Chapter 4

Symmetry

A "symmetry" of a plane figure F is an isometry that fixes F. If F is an equilateral triangle with centroid C, for example, there are six symmetries of F, one of which is the rotation $\rho_{C,120}$. In this chapter we observe that the set of symmetries of a given plane figure is a "group" under composition. The structure of these groups, called symmetry groups, encodes information pertaining to the "symmetry types" of plane figures. But unfortunately, geometrical information is often lost in the group structure. For example, a butterfly has line symmetry but no point symmetry whereas a yin-yang symbol has point symmetry but no line symmetry. Yet their symmetry groups, which contain very different symmetries, are isomorphic since both groups are cyclic of order two. Thus symmetry groups are an imperfect invariant, i.e., we cannot recover all symmetries of a plane figure from the structure of its symmetry group. Nevertheless, we can be sure that two plane figures with non-isomorphic symmetry groups have different symmetry types. Equivalently, two plane figures with the same symmetry type have isomorphic symmetry groups.

The Classification Theorem of Plane Isometries (Theorem 125) assures us that symmetries are always reflections, translations, rotations or glide reflections. Consequently, we can systematically identify all symmetries of a given plane figure. Now if we restrict our attention to those plane figures with "finitely generated" symmetry groups, there are exactly five classes of symmetry types: (1) asymmetrical patterns, (2) patterns with only bilateral symmetry, (3) rosettes, (4) frieze patterns and (5) wallpaper patterns. Quite surprisingly, there are exactly seven symmetry types of frieze patterns and seventeen symmetry types of wallpaper patterns. Although there are infinitely many symmetry types of rosettes, their symmetry is simple and easy to understand. Furthermore, it is interesting to note that two rosettes with different symmetries have non-isomorphic symmetry groups. So for rosettes, the symmetry group is a perfect invariant. We begin our discussion with what little group theory we need.

4.1 Groups of Isometries

In this section we introduce the group of isometries $\mathcal{I}$ and some of its subgroups.

Definition 127 *A non-empty set G equipped with a binary operation $*$ is a group if and only if the following properties are satisfied:*

1. *Closure: If $a, b \in G$, then $a * b \in G$.*
2. *Associativity: If $a, b, c \in G$, then $a * (b * c) = (a * b) * c$.*
3. *Identity: For all $a \in G$, there exists an element $e \in G$ such that $e * a = a * e = a$.*
4. *Inverses: For each $a \in G$, there exists $b \in G$ such that $a * b = b * a = e$.*

A group G is abelian (or commutative) if and only if for all $a, b \in G$, $a * b = b * a$.

Theorem 128 *The set $\mathcal{I}$ of all isometries is a group under function composition.*

Proof. The work has already been done. Closure was proved in Exercise 1.1.3; the fact that composition of isometries is associative is a special case of Exercise 1.1.4; the fact that ι acts as an identity element in $\mathcal{I}$ was proved in Exercise 1.1.5; and the existence of inverses was proved in Exercise 1.1.7. ■

Since two halfturns with distinct centers of rotation do not commute and halfturns are elements of $\mathcal{I}$, the group $\mathcal{I}$ is *non-abelian.* On the other hand, some subsets of $\mathcal{I}$ (the translations for example) contain commuting elements. When such a subset is a group in its own right, it is abelian.

Definition 129 *Let $(G, *)$ be a group and let H be a non-empty subset of G. Then H is a subgroup of G if and only if $(H, *)$ is a group, i.e., H is a group under the operation inherited from G.*

Given a non-empty subset H of a group G, is H itself a group under the operation in G? One could appeal to the definition and check all four properties, but it is sufficient to check just two.

Theorem 130 *Let $(G, *)$ be a group and let H be a non-empty subset of G. Then H is a subgroup of G if and only if the following two properties hold:*
a. Closure: If $a, b \in H$, then $a * b \in H$.
b. Inverses: For every $a \in H$, there exists $b \in H$ such that $a * b = b * a = e$.

Proof. If H is a subgroup of G, properties (a) and (b) hold by definition. Conversely, suppose that H is a non-empty subset of G in which properties (a) and (b) hold. *Associativity* is inherited from G, i.e., if $a, b, c \in H$, then as elements of G, $a * (b * c) = (a * b) * c$. *Identity:* Since $H \neq \varnothing$, choose an element

$a \in H$. Then $a^{-1} \in H$ since H has inverses by property (b). Furthermore, operation $*$ is closed in H by property (a) so that $a * a^{-1} \in H$. But $a * a^{-1} = e$ since a and a^{-1} are elements of G, so $e \in H$ as required. Therefore H is a subgroup of G. ■

Proposition 131 *The set $\mathcal{T}$ of all translations is an abelian group.*

Proof. Closure and commutativity follow from Proposition 33; the existence of inverses was proved in Exercise 7. Therefore $\mathcal{T}$ is an abelian subgroup of $\mathcal{I}$ by Theorem 130, and consequently $\mathcal{T}$ is an abelian group. ■

