Why Simply-Connected?
Math 311

Consider the vector field $\mathbf{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$. Let C_n be the unit circle parametrized by $x = \cos 2\pi t$ and $y = \sin 2\pi t$ with $0 \leq t \leq n$, where n is a positive integer. Note that C_n is a closed curve for each n since it begins and ends at $(1,0)$; we call n the *winding number* of C_n since n is the number of times the particle tracing out C_n winds around the origin. Now

$$\int_{C_n} \mathbf{F} \bullet d\mathbf{r} = \int_0^n (-\sin 2\pi t)(-\sin 2\pi t\ dt) + (\cos 2\pi t)(\cos 2\pi t\, dt) = \int_0^n dt = n,$$

so different values of n give different values for the line integral. Thus the vector field $\mathbf{F}$ fails to be conservative even though

$$\begin{aligned}\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} &= \frac{\left(x^2+y^2\right) - x(2x)}{\left(x^2+y^2\right)^2} - \frac{\left(x^2+y^2\right)(-1) - (-y)(2y)}{\left(x^2+y^2\right)^2} \\ &= \frac{x^2+y^2-2x^2+x^2+y^2-2y^2}{\left(x^2+y^2\right)^2} = 0.\end{aligned}$$

Theorem: *If* $\mathbf{F} = \langle M, N \rangle$ *is defined on a simply-connected region* R *and the component functions* M *and* N *have continuous first partials in* R, *then* $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$ *if and only if* $\mathbf{F}$ *is conservative in* R.

Here's why our observations above do not contradict this theorem: Since C_n encircles the origin, every simply-connected region R containing C_n contains the origin. But $\mathbf{F}$ is undefined at the origin, so the hypotheses of the theorem are not satisfied for any simply-connected region R containing C_n and the conclusion that $\mathbf{F}$ is conservative in R does not follow.