Chapter 3

Classification of Isometries

From our work in Chapter 1 we know that reflections, translations, glide reflections and rotations are isometries. Furthermore, the Fundamental Theorem tells us that every isometry is a product of three or fewer reflections. In this chapter we consider the following question: Is there a way to compose three reflections and obtain an isometry other than a reflection, translation, glide reflection or rotation? As we shall soon see, the surprising answer is no! We begin with a special case of the Angle Addition Theorem, which will be discussed in full generality later in this chapter.

3.1 The Angle Addition Theorem, part I

Rotations and translations are closely related. We shall observe that the product of two rotations with arbitrary centers is either a rotation or a translation. We begin by identifying the isometries that are involutions. Recall that the identity can be thought of as a rotation about an arbitrary point C through directed angle $\Theta \in 0^\circ$ or as a translation from a point P to itself, i.e., $\tau_{\mathbf{PP}}$.

Theorem 97 *Every involutory isometry is either a reflection or a halfturn.*

Proof. Reflections are involutory isometries, so consider an involutory isometry α that is not a reflection. We must show that α is a halfturn. Since $\alpha \neq \iota$, there exist distinct points P and P' such that $\alpha(P) = P'$. Apply α to both sides and obtain

$$\alpha^2(P) = \alpha(P').$$

Since α is an involution, $\alpha^2(P) = P$. Hence $\alpha(P') = P$ and we see that α interchanges the points P and P'. Let M be the midpoint of P and P'; then

$$PM = MP'.$$

I claim that M is a fixed point. Let $M' = \alpha(M)$. Since α is an isometry,

$$MP' = PM' = M'P'$$

in which case M' is equidistant from P and P' and lies on the perpendicular bisector of $\overline{PP'}$. But

$$PP' = PM + MP' = PM' + M'P'$$

so that P, M', and P' are collinear by the triangle inequality. Thus $PP' = 2PM'$ and M' is the midpoint of P and P'. It follows that

$$\alpha(M) = M' = M.$$

Since M is a fixed point, Theorem 91 tells us that α is either a non-identity rotation about the point M or a reflection in some line containing M. By assumption, α is not a reflection so it must be a non-identity rotation about the point M, i.e., $\alpha = \rho_{M,\Theta} \neq \iota$, where $\Theta \notin 0°$. By Proposition 71,

$$\alpha^2 = \rho^2_{M,\Theta} = \rho_{M,2\Theta}.$$

Since $\alpha^2 = \iota$ we have

$$\rho_{M,2\Theta} = \iota,$$

in which case $2\Theta \in 0° = 360°$ and $\Theta \in 180°$. Therefore the rotation is a halfturn as claimed. ■

We know from Proposition 55 that a halfturn φ_C fixes a line ℓ if and only if C is on line ℓ. But which lines are fixed by a general rotation?

Theorem 98 *Non-identity rotations that fix a line are halfturns.*

Proof. Let $\rho_{C,\Theta}$ be a non-identity rotation that fixes line ℓ, i.e., $\ell = \rho_{C,\Theta}(\ell)$. Let m be the line through C perpendicular to ℓ. By Corollary 85, there is a line n through C such that

$$\rho_{C,\Theta} = \sigma_n \circ \sigma_m$$

(see Figure 3.1).

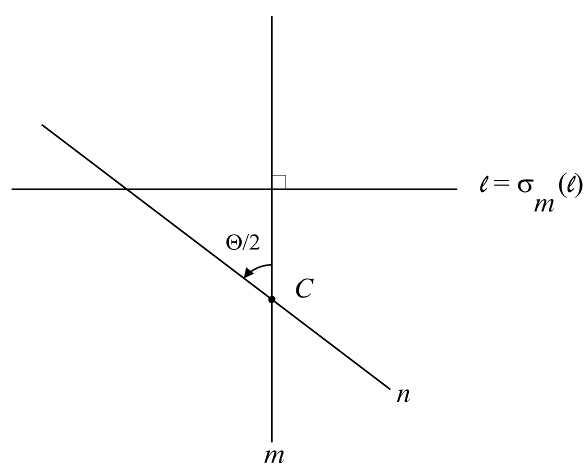


Figure 3.1.

Since ℓ and m are perpendicular, $\sigma_m(\ell) = \ell$. Therefore

$$\ell = \rho_{C,\Theta}(\ell) = (\sigma_n \circ \sigma_m)(\ell) = \sigma_n(\sigma_m(\ell)) = \sigma_n(\ell)$$

and σ_n fixes line ℓ. Either $n = \ell$ or $n \perp \ell$. But lines m and n intersect at point C and $m \perp \ell$. So we must have $n = \ell$. Therefore $\rho_{C,\Theta} = \sigma_\ell \circ \sigma_m$, which is a halfturn by Corollary 87. ■

A technique similar to the one used in Section 2.3 to transform a product of halfturns into a product of reflections in parallel lines can be applied to a pair of general rotations and gives part I of the *Angle Addition Theorem:*

Theorem 99 ***(The Angle Addition Theorem, part I)*** *Let Θ and Φ be real numbers such that $\Theta + \Phi \notin 0^\circ$. Then there is a unique point C such that*

$$\rho_{B,\Phi} \circ \rho_{A,\Theta} = \rho_{C,\Theta+\Phi}.$$

Proof. If $A = B$, then $C = A$ and the result was proved earlier in Proposition 71. So assume that $A \neq B$ and let $\ell = \overleftrightarrow{AB}$. By Corollary 85, there exist unique lines m and n passing through A and B, respectively, such that

$$\rho_{B,\Phi} = \sigma_n \circ \sigma_\ell \text{ and } \rho_{A,\Theta} = \sigma_\ell \circ \sigma_m$$

(see Figure 3.2).

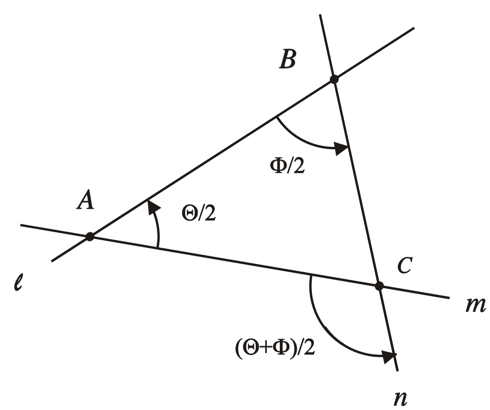


Figure 3.2.

The measure of the directed angle from m to ℓ is $\frac{1}{2}\Theta$ and the measure of the directed angle from ℓ to n is $\frac{1}{2}\Phi$. Since $\frac{1}{2}(\Theta^\circ+\Phi^\circ) < 180^\circ$, lines m and n intersect at some unique point C. The Exterior Angle Theorem applied to $\triangle ABC$ implies that the measure of the angle from m to n is $\frac{1}{2}(\Theta^\circ + \Phi^\circ)$. Hence

$$\rho_{B,\Phi} \circ \rho_{A,\Theta} = (\sigma_n \circ \sigma_\ell) \circ (\sigma_\ell \circ \sigma_m) = \sigma_n \circ \sigma_m = \rho_{C,\Theta+\Phi}.$$

■

Exercises

1. Let $O = \begin{bmatrix}0\\0\end{bmatrix}$ and $C = \begin{bmatrix}2\\0\end{bmatrix}$.
   a. Find equations of lines ℓ, m and n such that $\rho_{C,90} = \sigma_m \circ \sigma_n$ and $\rho_{O,90} = \sigma_\ell \circ \sigma_m$.

   b. Find xy-coordinates of the point D such that $\varphi_D = \rho_{O,90} \circ \rho_{C,90}$.

   c. Find xy-coordinates for the point E such that $\varphi_E = \rho_{C,90} \circ \rho_{O,90}$.

