THREE GREAT THEOREMS OF VECTOR CALCULUS

Green's Theorem: *Let R be a connected plane region whose boundary is a piecewise smooth simple closed curve C. If M and N are continuous with continuous first partials in some open region of the plane containing R, then*

$$\oint_C M dx + N dy \; = \; \iint\limits_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

Stokes's Theorem: *Let f be a differentiable real-valued function defined on a connected plane region R whose boundary is a piecewise smooth simple closed curve. Let S be the graph of $z = f(x, y)$ on domain R and let C be the boundary of S. If M, N and P are continuous with continuous first partials on some open region of space containing S, then*

$$\oint_C M dx + N dy + P dz \; = \; \iint\limits_S \left(\nabla \times \langle M, N, P \rangle \right) \bullet \; \mathbf{n} \; dS,$$

where $\mathbf{n}$ *denotes the upward unit normal** *on* S.

*To find the upward unit normal $\mathbf{n}$ on $S : z = f(x, y)$, define $g(x, y, z) = z - f(x, y)$ so that S is a level surface for g. Then $\nabla g = \left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right\rangle$ is normal to S with positive z-component. Normalize to obtain

$$\mathbf{n} = \frac{\left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right\rangle}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}}.$$

Note that $\mathbf{n} \; dS = \nabla g \; dA$.

Gauss's Theorem: *Let Q be a closed solid in space whose boundary is a closed piecewise smooth surface S. If M, N and P are continuous with continuous first partials on some open region of space containing Q*, then

$$\iint\limits_S \langle M, N, P \rangle \bullet \; \mathbf{n} \; dS \; = \; \iiint_Q \nabla \bullet \langle M, N, P \rangle \; dV,$$

where $\mathbf{n}$ *denotes the outward unit normal on* S.