LINE INTEGRALS IN 2 DIMENSIONS

Line integrals appear as one of two kinds: (1) As integrals of scalar functions with respect to arc length, referred to as "mass integrals" or (2) as integrals of vector fields with respect to position, referred to as "work integrals". In either case, we are given a smooth plane curve C.

MASS INTEGRALS: Let f be a real valued continuous function on an open region D of the plane containing C. (For space curves C, f is defined on some open space region Q containing C.) Think of C as a thin wire with non-homogeneous density and let $f(x, y)$ be the density of the wire at the point (x, y) on the curve C. We wish to evaluate

$$\int_C f(x, y)\ ds,$$

which gives the mass of the wire when f is its density.

1. Choose a parametrization of C: $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ with $a \leq t \leq b$.

2. Let $s(t)$ denote the arc length from $(x(a), y(a))$ to $(x(t), y(t))$ along the curve C. Calculate the speed (the rate of change of arc length with respect to time):

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\| = \|\langle x'(t), y'(t) \rangle\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

3. Separate variables:

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\ dt$$

4. Evaluate:

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2}\ dt$$

<u>WORK INTEGRALS:</u> Let $\mathbf{F}$ be a continuous vector field on an open region D of the plane containing C. (For space curves C, $\mathbf{F}$ is defined on some open space region Q containing C.) Think of $\mathbf{F}$ as a force acting on a particle constrained to the path C. We wish to evaluate

$$\int_C \mathbf{F} \bullet d\mathbf{r},$$

which gives the work done by $\mathbf{F}$ as it moves a particle from one end of C to the other.

1. Choose a parametrization of C: $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ with $a \leq t \leq b$.
2. Calculate the velocity:
$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \langle x'(t), y'(t) \rangle .$$
3. Separate variables:
$$d\mathbf{r} = \mathbf{r}'(t) \ dt$$
4. Evaluate:
$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_a^b \mathbf{F}(x(t), y(t)) \bullet \mathbf{r}'(t) \ dt$$

<u>Notation:</u>

Let $\mathbf{F}(x, y) = \langle M(x, y), N(x, y) \rangle$ and let $d\mathbf{r} = \langle dx, dy \rangle$. Then

$$\begin{aligned} \int_C \mathbf{F} \bullet d\mathbf{r} &= \int_C \langle M, N \rangle \bullet \langle dx, dy \rangle \\ &= \int_C M \ dx + N \ dy \end{aligned}$$

WORK INTEGRALS AS MASS INTEGRALS:

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\|\mathbf{r}'(t)\| = \mathbf{T}(t)\|\mathbf{r}'(t)\| = \mathbf{T}(t)\frac{ds}{dt}.$$

Thus:

$$d\mathbf{r} = \mathbf{T}(t)\ ds$$

so that

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C \mathbf{F} \bullet \mathbf{T}\ ds.$$

Note that $\mathbf{F} \bullet \mathbf{T}$ is a real valued function on the domain D. Thus we have transformed the work integral on the left to a mass integral on the right.

EVALUATING LINE INTEGRALS IN 2 DIMENSIONS

Let $\mathbf{F}(x,y) = \langle M(x,y), N(x,y)\rangle$ be a continuous vector field on some open region D containing a piece-wise smooth plane curve C. Given the right conditions, there are three ways to evaluate the line integral of $\mathbf{F}$ along C:

1. *The definition*: If C is parametrized by $\mathbf{r}(t) = \langle f(t), g(t)\rangle$ with $t \in [a,b]$, then

$$\int_C \mathbf{F}\bullet d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\ dt.$$

2. *The Fundamental Theorem of Line Integrals*: If C runs from $A(x_1, y_1)$ to $B(x_2, y_2)$ and $\nabla f(x,y) = \mathbf{F}(x,y)$ for all (x,y) in D, then

$$\int_C \mathbf{F}\bullet d\mathbf{r} = \int_A^B \mathbf{F}\bullet d\mathbf{r} = f(B) - f(A).$$

3. *Green's Theorem*: Suppose C is positively oriented simple closed and R is the region consisting of C and its interior. If $R \cap D$ is simply connected and $\frac{\partial N}{\partial x}$ and $\frac{\partial M}{\partial y}$ are continuous in D, then

$$\oint_C \mathbf{F}\bullet d\mathbf{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

Remark: Theoretically, one can always use the definition (1), but the Fundamental Theorem (2) and Green's Theorem (3) can often save you some work when they apply. Here's a procedure that you can use to decide which technique to apply:

Evaluate the expression $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$.

a. *If* $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = 0$: Find a potential function (antiderivative) f and apply the Fundamental Theorem (2).

b. *If* $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} \neq 0$: If the hypotheses of Green's Theorem are satisfied apply (3).

c. *Otherwise:* Definition (1) is your only option.