Proposition 132 *The set $\mathcal{R}_C$ of all rotations about a point C is an abelian group.*

Proof. The proof is left as an exercise for the reader. ■

Exercises

1. Prove that the set $\mathcal{R}_C$ of all rotations about a point C is an abelian group.

2. Prove that the set $\mathcal{E}$ of all even isometries is a non-abelian group.

3. Prove that the set $\mathcal{D}$ of all dilatations is a non-abelian group.

4.2 Groups of Symmetries

In this section we observe that the set of symmetries of a given plane figure F is a group, called the symmetry group of F. Consequently, symmetry groups are always subgroups of $\mathcal{I}$ (the group of all isometries).

Definition 133 *A plane figure is a non-empty subset of the plane.*

Definition 134 *Let F be a plane figure. An isometry α is a symmetry of F if and only if α fixes F.*

Theorem 135 *Let F be a plane figure. The set of all symmetries of F is a group, called the symmetry group of F.*

Proof. Let F be a plane figure and let $\mathcal{S} = \{\alpha : \alpha \text{ is a symmetry of } F\}$. Since the identity $\iota \in \mathcal{S}$, the set $\mathcal{S}$ is a non-empty subset of $\mathcal{I}$.

Closure: Let $\alpha, \beta \in \mathcal{S}$. By Exercise 1.1.3, the composition of isometries is an isometry. So it suffices to check that $\alpha \circ \beta$ fixes F. But since $\alpha, \beta \in \mathcal{S}$ we have $(\alpha \circ \beta)(F) = \alpha(\beta(F)) = \alpha(F) = F$.

Inverses: Let $\alpha \in \mathcal{S}$; we know that $\alpha^{-1} \in \mathcal{I}$ by Exercise 1.1.7; we must show that α^{-1} fixes F. But $\alpha^{-1}(F) = \alpha^{-1}(\alpha(F)) = \left(\alpha^{-1} \circ \alpha\right)(F) = F$ so that α^{-1} also fixes F. Thus $\alpha^{-1} \in \mathcal{S}$ whenever $\alpha \in \mathcal{S}$.

Therefore $\mathcal{S}$ is a group by Theorem 130. ■

Example 136 *(The Dihedral Group D_3) Let F denote an equilateral triangle positioned with its centroid at the origin and one vertex on the y-axis. There are exactly six symmetries of F, namely, the identity ι, two rotations ρ_{120} and ρ_{240} about the centroid, and three reflections σ_ℓ, σ_m and σ_n where ℓ, m and n have respective equations $\sqrt{3}x - 3y = 0$; $x = 0$; and $\sqrt{3}x + 3y = 0$ (see Figure 4.1).*

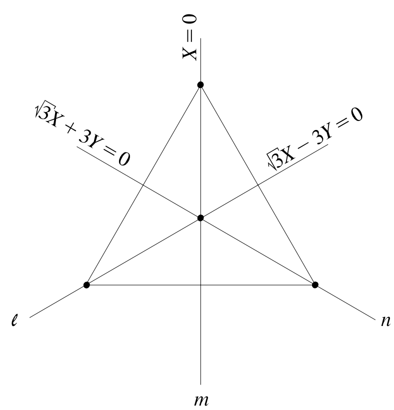


Figure 4.1: Lines of symmetry ℓ, m and n.

The multiplication table for composing these various symmetries appears in Table 4.1 below. Closure holds by inspection. Furthermore, since each row and column contains the identity element ι in exactly one position, each element has a unique inverse. By Theorem 130 these six symmetries form a group D_3 called the *Dihedral Group of order 6.*

$\circ$	ι	ρ_{120}	ρ_{240}	σ_ℓ	σ_m	σ_n
ι	ι	ρ_{120}	ρ_{240}	σ_ℓ	σ_m	σ_n
ρ_{120}	ρ_{120}	ρ_{240}	ι	σ_m	σ_n	σ_ℓ
ρ_{240}	ρ_{240}	ι	ρ_{120}	σ_n	σ_ℓ	σ_m
σ_ℓ	σ_ℓ	σ_n	σ_m	ι	ρ_{240}	ρ_{120}
σ_m	σ_m	σ_ℓ	σ_n	ρ_{120}	ι	ρ_{240}
σ_n	σ_n	σ_m	σ_ℓ	ρ_{240}	ρ_{120}	ι

Table 4.1: The Dihedral Group of Order 6

Look carefully at the upper left 4×4 block in Table 4.1 above. This is the multiplication table for the rotations $\{\iota, \rho_{120}, \rho_{240}\} \subset D_3$ (the identity is a rotation through angle 0); we shall denote this set by C_3. Once again we see that composition is closed in C_3 and the inverse of each element in C_3 is also in C_3. Therefore C_3 is a group; the symbol "C_3" stands for "*cyclic group of order 3*".