2. Let $O = \begin{bmatrix}0\\0\end{bmatrix}$ and $C = \begin{bmatrix}0\\1\end{bmatrix}$.
   a. Find equations of the lines ℓ, m and n such that $\varphi_O = \sigma_m \circ \sigma_\ell$ and $\rho_{C,120} = \sigma_n \circ \sigma_m$.

   b. Find xy coordinates for the point D and the angle of rotation Θ such that $\rho_{D,\Theta} = \rho_{C,120} \circ \varphi_O$.

   c. Find xy-coordinates for the point E and the angle of rotation Φ such that $\rho_{E,\Phi} = \rho_{C,120} \circ \rho_{O,60}$.

3. Let $A = \begin{bmatrix}4\\0\end{bmatrix}$ and $B = \begin{bmatrix}0\\4\end{bmatrix}$.
   a. Find equations of lines ℓ, m and n such that $\rho_{A,90} = \sigma_m \circ \sigma_\ell$ and $\rho_{B,120} = \sigma_n \circ \sigma_m$.

   b. Find xy coordinates for the point C and the angle of rotation Θ such that $\rho_{C,\Theta} = \rho_{B,120} \circ \rho_{A,90}$.

   c. Find xy-coordinates for the point D and the angle of rotation Φ such that $\rho_{D,\Phi} = \rho_{A,90} \circ \rho_{B,120}$.

4. If ℓ, m and n are the angle bisectors of a triangle, prove that the line p such that $\sigma_p = \sigma_n \circ \sigma_m \circ \sigma_\ell$ is perpendicular to a side of the triangle.

3.2 Parity

An isometry has even parity if it factors as an even number of reflections; otherwise it has odd parity. In this section we shall observe that every even isometry is the identity, a non-identity translation, or a non-identity rotation and every odd involutory isometry is a reflection. The case of non-involutory isometries with odd parity will be addressed in a later section.

Before we can make the notion of parity precise, we must be sure that a product of reflections with an even number of factors can never be simplified to an odd number of factors and visa-versa, which is the point of our next lemma and theorems that follow.

Lemma 100 *Given a point P and lines a and b, there exist lines c and d with c passing through P such that*

$$\sigma_b \circ \sigma_a = \sigma_d \circ \sigma_c.$$

Proof. If $a \| b$, let c be the line through P parallel to a and b. By Corollary 81, there exists a (unique) line d parallel to a, b, and c such that $\sigma_d = \sigma_b \circ \sigma_a \circ \sigma_c$. Multiplying both sides of this equation on the right by σ_c gives the result in this case. On the other hand, suppose that a and b intersect at a point C. Let $c = \overleftrightarrow{CP}$; then lines a, b, and c are concurrent at point C. By Corollary 86, there exists a (unique) line d passing through C such that $\sigma_d = \sigma_b \circ \sigma_a \circ \sigma_c$ (see Figure 3.3). Multiplying both sides of this equation on the right by σ_c gives the result in this case as well.

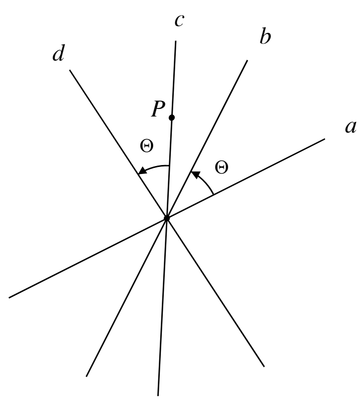


Figure 3.3.

■

Theorem 101 *A product of four reflections can always be simplified to a product of two reflections, i.e., given lines p, q, r, and s, there exist lines ℓ and m such that*

$$\sigma_s \circ \sigma_r \circ \sigma_q \circ \sigma_p = \sigma_m \circ \sigma_\ell.$$

Proof. Consider the product $\sigma_s \circ \sigma_r \circ \sigma_q \circ \sigma_p$. Let P be a point on line p. Apply Lemma 100 to the point P and given lines q and r to obtain lines q' and r' with q' passing through P such that

$$\sigma_r \circ \sigma_q = \sigma_{r'} \circ \sigma_{q'}.$$

Next apply Lemma 100 to the point P and lines r' and s to obtain lines r'' and m with r'' passing through P such that

$$\sigma_s \circ \sigma_{r'} = \sigma_m \circ \sigma_{r''}$$

(see Figure 3.4).

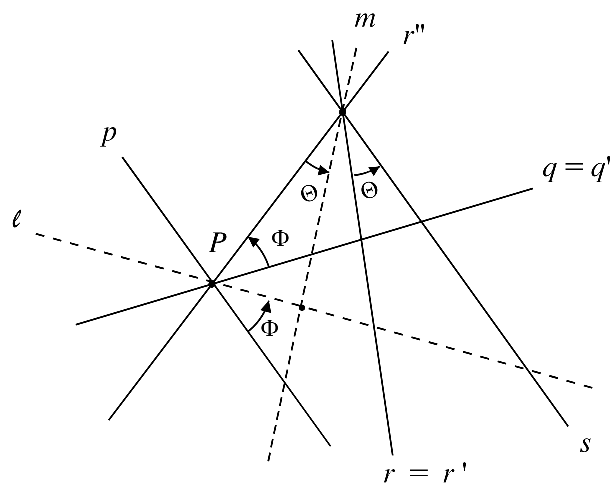


Figure 3.4.

Since p, q', and r'' are concurrent at point P, Corollary 86 tells us that there exists a (unique) line ℓ passing through P such that

$$\sigma_\ell = \sigma_{r''} \circ \sigma_{q'} \circ \sigma_p.$$

Therefore,

$$\sigma_s \circ \sigma_r \circ \sigma_q \circ \sigma_p = \sigma_s \circ \sigma_{r'} \circ \sigma_{q'} \circ \sigma_p = \sigma_m \circ \sigma_{r''} \circ \sigma_{q'} \circ \sigma_p = \sigma_m \circ \sigma_\ell.$$

■

By repeatedly applying Theorem 101 to an even length string of reflections, we can reduce the string to two reflections. On the other hand, repeated applications of Theorem 101 to an odd length string of reflections reduces it to either one or a product of three reflections. Therefore we need:

Theorem 102 *A product of two reflections never equals one or a product of three reflections.*

Proof. The product of two reflections is the identity, a rotation, or a translation, which fix every point, exactly one point, or no points, respectively. On the other hand, a reflection fixes every point on its reflecting line and no other point. So the product of two reflections can never equal one reflection. Now arguing indirectly, suppose there exist lines p, q, r, s, and t such that

$$\sigma_r \circ \sigma_q \circ \sigma_p = \sigma_s \circ \sigma_t.$$

Multiplying both sides of this equation on the left by σ_s gives

$$\sigma_s \circ \sigma_r \circ \sigma_q \circ \sigma_p = \sigma_t.$$

But by Theorem 101, there exist lines ℓ and m such that

$$\sigma_m \circ \sigma_\ell = \sigma_s \circ \sigma_r \circ \sigma_q \circ \sigma_p = \sigma_t,$$

which contradicts our earlier conclusion that two reflections can never equal one reflection. Therefore a product of three reflections can never equal a product of two reflections. ■

Corollary 103 *Every product of reflections with an even number of factors is a product of two reflections. Every product of reflections with an odd number of factors is either a single reflection or a product of three reflections.*

Definition 104 *An isometry α is even if and only if α is a product of an even number of reflections; otherwise α is odd.*

In light of Corollary 103, Theorem 78, and Theorem 83, we see that every even isometry is the identity, a non-identity translation, or a non-identity rotation.

