MOMENTS AND CENTER OF MASS

Math 311

Consider a point-mass m at position x on a horizontal coordinate line. When gravity acts on m, the weight mg applies a torque at the origin:

$$\|\tau\| = \|(x\mathbf{i}) \times (-mg\mathbf{j})\| = |x| \, (mg) \sin 90 = m \, |x| \, g.$$

This rotational force at the origin is closely related to the *first moment of* m.

Definition 1 *The* <u>*first moment*</u> *of a point-mass* m *at position* x *on a horizontal coordinate line is defined to be*

$$M = mx.$$

Note that

$$M = \begin{cases} \|\tau\| / g, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -\|\tau\| / g, & \text{if } x < 0. \end{cases}$$

Thus first moment measures the *tendency* of the coordinate line to rotate about the origin. A positive first moment indicates the potential for clockwise rotation, a negative first moment indicates the potential for a counterclockwise rotation. Note that

$$\overline{x} = \frac{M}{m} = x$$

is the balance point, i.e., *center of mass.*

Definition 2 *The* <u>*first moment*</u> *of a system of point-masses* $m_1, ..., m_k$ *at positions* $x_1, \ldots, x_k$ *on a (horizontal) coordinate line is defined to be*

$$M = m_1 x_1 + \cdots + M_k x_k;$$

the <u>*center of mass*</u> *is given by*

$$\overline{x} = \frac{M}{m} = \frac{m_1 x_1 + \cdots + M_k x_k}{m_1 + \cdots + m_k}.$$

This measures the *net* tendency of the coordinate line to rotate about the origin.

Now consider a flat plate of negligible thickness (a lamina) with non-constant but continuously varying density ρ. We wish to determine the tendency of the the plate to rotate about the x and y axes and the xy-coordinates of its center of mass. Model the plate as a closed and bounded region R in the plane on which there is defined a continuous density function $\rho = \rho(x, y)$. Choose an inner partition $\mathcal{P} = \{R_1, \ldots, R_k\}$ of R and evaluation points $(u_i, v_i) \in R_i$. When $\|\mathcal{P}\|$ is small, the density $\rho(x, y) \approx \rho(u_i, v_i)$ at each point $(x, y) \in R_i$. So the mass of R_i is approximately

$$m_i \approx \rho(u_i, v_i) \Delta A_i$$

and the total mass of the plate is approximately

$$m \approx \sum_{i=1}^{k} \rho\left(u_i, v_i\right) \Delta A_i.$$

Thus

$$m = \lim_{\|\mathcal{P}\| \to 0} \sum_{i=1}^{k} \rho\left(u_i, v_i\right) \Delta A_i = \iint_R \rho\left(x, y\right)\ dA.$$

Furthermore, the system of point-masses $m_1, \ldots, m_k$ at horizontal positions $u_1, \ldots, u_k$ has first moment

$$M_y = \sum_{i=1}^{k} m_i u_i = \sum_{i=1}^{k} \rho\left(u_i, v_i\right) u_i \Delta A_i$$

about the y-axis and at vertical positions $v_1, \ldots, v_k$ has first moment

$$M_x = \sum_{i=1}^{k} m_i v_i = \sum_{i=1}^{k} \rho\left(u_i, v_i\right) v_i \Delta A_i$$

about the x-axis. Again, let $\|\mathcal{P}\| \to 0$, then the first moments of the lamina R are exactly

$$M_y = \lim_{\|\mathcal{P}\| \to 0} \sum_{i=1}^{k} \rho\left(u_i, v_i\right) u_i \Delta A_i = \iint_R \rho\left(x, y\right) x\ dA$$

and

$$M_x = \lim_{\|\mathcal{P}\| \to 0} \sum_{i=1}^{k} \rho\left(u_i, v_i\right) v_i \Delta A_i = \iint_R \rho\left(x, y\right) y\ dA.$$

Definition 3 *The xy-coordinates of the* <u>*center of mass*</u> *are*

$$\overline{x} \approx \frac{M_y}{m} \text{ and } \overline{y} \approx \frac{M_x}{m}.$$