CURVATUVE IN 3-SPACE

Definition: *Let* $\mathbf{r}(s) = \langle f(s), g(s), h(s)\rangle$ *be a unit speed curve. The* *curvature at* s *is defined to be*

$$K(s) = \|\mathbf{T}'(s)\|.$$

Frenet's Formula: Since the unit tangent vector $\mathbf{T}(s) = \mathbf{r}'(s)$ and the unit normal vector $\mathbf{N}(s) = \mathbf{T}'(s) / \|\mathbf{T}'(s)\| = \mathbf{T}'(s)/K(s)$, we have

$$\mathbf{T}' = K\mathbf{N}.$$

Example: Find the curvature of the circular helix

$$\mathbf{r}(t) = \langle a\cos t, a\sin t, bt\rangle.$$

First we must reparametrize with unit speed.

$$\mathbf{r}'(t) = \langle -a\sin t, a\cos t, b\rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{a^2+b^2}.$$

The arc length function is

$$s(t) = \int_0^t \sqrt{a^2+b^2}\,du = \sqrt{a^2+b^2}\,t.$$

Solving for t in terms of s gives

$$t = \frac{s}{\sqrt{a^2+b^2}}.$$

Therefore

$$\mathbf{r}(s) = \left\langle a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{bs}{\sqrt{a^2+b^2}}\right\rangle$$

is parametrized with unit speed. Now

$$\mathbf{T}(s) = \mathbf{r}'(s) = \frac{1}{\sqrt{a^2+b^2}}\left\langle -a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), b\right\rangle$$

so that

$$\mathbf{T}'(s) = \frac{a}{a^2+b^2}\left\langle -\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), -\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0\right\rangle$$

and

$$\mathbf{N}(s) = \left\langle -\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), -\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0\right\rangle.$$

By Frenet's formula,

$$K(s) = \frac{a}{a^2+b^2}.$$

Note that if $b = 0$, the helix collapses to a circle of radius a in the xy-plane and the curvature becomes $K(s) = \frac{1}{a}$.