Theorem 105 *Every even involutory isometry is a halfturn; every odd involutory isometry is a reflection.*

Proof. By Theorem 97, every involutory isometry is a halfturn or a reflection. As noted above, every even isometry is a translation or a rotation. Therefore every even involutory isometry is a halfturn. Furthermore, Corollary 103 tells us that every odd isometry is either a single reflection or a product of three reflections. Therefore every odd involutory isometry is a reflection. ■

The notion of parity is closely related to "orientation", which we now define. Choose an ordered basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for the plane $\mathbb{R}^2$ as a vector space over $\mathbb{R}$ and position $\mathbf{v}_1$ and $\mathbf{v}_2$ with their initial points at the origin O. Let A and B be their respective terminal points and let $\Theta^\circ = m\angle AOB$. (see Figure 3.5).

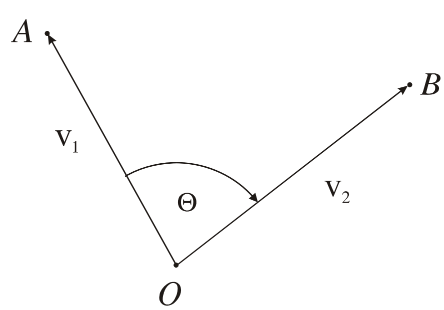


Figure 3.5.

Definition 106 *An orientation of the plane is a choice of ordered basis* $\{\mathbf{u}, \mathbf{v}\}$. *Given an orientation* $\{\mathbf{u}, \mathbf{v}\}$, *let* Θ° *be the measure of the (undirected) angle with initial side* $\mathbf{u}$ *and terminal side* $\mathbf{v}$. *The orientation is negative if* $-180^\circ < \Theta^\circ < 0^\circ$ *and positive if* $0^\circ < \Theta^\circ < 180^\circ$.

Theorem 107 *The sign of an orientation* $\left\{\mathbf{u} = \left[\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right], \mathbf{v} = \left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right]\right\}$ *is the sign of the determinant*

$$\det [\mathbf{u}|\mathbf{v}] = u_1 v_2 - u_2 v_1.$$

Proof. Let α and β be respective direction angles for $\mathbf{u}$ and $\mathbf{v}$; then $\mathbf{u} = \left[\begin{smallmatrix} \|\mathbf{u}\| \cos \alpha \\ \|\mathbf{u}\| \sin \alpha \end{smallmatrix}\right]$, $\mathbf{v} = \left[\begin{smallmatrix} \|\mathbf{v}\| \cos \beta \\ \|\mathbf{v}\| \sin \beta \end{smallmatrix}\right]$ and the (undirected) angle from $\mathbf{u}$ to $\mathbf{v}$ is $(\beta - \alpha)^\circ$. Hence

$$\begin{aligned} a_1 b_2 - b_1 a_2 &= (\|\mathbf{u}\| \cos \alpha)(\|\mathbf{v}\| \sin \beta) - (\|\mathbf{v}\| \cos \beta)(\|\mathbf{u}\| \sin \alpha) \\ &= \|\mathbf{u}\| \|\mathbf{v}\| (\sin \beta \cos \alpha - \cos \beta \sin \alpha) \\ &= \|\mathbf{u}\| \|\mathbf{v}\| \sin (\beta - \alpha), \end{aligned}$$

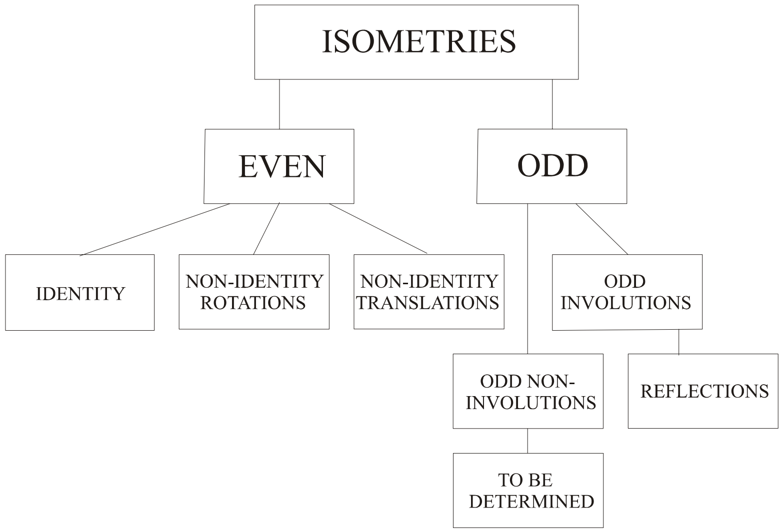
which is negative if and only if $-180^\circ < (\beta - \alpha)^\circ < 0^\circ$ and positive if and only if $0^\circ < (\beta - \alpha)^\circ < 180^\circ$. ■

Definition 108 *A transformation* α *is orientation preserving if the orientations* $\{\mathbf{u}, \mathbf{v}\}$ *and* $\{\alpha(\mathbf{u}), \alpha(\mathbf{v})\}$ *have the same sign; otherwise* α *is orientation reversing.*

The following fact is intuitively obvious. We leave the proof to the reader as a series of exercises in this and the next sections (see Exercises 10, 11, 9 and 10).

Proposition 109 *Even isometries preserve orientation; odd isometries reverse orientation.*

Here is a chart that summarizes the results obtained so far:



Exercises

In problems 1-5, the respective equations of lines p, q, r and s are given. In each case, find lines ℓ and m such that $\sigma_m \circ \sigma_\ell = \sigma_s \circ \sigma_r \circ \sigma_q \circ \sigma_p$ and identify the isometry $\sigma_m \circ \sigma_\ell$ as a translation, a rotation or the identity.

1. $Y = X$, $Y = X + 1$, $Y = X + 2$ and $Y = X + 3$.

2. $Y = X$, $Y = X + 1$, $Y = X + 2$ and $Y = -X$.

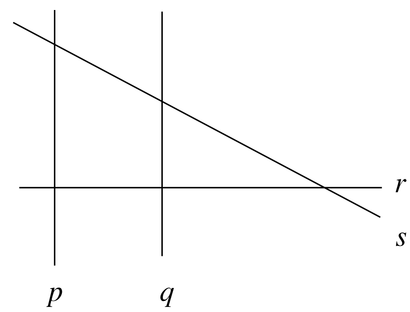
3. $X = 0$, $Y = 0$, $Y = X$ and $Y = -X$.

4. $X = 0$, $Y = 0$, $Y = 2$, and $X = 2$.

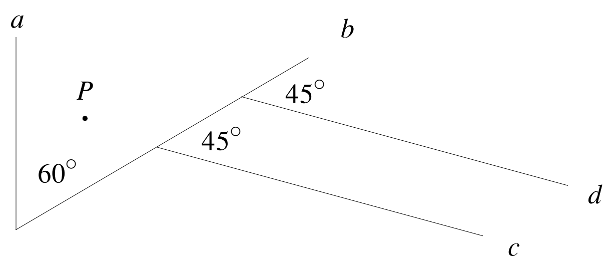
5. $X = 0$, $Y = 0$, $X = 1$ and $Y = X + 2$.

6. $X = 0$, $Y = 0$, $Y = X + 1$ and $Y = -X + 2$.

7. In the diagram below, use a MIRA to construct lines ℓ and m such that $\sigma_m \circ \sigma_\ell = \sigma_s \circ \sigma_r \circ \sigma_q \circ \sigma_p$.



8. Consider the following diagram:



   a. Use a MIRA to construct the point Q such that $\rho_{Q,\Theta} = \sigma_b \circ \varphi_P \circ \sigma_a$ and find the rotation angle Θ.

   b. Use a MIRA to construct the point R such that $\rho_{R,\Phi} = \sigma_d \circ \sigma_c \circ \sigma_b \circ \sigma_a$ and find the rotation angle Φ.

