0 to t, the arc length $s(t)$ is given by

$$s(t) = \int_{t_0}^{t} \|\mathbf{r}'(u)\| \ du.$$

By the Fundamental Theorem of Calculus (see Swokowski, Olinick and Pence Theorem 4.30, part I, p.397), we know that $s'(t) = \|\mathbf{r}'(t)\|$. But $\mathbf{r}$ is smooth so $\|\mathbf{r}'(t)\| > 0$, and the arc length function $s = s(t)$ is strictly increasing. Hence s is one-to-one and has a differentiable inverse by the Inverse Function Theorem (see Swokowski, Olinick and Pence Corollary 6.8, p.521). This means that we can always solve for t in terms of s, obtain a differentiable function $t = t(s)$ and reparametrize C by arc length s in the following way:

$$\mathbf{r}(s) = \langle f(t(s)), g(t(s))\rangle.$$

Thus,

$$\mathbf{r}'(s) = \langle f'(t(s)), g'(t(s))\rangle\, t'(s).$$

Applying the Inverse Function Theorem and taking norms gives

$$\|\mathbf{r}'(s)\| = \|\langle f'(t), g'(t)\rangle\| \frac{1}{|s'(t)|} = \|\mathbf{r}'(t)\| \frac{1}{\|\mathbf{r}'(t)\|} = 1.$$

□

Theorem 2 *Let C be a plane curve with smooth parametrization $\mathbf{r}(t) = \langle f(t), g(t)\rangle$, $t \in (a,b)$, whose component functions f and g have second derivatives. Then the curvature $K(t)$ at the point $\mathbf{r}(t)$ is given by*

$$K(t) = \frac{|f'(t)\, g''(t) - g'(t)\, f''(t)|}{\|\mathbf{r}'(t)\|^3}.$$

Proof: By Theorem 1 above, there is a differentiable function $t = t(s)$ such that $\mathbf{r}(s) = \langle f(t(s)), g(t(s)) \rangle$ is a unit speed reparametrization of C. Let $\theta(s)$ Denote the angle between $\mathbf{r}'(s)$ and the unit coordinate vector $\mathbf{i}$. Then by definition, the curvature $K(s) = |\theta'(s)|$. Now define $\phi(t) = \theta(s(t))$, where $s = s(t)$ is the arc length function, and differentiate with respect to t. Then

$$\phi'(t) = \theta'(s(t))\, s'(t) = \theta'(s(t)) \cdot \|\mathbf{r}'(t)\|$$

and hence

$$|\theta'(s(t))| = \frac{1}{\|\mathbf{r}'(t)\|} |\phi'(t)|$$

so that the curvature K at time t is

$$K(t) = \frac{1}{\|\mathbf{r}'(t)\|} |\phi'(t)|. \tag{1}$$

The slope $m(t)$ of the tangent line to curve C at point $\mathbf{r}(t)$ is given by

$$m(t) = \frac{g'(t)}{f'(t)} = tan[\phi(t)]$$

or

$$tan^{-1}\left[\frac{g'(t)}{f'(t)}\right] = \phi(t),$$

as long as $f'(t) \neq 0$. Therefore

$$\begin{aligned} \phi'(t) &= \frac{1}{1 + [g'(t)/f'(t)]^2} \cdot \frac{f'(t)\, g''(t) - f''(t)\, g'(t)}{[f'(t)]^2} \\ &= \frac{f'(t)\, g''(t) - f''(t)\, g'(t)}{\|\mathbf{r}'(t)\|^2}. \end{aligned}$$

From formula (1) we obtain

$$K(t) = \frac{|f'(t)\, g''(t) - f''(t)\, g'(t)|}{\|\mathbf{r}'(t)\|^3}. \tag{2}$$

On the other hand, if $f'(t) = 0$, then $\|\mathbf{r}'(t)\| = |g'(t)| \neq 0$ since $\mathbf{r}$ is smooth. In general, the cosine of the angle $\phi(t)$ between the unit tangent vector $\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$ and the unit coordinate vector $\mathbf{i}$ is given by

$$\cos[\phi(t)] = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \bullet \mathbf{i} = \frac{f'(t)}{s'(t)},$$

where $s = s(t)$ is the arc length function. Differentiating implicitly gives

$$-\sin[\phi(t)]\,\phi'(t) = \frac{s'(t)\,f''(t) - f'(t)\,s''(t)}{[s'(t)]^2}.$$

In particular, when $f'(t) = 0$, $\phi(t) = \frac{\pi}{2}$ in which case formula (1) becomes

$$\begin{aligned} K(t) &= \frac{1}{|g'(t)|}\,|\phi'(t)| = \frac{1}{|g'(t)|}\frac{|s'(t)\,f''(t)|}{[s'(t)]^2} \\ &= \frac{1}{|g'(t)|}\left|\frac{f''(t)}{g'(t)}\right| = \frac{|f''(t)|}{[g'(t)]^2}, \end{aligned}$$

which is a special case of formula (2).

□