Solution to Problem #61, p. 994
Math 261 Fall, 2003

Problem: *Construct the function* $d(x, y)$ *giving the distance from a point* (x, y, z) *on the paraboloid* $z = 4 - x^2 - y^2$ *to the point* $(3, -2, 1)$. *Then determine the point that minimizes* $d(x, y)$.

Solution: Let's use the method of Lagrange Multipliers. Instead of $d(x, y)$, let's minimize the square of the distance from (x, y, z) to $(3, -2, 1)$ (this avoids nasty square roots with messy derivatives):

$$f(x, y, z) = [d(x, y, z)]^2 = (x-3)^2 + (y+2)^2 + (z-1)^2.$$

The domain of the function f is all of space $\mathbb{R}^3$. To restrict f to the paraobloid, define the function $g(x, y, z) = x^2 + y^2 + z$; then the paraboloid $x^2 + y^2 + z = 4$ is the level surface of height 4 for g. Our objective is to minimize f along the paraboloid $g(x, y, z) = 4$. Applying the method of Lagrange multipliers we set $\nabla f = \lambda \nabla g$ and get:

$$\langle 2x-6, 2y+4, 2z-2 \rangle = \lambda \langle 2x, 2y, 1 \rangle.$$

Multiplying both sides by $\frac{1}{2}$ gives

$$\langle x-3, y+2, z-1 \rangle = \lambda \left\langle x, y, \frac{1}{2} \right\rangle.$$

Equating components gives the following non-linear system of four equations in four unknowns

$$\begin{aligned} x^2 + y^2 + z &= 4 \\ x - 3 &= \lambda x \\ y + 2 &= \lambda y \\ z - 1 &= \tfrac{\lambda}{2} \end{aligned}$$

Now if $\lambda = 1$, then $x - 3 = x$ and $-3 = 0$, which is a contradiction, so $\lambda \neq 1$. Therefore the system above can be rewritten as

$$\begin{aligned} x^2 + y^2 + z &= 4 \\ x &= \tfrac{3}{1-\lambda} \\ y &= \tfrac{-2}{1-\lambda} \\ z &= \tfrac{\lambda+2}{2}. \end{aligned}$$

Substituting for x, y, z in the first equation gives

$$\left(\frac{3}{1-\lambda}\right)^2 + \left(\frac{-2}{1-\lambda}\right)^2 + \frac{\lambda+2}{2} = 4;$$

and multiplying both sides by $2(1-\lambda)^2$ to clear fractions gives

$$\begin{aligned} (1-\lambda)^2(\lambda+2) + 26 &= 8(1-\lambda)^2 \\ \lambda^3 + 2\lambda^2 - 2\lambda^2 - 4\lambda + \lambda + 2 + 26 &= 8\lambda^2 + 16\lambda + 8 \\ \lambda^3 - 8\lambda^2 + 13\lambda + 20 &= 0 \end{aligned}$$

This cubic equation has one real root. Using Newton's method we obtain $\lambda \approx -.9361$, in which case

$$\begin{aligned} x &\approx \tfrac{3}{1-(-.9361)} \approx 1.5495 \\ y &\approx \tfrac{-2}{1-(-.9361)} \approx -1.033 \\ z &\approx \tfrac{-.9361+2}{2} \approx .5319 \end{aligned}$$

Note that $1.5495^2 + 1.033^2 + .5319 = 3.9999$, so this answer checks out. Finally, the minimim distance

$$d \approx \sqrt{(1.5495-3)^2 + (-1.033+2)^2 + (.5319-1)^2} \approx 1.805.$$