9. Prove that an even isometry fixing two distinct points is the identity.

10. Prove that a translation preserves orientation.

11. Prove that the reflection in a line through the origin reverses orientation.

3.3 The Geometry of Conjugation

In this section we give geometrical meaning to algebraic conjugation, which plays a key role in the classification of isometries. You've seen conjugation before. For example, to rationalize the denominator of $\frac{1}{3+\sqrt{2}}$ we *conjugate by* $3-\sqrt{2}$, i.e., we multiply by $3-\sqrt{2}$ and its multiplicative inverse:

$$\begin{aligned} \frac{1}{3+\sqrt{2}} &= \left(3-\sqrt{2}\right)\left(\frac{1}{3+\sqrt{2}}\right)\left(3-\sqrt{2}\right)^{-1} \\ &= \left(3-\sqrt{2}\right)\left(\frac{1}{3+\sqrt{2}}\right)\left(\frac{1}{3-\sqrt{2}}\right) = \frac{3-\sqrt{2}}{7}. \end{aligned} \tag{3.1}$$

However, you are probably used to seeing conjugation by $3-\sqrt{2}$ expressed as multiplication by the fraction $\frac{3-\sqrt{2}}{3-\sqrt{2}}$ rather than the seemingly awkward expression given above. We are free to use the familiar fraction form precisely because *multiplication of real numbers is commutative*; thus we can interchange the positions of real numbers $3-\sqrt{2}$ and $\frac{1}{3+\sqrt{2}}$ above and multiply. But when multiplication non-commutative, as in the study of isometries in which multiplication is function composition, conjugation takes the form in (3.1) above.

Definition 110 *Let α and β be isometries. The conjugate of α by β is the isometry*

$$\beta \circ \alpha \circ \beta^{-1}.$$

In general, $\alpha = \beta \circ \alpha \circ \beta^{-1}$ if and only if α and β commute i.e., $\alpha \circ \beta = \beta \circ \alpha$.

We saw an important example of conjugation in Chapter 1 when we derived the equations for a general rotation about a point C other than the origin O by conjugating a rotation about O by the translation from O to C (see the proof of Theorem 69). Precisely, $\rho_{C,\Theta}$ is obtained by

1. *translating from C to O followed by*
2. *rotating about O through angle Θ followed by*
3. *translating from O to C.*

Thus

$$\rho_{C,\Theta} = \tau_{\mathbf{OC}} \circ \rho_{O,\Theta} \circ \tau_{\mathbf{OC}}^{-1}, \tag{3.2}$$

i.e., $\rho_{C,\Theta}$ *is the conjugate of* $\rho_{O,\Theta}$ *by* $\tau_{\mathbf{OC}}$. Indeed, since rotations and translation do not commute, the equations for rotations about C have a form quite different from those for rotations about O.

Here are some algebraic properties of conjugation.

Theorem 111

a. *The square of a conjugate is the conjugate of the square.*

b. *The conjugate of an involution is an involution.*

Proof. Let α and β be isometries.

(a). $\left(\alpha \circ \beta \circ \alpha^{-1}\right)^2 = \alpha \circ \beta \circ \alpha^{-1} \circ \alpha \circ \beta \circ \alpha^{-1} = \alpha \circ \beta^2 \circ \alpha^{-1}$.

(b). If β is an involution then $\alpha \circ \beta \circ \alpha^{-1} \neq \iota$ (otherwise $\alpha \circ \beta \circ \alpha^{-1} = \iota$ implies $\beta = \alpha^{-1} \circ \alpha = \iota$). By part (a), $\left(\alpha \circ \beta \circ \alpha^{-1}\right)^2 = \alpha \circ \beta^2 \circ \alpha^{-1} = \alpha \circ \iota \circ \alpha^{-1} = \alpha \circ \alpha^{-1} = \iota$ and $\alpha \circ \beta \circ \alpha^{-1}$ is an involution. ■

Here are some geometrical consequences.

Proposition 112 *Conjugation preserves parity, i.e., if α and β are isometries, then β and $\alpha \circ \beta \circ \alpha^{-1}$ have the same parity; both either preserve orientation or reverse orientation.*

Proof. By Theorem 92 we know that α factors as a product of reflections. Since the inverse of a product is the product of the inverses in reverse order, α and α^{-1} have the same parity and together contribute an even number of factors to every factorization of $\alpha \circ \beta \circ \alpha^{-1}$ as a product of reflections. Therefore the parity of an isometry β is the same as the parity of $\alpha \circ \beta \circ \alpha^{-1}$. The fact that β and $\alpha \circ \beta \circ \alpha^{-1}$ both either preserve orientation or reverse orientation follows immediately from Proposition 109. ■

Theorem 113 *Let α be an isometry.*

a. *Every conjugate of a halfturn is a halfturn. Furthermore, if P is any point, then*
$$\alpha \circ \varphi_P \circ \alpha^{-1} = \varphi_{\alpha(P)}.$$

b. *Every conjugate of a reflection is a reflection. Furthermore, if ℓ is any line, then*
$$\alpha \circ \sigma_\ell \circ \alpha^{-1} = \sigma_{\alpha(\ell)}.$$

Proof. (a) Since φ_P is even by Corollary 87, so is $\alpha \circ \varphi_P \circ \alpha^{-1}$ by Proposition 112. Furthermore, $\alpha \circ \varphi_P \circ \alpha^{-1}$ is an involution by Theorem 111. By Theorem 97, involutory isometries are either reflections (which are odd) or halfturns (which are even), so $\alpha \circ \varphi_P \circ \alpha^{-1}$ is a halfturn. To locate its center, observe that

$$\left(\alpha \circ \varphi_P \circ \alpha^{-1}\right)(\alpha(P)) = \left(\alpha \circ \varphi_P \circ \alpha^{-1} \circ \alpha\right)(P) = (\alpha \circ \varphi_P)(P) = \alpha(P)$$

so $\alpha(P)$ is a fixed point for the halfturn $\alpha \circ \varphi_P \circ \alpha^{-1}$ and is consequently its center, i.e.,
$$\alpha \circ \varphi_P \circ \alpha^{-1} = \varphi_{\alpha(P)}.$$

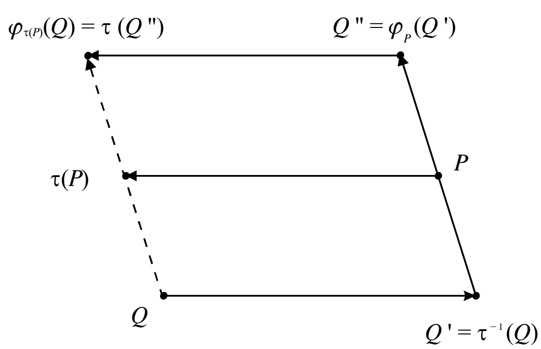


Figure 3.6: Conjugation of φ_P by τ.

(b) Similarly, since σ_ℓ is odd, so is $\alpha \circ \sigma_\ell \circ \alpha^{-1}$ has odd parity by Proposition 112. Furthermore $\alpha \circ \sigma_\ell \circ \alpha^{-1}$ is an involution by Theorem 111. By Theorem 97 involutory isometries are either halfturns or reflections, so $\alpha \circ \sigma_\ell \circ \alpha^{-1}$ is a reflection. To determine its reflecting line, observe that for any point P on ℓ

$$\left(\alpha \circ \sigma_\ell \circ \alpha^{-1}\right)(\alpha(P)) = \left(\alpha \circ \sigma_\ell \circ \alpha^{-1} \circ \alpha\right)(P) = (\alpha \circ \sigma_\ell)(P) = \alpha(P)$$

so the line $\alpha(\ell)$ is fixed pointwise by the reflection $\alpha \circ \sigma_\ell \circ \alpha^{-1}$. Consequently, $\alpha(\ell)$ is the reflecting line and

$$\alpha \circ \sigma_\ell \circ \alpha^{-1} = \sigma_{\alpha(\ell)}.$$

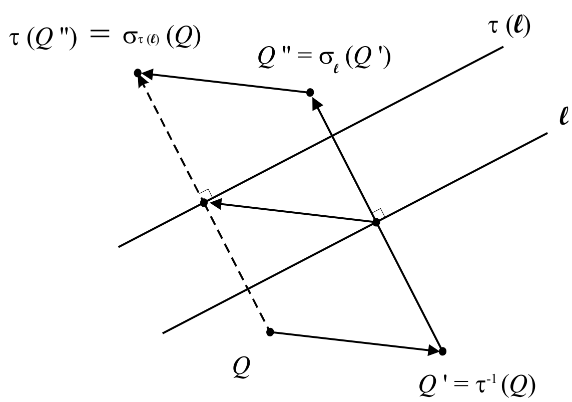


Figure 3.7: Conjugation of σ_ℓ by τ.

