UNIT NORMALS ON A SURFACE
Math 311

There are two ways to define a surface S in space:

1. As the graph of a function $f(x, y)$ or
2. As a level surface of some function $g(x, y, z) = 0$.

In the later case we say that S is *implicitly defined by* g. A unit normal vector field $\mathbf{n}$ on S is given by

$$\mathbf{n} = \frac{\nabla g}{\|\nabla g\|} = \frac{\langle g_x, g_y, g_z\rangle}{\sqrt{g_x^2 + g_y^2 + g_z^2}}.$$

If S is a closed surface such as a sphere, all vectors of $\mathbf{n}$ point inward or outward (one or the other!) depending on the function g. If we desire an outward normal, we need to sketch the surface, picture $\mathbf{n}$ at some point on the surface, and if necessary, reverse the sign of $\mathbf{n}$.

Example: One way to define the sphere is by setting

$$g(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0.$$

A unit normal vector field $\mathbf{n}$ is given by

$$\mathbf{n} = \frac{\langle 2x, 2y, 2z\rangle}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{\langle x, y, z\rangle}{\sqrt{x^2 + y^2 + z^2}} = \left\langle \frac{x}{a}, \frac{y}{b}, \frac{z}{c} \right\rangle.$$

Note that the vectors of $\mathbf{n}$ point radially outward from the origin with their initial points on the sphere, so $\mathbf{n}$ is the outward unit normal. On the other hand, if we set

$$g(x, y, z) = a^2 - x^2 - y^2 - z^2 = 0,$$

then

$$\mathbf{n} = \frac{\langle -2x, -2y, -2z\rangle}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{-\langle x, y, z\rangle}{\sqrt{x^2 + y^2 + z^2}} = -\left\langle \frac{x}{a}, \frac{y}{b}, \frac{z}{c} \right\rangle$$

is the inward unit normal.

When S is the graph of a function $f(x, y)$, we find $\mathbf{n}$ just as we did above by setting

$$g(x, y, z) = z - f(x, y) = 0.$$

Then

$$\mathbf{n} = \frac{\nabla g}{\|\nabla g\|} = \frac{\langle -f_x, -f_y, 1\rangle}{\sqrt{f_x^2 + f_y^2 + 1}}.$$

Note that the z component of $\mathbf{n}$ is positive, so the vectors of $\mathbf{n}$ point upward. This is the *upward unit normal on* S. If instead we set

$$g(x,y,z) = f(x,y) - z = 0$$

we obtain the downward unit normal

$$\mathbf{n} = \frac{\nabla g}{\|\nabla g\|} = \frac{\langle f_x, f_y, -1\rangle}{\sqrt{f_x^2 + f_y^2 + 1}}.$$

Example: The graph of the function $f(x,y) = x^2 - y^2$ is a hyperbolic paraboloid S. To find the upward unit normal vector field on S, set

$$g(x,y,z) = z - x^2 + y = 0.$$

Then the upward unit normal on S is given by

$$\mathbf{n} = \frac{\nabla g}{\|\nabla g\|} = \frac{\langle -y, -x, 1\rangle}{\sqrt{x^2 + y^2 + 1}}.$$