THE DERIVATIVE

The geometrical idea underlying the derivative in single-variable calculus is slope, i.e., the limit of slopes of secant lines is the slope of the tangent line. Since slope is only defined in the plane, we need to rethink the derivative with an eye towards generalizing to higher dimensions.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a. Then by definition

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

or equivalently

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} - f'(a) = 0.$$

We can rewrite this as

$$\lim_{x \to a} \frac{f(x) - [f(a) + f'(a)(x - a)]}{x - a} = 0. \tag{1}$$

Let

$$T(x) = f(a) + f'(a)(x - a)$$

and substitute in (1) to obtain

$$\lim_{x \to a} \frac{f(x) - T(x)}{|x - a|} = 0,$$

where the absolute values in the denominator are permitted since the limit is zero. Observe that T is the linear function whose graph is the tangent line to the graph of f at the point $(a, f(a))$.

Conversely, consider any (non-vertical) line ℓ passing through $(a, f(a))$ with slope m_ℓ and let

$$T_\ell(x) = f(a) + m_\ell(x - a).$$

Then T_ℓ is the linear function whose graph is line ℓ. Now suppose that

$$\lim_{x \to a} \frac{f(x) - T_\ell(x)}{|x - a|} = 0.$$

Then by substituting for T_ℓ we have

$$0 = \lim_{x\to a} \frac{f(x) - [f(a) + m_\ell(x-a)]}{|x-a|} = (\pm 1)\left[\lim_{x\to x_0} \frac{f(x)-f(a)}{x-a} - m_\ell\right].$$

In either case,

$$m_\ell = \lim_{x\to a} \frac{f(x)-f(a)}{x-a},$$

which is to say that f is differentiable at a and $f'(a) = m_\ell$. Therefore, of all linear functions T_ℓ whose graphs ℓ pass through $(a, f(a))$, only T (whose graph is the tangent line) has the property that

$$\lim_{x\to a} \frac{f(x) - T(x)}{|x-a|} = 0.$$

Let's reformulate the definition of the derivative in these terms:

Definition 1 *Let $\mathcal{U}$ be an open subset of $\mathbb{R}$ and let $f : \mathcal{U} \to \mathbb{R}$. Then f is differentiable at $a \in \mathcal{U}$ if and only if there is a linear function*

$$T(x) = f(a) + m(x-a)$$

such that

$$\lim_{x\to a} \frac{f(x) - T(x)}{|x-a|} = 0.$$

If f is differentiable at a, the number m is called the derivative of f at a and is denoted by $f'(a)$. If f is differentiable at each point $x \in \mathcal{U}$, we say that f is differentiable on $\mathcal{U}$.

This definition generalizes easily to 2 (and $n \geq 2$) dimensions.

Definition 2 *Let $\mathcal{U}$ be an open subset of $\mathbb{R}^2$ and let $f : \mathcal{U} \to \mathbb{R}$. Then f is differentiable at $(a,b) \in \mathcal{U}$ if and only if there is a linear function*

$$T(x,y) = f(a,b) + \langle m_1, m_2\rangle \bullet \langle x-a, y-b\rangle$$

such that

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - T(x,y)}{\|\langle x-a, y-b\rangle\|} = 0.$$

If f is differentiable at (a,b), the constant vector $\langle m_1, m_2\rangle$ is the total derivative of f at (a,b) and is denoted by $f'(a,b)$; the function T is called the tangent approximation to f at (a,b). If f is differentiable at each point in $\mathcal{U}$, we say that f is differentiable on $\mathcal{U}$.

Theorem 3 *Let $\mathcal{U}$ be an open subset of $\mathbb{R}^2$ and let $f : \mathcal{U} \to \mathbb{R}$ be differentiable at (a, b). Then the derivative of f at (a, b) agrees with the gradient of f at (a, b), i.e.,*

$$f'(a,b) = \nabla f(a,b) = \langle f_x(a,b), f_y(a,b)\rangle .$$

Proof: Since f is differentiable, there is a linear function

$$T(x,y) = f(a,b) + \langle m_1, m_2\rangle \bullet \langle x-a, y-b\rangle$$

such that

$$0 = \lim_{(x,y)\to(a,b)} \frac{f(x,y) - T(x,y)}{\|\langle x-a, y-b\rangle\|}.$$

Restricting to the line $y = b$, this limit becomes

$$\begin{aligned} 0 &= \lim_{x\to a} \frac{f(x,b) - [f(a,b) + m_1(x-a)]}{\sqrt{(x-a)^2}} \\ &= \lim_{x\to a} \left[\frac{f(x,b) - f(a,b)}{|x-a|} - m_1 \frac{x-a}{|x-a|}\right] \\ &= (\pm 1)\left[\lim_{x\to a} \frac{f(x,b) - f(a,b)}{x-a} - m_1\right]. \end{aligned}$$

Therefore

$$m_1 = \lim_{x\to a} \frac{f(x,b) - f(a,b)}{x-a} = f_x(a,b).$$

The same argument with $y \to b$ while restricting to the line $x = a$ gives the second component.

$\square$

Now suppose that if f is differentiable at (a, b). The graph of the equation

$$z = f(a,b) + \langle f_x(a,b), f_y(a,b)\rangle \bullet \langle x-a, y-b\rangle \tag{2}$$

is the graph of the tangent approximating function T at (a, b). By applying the dot product and rearranging terms we obtain the equation of a plane

$$f_x(a,b)\,x + f_y(a,b)\,y - z = c,$$

where $c = f_x(a,b)\,a + f_y(a,b)\,b - f(a,b)$ is a constant. Thus equation (2) is the equation of the plane tangent to the graph of f at $(a, b, f(a, b))$.

Theorem 4 *If* $f : \mathcal{U} \to \mathbb{R}$ *is differentiable at* (a, b), *then* f *is continuous at* (a, b).

Proof: Since f is differentiable at (a, b), it is defined at (a, b). We must show that $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{(x,y)\to(a,b)} [T(x, y) + f(x, y) - T(x, y)]$$

$$= \lim_{(x,y)\to(a,b)} [f(a, b) + \langle f_x(a, b), f_y(a, b)\rangle \bullet \langle x - a, y - b\rangle + f(x, y) - T(x, y)]$$

$$= f(a, b) + \langle f_x(a, b), f_y(a, b)\rangle \bullet \langle 0, 0\rangle + \lim_{(x,y)\to(a,b)} [f(x, y) - T(x, y)]$$

$$= f(a, b) + \lim_{(x,y)\to(a,b)} \left[\tfrac{f(x,y)-T(x,y)}{\|\langle x-a,y-b\rangle\|} \|\langle x - a, y - b\rangle\|\right]$$

$$= f(a, b) + \lim_{(x,y)\to(a,b)} \tfrac{f(x,y)-T(x,y)}{\|\langle x-a,y-b\rangle\|} \cdot \lim_{(x,y)\to(a,b)} \|\langle x - a, y - b\rangle\|$$

$$= f(a, b) + 0 \cdot 0 = f(a, b).$$