■

Theorem 114 *Every conjugate of a translation is a translation. If α is an isometry, A and B are points, $A' = \alpha(A)$, and $B' = \alpha(B)$, then*

$$\alpha \circ \tau_{\mathbf{AB}} \circ \alpha^{-1} = \tau_{\mathbf{A'B'}}.$$

Proof. Let M be the midpoint of A and B; then

$$\tau_{\mathbf{AB}} = \varphi_M \circ \varphi_A$$

by Theorem 56. Since α is an isometry, $M' = \alpha(M)$ is the midpoint of $A' = \alpha(A)$ and $B' = \alpha(B)$. So again by Theorem 56,

$$\tau_{\mathbf{A'B'}} = \varphi_{M'} \circ \varphi_{A'}.$$

Therefore, by Theorem 113 (part a) we have

$$\begin{aligned}\alpha \circ \tau_{\mathbf{AB}} \circ \alpha^{-1} &= \alpha \circ (\varphi_M \circ \varphi_A) \circ \alpha^{-1} \\ &= (\alpha \circ \varphi_M \circ \alpha^{-1}) \circ (\alpha \circ \varphi_A \circ \alpha^{-1}) \\ &= \varphi_{M'} \circ \varphi_{A'} = \tau_{\mathbf{A'B'}}.\end{aligned}$$

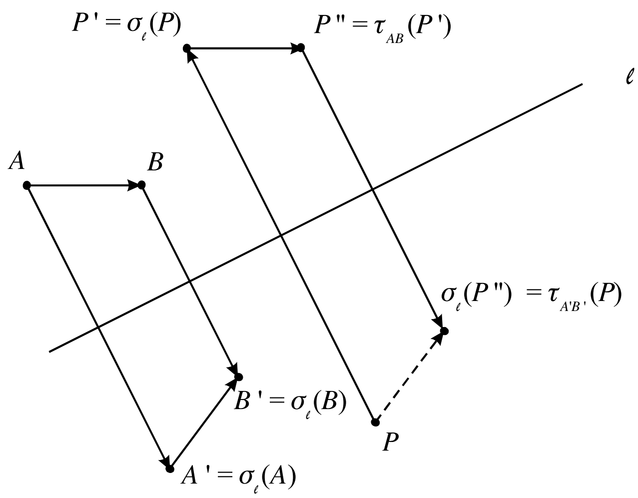


Figure 3.8: Conjugation of $\tau_{\mathbf{AB}}$ by σ_ℓ.

■

Our next theorem generalizes the remarks related to (3.2) above:

Theorem 115 *The conjugate of a rotation is a rotation. If α is an isometry, C is a point and $\Theta \in \mathbb{R}$, then*

$$\alpha \circ \rho_{C,\Theta} \circ \alpha^{-1} = \begin{cases} \rho_{\alpha(C),\Theta} & \text{if } \alpha \text{ is even} \\ \rho_{\alpha(C),-\Theta} & \text{if } \alpha \text{ is odd} \end{cases}.$$

Proof. Since α is the a product of three or fewer reflections, we consider each of the three possible factorizations of α separately.

Case 1: Let r be a line and let $\alpha = \sigma_r$. Then $\alpha = \alpha^{-1}$ is odd and we must show that $\sigma_r \circ \rho_{C,\Theta} \circ \sigma_r = \rho_{\sigma_r(C),-\Theta}$. Let m be the line through C perpendicular to r. By Corollary 85, there is a (unique) line n passing through C such that

$$\rho_{C,\Theta} = \sigma_n \circ \sigma_m$$

(see Figure 3.10).

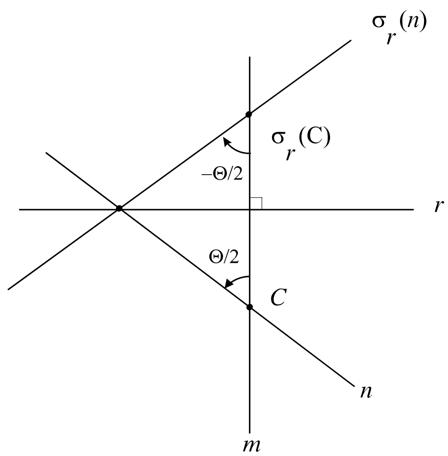


Figure 3.9.

Hence

$$\sigma_r \circ \rho_{C,\Theta} \circ \sigma_r = \sigma_r \circ \sigma_n \circ \sigma_m \circ \sigma_r = (\sigma_r \circ \sigma_n \circ \sigma_r) \circ (\sigma_r \circ \sigma_m \circ \sigma_r)$$

and Theorem 113 we have

$$(\sigma_r \circ \sigma_n \circ \sigma_r) \circ (\sigma_r \circ \sigma_m \circ \sigma_r) = \sigma_{\sigma_r(n)} \circ \sigma_{\sigma_r(m)}. \tag{3.3}$$

Since $m \perp r$ we know that

$$\sigma_r(m) = m.$$

Furthermore, $C = m \cap n$ so

$$\sigma_r(C) = m \cap \sigma_r(n).$$

Since the measure of the directed angle from m to n is $\frac{1}{2}\Theta$, the measure of the directed angle from m to $\sigma_r(n)$ is $-\frac{1}{2}\Theta$. So by Theorem 82 the right-hand side in (3.3) becomes

$$\sigma_{\sigma_r(n)} \circ \sigma_m = \rho_{\sigma_r(C),-\Theta}$$

and we conclude that

$$\sigma_r \circ \rho_{C,\Theta} \circ \sigma_r = \rho_{\sigma_r(C),-\Theta}. \tag{3.4}$$

Case 2: Let r and s be lines. Since $\alpha = \sigma_s \circ \sigma_r$ is even, we must show that $\alpha \circ \rho_{C,\Theta} \circ \alpha^{-1} = \rho_{\alpha(C),\Theta}$. But two successive applications of (3.4) give the desired result:

$$\begin{aligned} \alpha \circ \rho_{C,\Theta} \circ \alpha^{-1} &= (\sigma_s \circ \sigma_r) \circ \rho_{C,\Theta} \circ (\sigma_s \circ \sigma_r)^{-1} \\ &= \sigma_s \circ (\sigma_r \circ \rho_{C,\Theta} \circ \sigma_r) \circ \sigma_s \\ &= \sigma_s \circ \rho_{\sigma_r(C),-\Theta} \circ \sigma_s = \rho_{\alpha(C),\Theta}. \end{aligned}$$

Case 3: Let r, s, and t be lines. Since $\alpha = \sigma_t \circ \sigma_s \circ \sigma_r$ is odd, we must show that $\alpha \circ \rho_{C,\Theta} \circ \alpha^{-1} = \rho_{\alpha(C),-\Theta}$. This time, three successive applications of (3.4) in the manner of Case 2 give the result, as the reader can easily check. ■

Example 116 *Look again at the discussion on the equations for general rotations above. Equation (3.2) indicates that* $\tau_{\mathbf{OC}} \circ \rho_{O,\Theta} \circ \tau_{\mathbf{OC}}^{-1} = \rho_{C,\Theta}$. *Since* $\tau_{\mathbf{OC}}$ *is even and* $\tau_{\mathbf{OC}}(O) = C$ *we have*

$$\tau_{\mathbf{OC}} \circ \rho_{O,\Theta} \circ \tau_{\mathbf{OC}}^{-1} = \rho_{\tau_{\mathbf{OC}}(O),\Theta},$$

which confirms the conclusion of Theorem 115.

Let's apply these techniques to draw some interesting geometrical conclusions.

Theorem 117 *Two non-identity commuting rotations have the same center of rotation.*

Proof. Let α and $\rho_{C,\Theta}$ be non-identity commuting rotations. Since α is even, apply Theorem 115 to conjugate $\rho_{C,\Theta}$ by α and obtain

$$\alpha \circ \rho_{C,\Theta} \circ \alpha^{-1} = \rho_{\alpha(C),\Theta}.$$

Since α and $\rho_{C,\Theta}$ commute by assumption,

$$\rho_{\alpha(C),\Theta} = \alpha \circ \rho_{C,\Theta} \circ \alpha^{-1} = \rho_{C,\Theta} \circ \alpha \circ \alpha^{-1} = \rho_{C,\Theta}.$$

Therefore $\alpha(C) = C$. Since α is a non-identity rotation, it has exactly one fixed point, namely C, which is its center of rotation. Therefore α and $\rho_{C,\Theta}$ have the same center. ■

Theorem 118 *Let m and n be lines. Then the reflections σ_m and σ_n commute if and only if $m = n$ or $m \perp n$.*

Proof. If σ_m and σ_n commute, then

$$\sigma_m \circ \sigma_n = \sigma_n \circ \sigma_m$$

and equivalently

$$\sigma_n \circ \sigma_m \circ \sigma_n = \sigma_m. \tag{3.5}$$

By Theorem 113,

$$\sigma_n \circ \sigma_m \circ \sigma_n = \sigma_{\sigma_n(m)}. \tag{3.6}$$

By combining (3.5) and (3.6) we obtain

$$\sigma_m = \sigma_{\sigma_n(m)}.$$

Therefore $m = \sigma_n(m)$ in which case $n = m$ or $n \perp m$. Conversely, if $n = m$ or $n \perp m$, then $\sigma_n(m) = m$ in which case $\sigma_m = \sigma_{\sigma_n(m)}$. By Theorem 113 we have

$$\sigma_n \circ \sigma_m \circ \sigma_n = \sigma_{\sigma_n(m)}.$$

Substituting gives

$$\sigma_n \circ \sigma_m \circ \sigma_n = \sigma_m,$$

which is equivalent to

$$\sigma_m \circ \sigma_n = \sigma_n \circ \sigma_m.$$

■

Exercises

1. Given a line a and a point B off a, construct the line b such that $\varphi_B \circ \sigma_a \circ \varphi_B = \sigma_b$.

2. Given a line b and a point A off b, construct the point B such that $\sigma_b \circ \varphi_A \circ \sigma_b = \varphi_B$.

3. Given a line b and a point A such that $\sigma_b \circ \varphi_A \circ \sigma_b = \varphi_A$, prove that A lies on b.

4. Let A and B be distinct points and let c be a line. Prove that $\tau_{A,B} \circ \sigma_c = \sigma_c \circ \tau_{A,B}$ if and only if $\tau_{A,B}(c) = c$.

5. Let A and B be distinct points. Prove that if $\Theta + \Phi \notin 0^\circ$, $\rho_{B,\Phi} \circ \rho_{A,\Theta} = \rho_{C,\Theta+\Phi}$ and $\rho_{A,\Theta} \circ \rho_{B,\Phi} = \rho_{D,\Theta+\Phi}$, then $D = \sigma_{\overleftrightarrow{AB}}(C)$.

6. Let a be a line and let B be a point off a. Prove that $\varphi_B \circ \sigma_a \circ \varphi_B \circ \sigma_a \circ \varphi_B \circ \sigma_a \circ \varphi_B$ is a reflection in some line parallel to a.

7. Let A be a point and let τ be a non-identity translation τ. Prove that $\varphi_A \circ \tau \neq \tau \circ \varphi_A$.

8. Let A be a point and let c be a line. Prove that $\gamma_c \circ \varphi_A \neq \varphi_A \circ \gamma_c$, for every glide reflection γ_c with axis c.

9. Using Exercises 10 and 11, prove that a reflection reverses orientation.

10. Using Exercise 9, prove Proposition 109: Even isometries preserve orientation; odd isometries reverse orientation.

11. Complete the proof of Theorem 115 by proving Case 3.

3.4 The Angle Addition Theorem

In this section we answer the following questions:

1. *What is the result of composing two rotations with different centers the sum of whose rotation angles is a multiple of 360?*
2. *What is the result of composing a translation and a non-identity rotation (in either order)?*

These results will give us the second and third parts of the Angle Addition Theorem.

Let's summarize the facts about even isometries we've discovered so far. By Theorems 78, 83 and repeated applications of Theorem 101, we know that every even isometry is either a translation or a rotation (we think of the identity as a trivial rotation or translation). Furthermore, the product of two translations is a translation (Proposition 33), the product of two halfturns is a translation (Theorem 51), the product of two rotations with the same center is a rotation (Proposition 71), and the product of two rotations with different centers is a rotation as long as the sum of the rotation angles is not a multiple of 360 (Theorem 99).

Consider a product of two rotations $\rho_{D,\Phi} \circ \rho_{C,\Theta}$ about distinct centers $C \neq D$ and suppose that $\Theta + \Phi \in 0°$. Let $\ell = \overleftrightarrow{CD}$. By Corollary 85, there exist unique lines m and n passing through C and D, respectively, such that

$$\rho_{C,\Theta} = \sigma_\ell \circ \sigma_m \text{ and } \rho_{D,\Phi} = \sigma_n \circ \sigma_\ell,$$

where the measure of the directed angle from m to ℓ is $\frac{1}{2}\Theta$ and the measure of the directed angle from ℓ to n is $\frac{1}{2}\Phi$. But $\Theta° + \Phi° = 360°$, so that $\frac{1}{2}\Theta° + \frac{1}{2}\Phi° = 180°$. Therefore the measure of the directed angle from ℓ to n and from ℓ to m are both elements of $\frac{1}{2}\Phi°$ (see Figure 3.10).

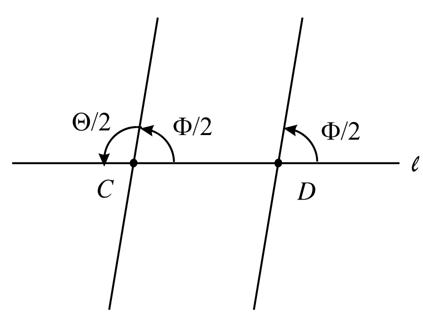


Figure 3.10: $\rho_{D,\Phi} \circ \rho_{C,\Theta}$ is a translation.

Thus $m \| n$ since ℓ is transverse to m and n with equal corresponding angles and

$$\rho_{D,\Phi} \circ \rho_{C,\Theta} = (\sigma_n \circ \sigma_\ell) \circ (\sigma_\ell \circ \sigma_m) = \sigma_n \circ \sigma_m$$

is a translation by Theorem 78. We have proved:

Theorem 119 ***(The Angle Addition Theorem, part II):*** *If C and D are distinct points and $\Theta^\circ + \Phi^\circ = 0^\circ$ then $\rho_{D,\Phi} \circ \rho_{C,\Theta}$ is a translation.*

Finally, consider the composition (in either order) of a translation τ and a non-identity rotation $\rho_{C,\Theta}$.

Theorem 120 ***(The Angle Addition Theorem, part III):*** *The composition of a non-identity rotation of Θ° and a translation (in either order) is a rotation of Θ°.*

Proof. Let ℓ be the line through C perpendicular to the direction of translation; let m and n be the unique lines such that $\rho_{C,\Theta} = \sigma_\ell \circ \sigma_m$ with $C = \ell \cap m$ and $\tau = \sigma_n \circ \sigma_\ell$ with $n \| \ell$ and m a transversal Thus $\tau \circ \rho_{C,\Theta} = \sigma_n \circ \sigma_\ell \circ \sigma_\ell \circ \sigma_m = \sigma_n \circ \sigma_m$ is the rotation about the point $B = m \cap n$ through angle Θ since the angles from m to ℓ and from m to n are corresponding. A similar argument for the composition $\rho_{C,\Theta} \circ \tau$ is left to the reader. ■

We conclude by gathering together the various parts of the Angle Addition Theorem:

Theorem 121 ***(The Angle Addition Theorem)***

a. *A rotation of Θ° followed by a rotation of Φ° is a rotation of $\Theta^\circ + \Phi^\circ$ unless $\Theta^\circ + \Phi^\circ = 0^\circ$, in which case the composition is a translation.*

b. *A translation followed by a non-identity rotation of Θ° is a rotation of Θ°.*

c. *A non-identity rotation of Θ° followed by a translation is a rotation of Θ°.*

d. *A translation followed by a translation is a translation.*

Exercises

1. Given distinct points A and B, use a MIRA to construct a point P such that $\rho_{A,60} = \tau_{P,B} \circ \rho_{B,60}$.

2. Given distinct non-collinear points A, B and C, use a MIRA to construct the point D such that $\rho_{D,60} = \tau_{A,B} \circ \rho_{C,60}$.

3. Let C be a point and let τ be a translation. Prove that there is a point R such that $\varphi_C \circ \tau = \varphi_R$.

4. Complete the proof of Theorem 120: Given a translation τ and a non-identity rotation $\rho_{C,\Theta}$, prove there is a point B such that $\rho_{C,\Theta} \circ \tau = \rho_{B,\Theta}$.

3.5 The Classification Theorem

The Fundamental Theorem proved in chapter 2, tells us that a transformation is an isometry if and only if it is a product of three or fewer reflections. Thus the classification of isometries reduces to analyzing the isometries that arise from the various configurations of three or fewer lines in the plane. Two lines are either parallel or intersecting; reflecting in two lines is either a translation or a rotation. Three lines can be either parallel, concurrent, or neither parallel nor concurrent. Since reflecting in three parallel or concurrent lines is a single reflection, the only issue that remains is to identify those isometries that arise as products of reflections in three non-parallel non-concurrent lines.

The main result of this section is the fact that an isometry is a glide reflection if and only if it factors as a product of reflections in three non-parallel non-concurrent lines. This key fact leads us to the conclusion that *every* isometry is either a reflection, a translation, a glide reflection or a rotation, and completely settles the classification problem. This result is profoundly significant—mathematics par excellence! Indeed, the goal of all mathematical inquiry is to classify the objects studied, and having done so is cause for great celebration.

Recall that an isometry γ is defined to be a glide reflection if and only if there exist if there exists a line c and a non-identity translation τ fixing c such that $\gamma = \sigma_c \circ \tau$. In light of Theorem 78, γ is a glide reflection if and only if there exist distinct parallel lines a and b with common perpendicular c such that $\gamma = \sigma_c \circ \sigma_b \circ \sigma_a$. In fact, a glide reflection can be thought of in many different ways:

Theorem 122 *For a non-identity isometry γ, the following are equivalent:*

a. *γ is a glide reflection.*

b. *γ is a non-identity translation fixing some line c followed by a reflection c, or vise versa.*

c. *γ is a reflection in some line a followed by a halfturn about some point B off a. The axis of γ is the line through B perpendicular to a.*

d. *γ is a halfturn about some point A followed by a reflection in some line b off A. The axis of γ is the line through A perpendicular to b.*

Proof. Statement (a) and the first statement in (b) are equivalent by definition. To establish the equivalence of the two statements in (b), choose distinct parallels a and b and a common perpendicular c such that $\gamma = \sigma_c \circ \sigma_b \circ \sigma_a$. Set $A = a \cap c$ and $B = b \cap c$; then $\gamma = \sigma_c \circ \tau^2_{\mathbf{AB}}$, where $\mathbf{AB}$ is a non-zero vector in the direction of c. But

$$\gamma = \sigma_c \circ \sigma_b \circ \sigma_a = \sigma_b \circ \sigma_c \circ \sigma_a = \sigma_b \circ \sigma_a \circ \sigma_c = \tau^2_{\mathbf{AB}} \circ \sigma_c.$$

To show that statement (a) implies statements (c) and (d), note that

$$\varphi_A = \sigma_a \circ \sigma_c = \sigma_c \circ \sigma_a \quad \text{and} \quad \varphi_B = \sigma_b \circ \sigma_c = \sigma_c \circ \sigma_b.$$

Then for a, b, c, A, B as above, we have

$$\gamma = \sigma_c \circ \sigma_b \circ \sigma_a = \varphi_B \circ \sigma_a,$$

i.e., γ is a reflection in some line a perpendicular to c followed by a halfturn about some point B off a (see Figure 3.11), and furthermore,

$$\gamma = \sigma_c \circ \sigma_b \circ \sigma_a = \sigma_b \circ \sigma_c \circ \sigma_a = \sigma_b \circ \sigma_c \circ \sigma_a = \sigma_b \circ \varphi_A,$$

i.e., γ is a halfturn about some point A followed by a reflection in some line b perpendicular to c off A.

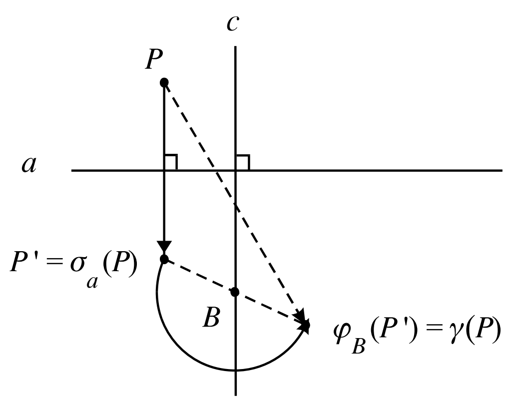


Figure 3.11: $\gamma = \varphi_B \circ \sigma_a$.

Conversely, if $\gamma = \varphi_B \circ \sigma_a$ with point B off a, let c be the line through B perpendicular to a, let b be the line perpendicular to c at B and let $A = a \cap c$. Then a and b are distinct since B is on b and off a, and

$$\varphi_B \circ \sigma_a = \sigma_c \circ \sigma_b \circ \sigma_a = \sigma_c \circ \tau^2_{\mathbf{AB}}$$

is a glide reflection by definition. Furthermore, if $\gamma = \sigma_b \circ \varphi_A$ with line b off A, let c be the line through A perpendicular to b, let a be the line perpendicular to c at A and let $B = b \cap c$. Then

$$\sigma_b \circ \varphi_A = \sigma_b \circ \sigma_a \circ \sigma_c = \tau^2_{\mathbf{AB}} \circ \sigma_c$$

is a glide reflection since the second statement in (b) implies (a). ∎

Theorem 123 *Let* γ *be a glide reflection with axis* c *and glide vector* $\mathbf{v}$.

a. γ^{-1} is a glide reflection with axis c and glide vector $-\mathbf{v}$.

b. *If* τ *is any translation fixing* c, *then* $\tau \circ \gamma = \gamma \circ \tau$.

c. γ^2 *is a non-identity translation.*

Proof. The proof of statement (a) is left to the reader.
(b) If τ is the identity it fixes c and commutes with γ. So assume that τ is a non-identity translation fixing c. Let A be a point on c; then $B = \tau(A) \neq A$ is a point on c and $\tau = \tau_{\mathbf{AB}}$. Thus $\sigma_c \circ \tau$ is a glide reflection, and by Theorem 122 (part a),

$$\sigma_c \circ \tau = \tau \circ \sigma_c. \tag{3.7}$$

On the other hand, by definition there exists a non-identity translation τ' fixing c such that $\gamma = \sigma_c \circ \tau'$. By Proposition 33, any two translations commute. This fact together with equation (3.7) gives

$$\gamma \circ \tau = \sigma_c \circ \tau' \circ \tau = \sigma_c \circ \tau \circ \tau' = \tau \circ \sigma_c \circ \tau' = \tau \circ \gamma.$$

(c) Write $\gamma = \sigma_c \circ \tau'$, where τ' is some non-identity translation that fixes c. By Theorem 122 (part a) we have

$$\gamma^2 = (\sigma_c \circ \tau')^2 = \sigma_c \circ \tau' \circ \sigma_c \circ \tau' = \sigma_c \circ \sigma_c \circ \tau' \circ \tau' = (\tau')^2,$$

which is the non-identity translation whose vector is twice the vector of τ'. ■

Now if $\gamma = \sigma_r \circ \sigma_q \circ \sigma_p$ is a glide reflection, then γ is *not* a reflection and lines p, q, and r are neither concurrent nor parallel (see Corollary 81 and Corollary 86). The converse is also true:

Theorem 124 *An isometry* $\gamma = \sigma_r \circ \sigma_q \circ \sigma_p$ *is a glide reflection if and only if lines* p, q *and* r *are neither concurrent nor parallel.*

Proof. Implication ($\Rightarrow$) was settled in the preceding remark. To prove the converse, assume that lines p, q, and r are neither concurrent nor parallel and show that $\gamma = \sigma_r \circ \sigma_q \circ \sigma_p$ is a glide reflection. We consider two cases:
Case 1: Suppose that lines p and q intersect at point Q. Since p, q, and r are not concurrent, Q lies off r. Let P be the foot of the perpendicular from Q to r, and let m be the line through P and Q. By Corollary 86, there is a (unique) line ℓ passing through Q such that

$$\sigma_q \circ \sigma_p = \sigma_m \circ \sigma_\ell.$$

Since $p \neq q$, $\ell \neq m$ and P (which is distinct from Q and lies on m) lies off ℓ. Thus

$$\gamma = \sigma_r \circ \sigma_q \circ \sigma_p = \sigma_r \circ \sigma_m \circ \sigma_\ell = \varphi_P \circ \sigma_\ell$$

with P off ℓ is a glide reflection by Theorem 122 (part b).
Case 2: Suppose that lines p and q are parallel. Then r is not parallel to either p or q and must intersect q at some point, call it Q. Consider the composition $\sigma_p \circ \sigma_q \circ \sigma_r$. By Case 1 above, there is some point P off some line ℓ such that

$$\sigma_p \circ \sigma_q \circ \sigma_r = \varphi_P \circ \sigma_\ell.$$

Hence,

$$\gamma = \sigma_r \circ \sigma_q \circ \sigma_p = (\sigma_p \circ \sigma_q \circ \sigma_r)^{-1} = (\varphi_P \circ \sigma_\ell)^{-1} = \sigma_\ell \circ \varphi_P$$

with point P off ℓ is once again a glide reflection by Theorem 122 (part c). ■

We can now obtain the long awaited classification theorem:

Theorem 125 ***(Classification of Plane Isometries)*** *Every non-identity isometry is exactly one of the following: a translation, a rotation, a reflection, or a glide reflection.*

Proof. By Corollary 103, every isometry is either even or odd. By Theorem 88, every even isometry is either a translation or a rotation. By Corollary 81, Corollary 86, and Theorem 124, every odd isometry is either a reflection or a glide reflection. ■

We can now determine the conjugates of a glide reflection.

Theorem 126 *The conjugate of a glide reflection is a glide reflection. If α is an isometry and γ is a glide reflection with axis c, then $\alpha \circ \gamma \circ \alpha^{-1}$ is a glide reflection with axis $\alpha(c)$.*

Proof. Suppose that γ is a glide reflection with axis c and let α be any isometry. Consider the line $\alpha(c)$; then

$$\left(\alpha \circ \gamma \circ \alpha^{-1}\right)(\alpha(c)) = \left(\alpha \circ \gamma \circ \alpha^{-1} \circ \alpha\right)(c) = (\alpha \circ \gamma)(c) = \alpha(\gamma(c)) = \alpha(c)$$

so that $\alpha \circ \gamma \circ \alpha^{-1}$ fixes line $\alpha(c)$. Since γ is an odd isometry, so is $\alpha \circ \gamma \circ \alpha^{-1}$. By Theorem 125, every odd isometry is either a reflection or a glide reflection. If $\alpha \circ \gamma \circ \alpha^{-1}$ were a reflection it would be an involution. But by Theorem 123 (part b), γ^2 is a non-identity translation τ. Therefore

$$\left(\alpha \circ \gamma \circ \alpha^{-1}\right)^2 = \alpha \circ \gamma \circ \alpha^{-1} \circ \alpha \circ \gamma \circ \alpha^{-1} = \alpha \circ \gamma^2 \circ \alpha^{-1} = \alpha \circ \tau \circ \alpha^{-1}$$

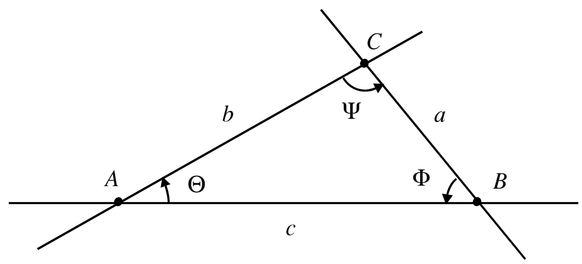
is a non-identity translation by Theorem 114. Hence $\alpha \circ \gamma \circ \alpha^{-1}$ is not an involution and consequently is not be a reflection. We conclude that $\alpha \circ \gamma \circ \alpha^{-1}$ is a glide reflection with axis $\alpha(c)$. ■

Exercises

1. Identify the isometries that are dilatations and explain.

2. Given a translation τ with vector $\mathbf{v}$, find the glide vector and axis of a glide-reflection γ such that $\gamma^2 = \tau$.

3. Let a, b and c be three non-parallel non-concurrent lines intersecting at points A, B and C as shown in the diagram below. Problems (b)-(d) use the result in part (a).



a. Prove that the axis of the glide reflection $\gamma = \sigma_a \circ \sigma_b \circ \sigma_c$ contains the feet of the perpendiculars to a and c from A and C, respectively.

b. Use a MIRA and the result in part (a) to construct the axis of γ and its glide vector $\mathbf{v}$.

c. Use a MIRA and the result in part (a) to construct the axis of $\gamma' = \sigma_c \circ \sigma_a \circ \sigma_b$ and its glide vector $\mathbf{v}'$.

d. Use a MIRA and the result in part (a) to construct the axis of $\gamma'' = \sigma_b \circ \sigma_c \circ \sigma_a$ and its glide vector $\mathbf{v}''$.

e. Show that $\rho_{C,2\Psi} \circ \rho_{B,2\Phi} \circ \rho_{A,2\Theta} = \gamma^2$ but $\rho_{A,2\Theta} \circ \rho_{B,2\Phi} \circ \rho_{C,2\Psi} = \iota$.

4. Prove Theorem 123 (a): If γ is a glide reflection with axis c and glide vector $\mathbf{v}$, then γ^{-1} is a glide reflection with axis c and glide vector $-\mathbf{v}$.

5. Let A and B be distinct points and let γ_c be a glide reflection with axis c. Prove that $\tau_{A,B} \circ \gamma_c = \gamma_c \circ \tau_{A,B}$ if and only if $\tau_{A,B}(c) = c$.