Definition 137 *A plane figure F has point symmetry if and only if some (non-identity) rotation is a symmetry of F. The center of a (non-identity) rotational symmetry of F is called a point of symmetry for F.*

Definition 138 *A plane figure F has line symmetry if and only if some reflection is a symmetry of F. The reflecting line of a reflection symmetry of F is called a line of symmetry for F.*

Definition 139 *A plane figure F has bilateral symmetry if and only if F has a unique line of symmetry and no points of symmetry.*

Corollary 140 *The two symmetries of a figure with bilateral symmetry form a group denoted by D_1. The four symmetries of a non-square rhombus form a group denoted by D_2. For $n \geq 3$, the $2n$ symmetries of a regular n-gon form a group denoted by D_n.*

Proof. A plane figure with bilateral symmetry has one line of symmetry and one rotational symmetry (the identity). A non-square rhombus has two lines of symmetry and two rotational symmetries about the centroid (including the identity). If $n \geq 3$, a regular n-gon has n lines of symmetry and n rotational symmetries about the centroid (including the identity). These sets of symmetries form groups by Theorem 135. ■

Definition 141 *For $n \geq 1$, the group D_n is called the dihedral group of order $2n$.*

Definition 142 *Let G be a group, let $a \in G$, and define $a^0 = \iota$. The group G is cyclic if and only if for all $a \in G$, there is an element $g \in G$ (called a generator) such that $a = g^k$ for some $k \in \mathbb{Z}$. A cyclic group with $n < \infty$ elements is said to be cyclic of order n. A cyclic group G with infinitely many elements is said to be infinite cyclic.*

Example 143 *Observe that the elements of $C_3 = \{\iota, \rho_{120}, \rho_{240}\}$ can be obtained as powers of either ρ_{120} or ρ_{240}. For example,*

$$\rho_{120} = \rho_{120}^1;\ \rho_{240} = \rho_{120}^2;\ \text{and } \iota = \rho_{120} \circ \rho_{240} = \rho_{120}^3.$$

We say that ρ_{120} and ρ_{240} "generate" C_3. Also observe that this 4×4 block is symmetric with respect to the upper-left-to-lower-right diagonal. This indicates that C_3 is an abelian group. More generally, let C be a point, let n be a positive integer and let $\theta = \frac{360}{n}$. Then for each integer k, $\rho_{C,\theta}^k = \rho_{C,k\theta}$ and $\rho_{C,\theta}^n = \rho_{C,360} = \iota$. Therefore the group of rotations generated by $\rho_{C,\theta}$ is cyclic with n elements and is denoted by C_n.

If C is the centroid of a regular n-gon with $n \geq 3$, the finite cyclic group of rotations C_n introduced in Example 143 is the abelian subgroup of rotations in D_n. On the other hand, C_n can be realized as the symmetry group of a $3n$-gon constructed as follows: For $n = 4$, cut a square out of paper and draw its diagonals, thereby subdividing the square into four congruent isosceles right triangles with common vertex at the centroid of the square. From each of the four vertices, cut along the diagonals stopping midway between the vertices and the centroid. With the square positioned so that its edges are vertical or horizontal, fold the triangle at the top so that its right-hand vertex aligns with the centroid of the square. Rotate the paper 90° and fold the triangle now at the top in a similar way. Continue rotating and folding until you have what looks like a flattened pinwheel with four paddles (see Figure 4.2). The outline of this flattened pinwheel is a dodecagon (12-gon) whose symmetry group is C_4 generated by either ρ_{90} or ρ_{270}. For a general n, one can construct a $3n$-gon whose symmetry group is cyclic of order n by cutting and folding a regular n-gon in a similar way to obtain a pinwheel with n paddles.

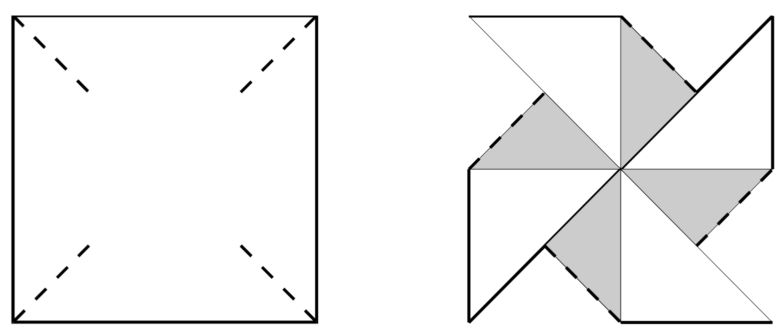


Figure 4.2: A polygon whose symmetry group is cyclic of order 4.

Example 144 *Let τ be a non-identity translation; let P be any point and let $Q = \tau(P)$. Then $\tau = \tau_{\mathbf{PQ}}$ and $\tau^2 = \tau_{\mathbf{PQ}} \circ \tau_{\mathbf{PQ}} = \tau_{2\mathbf{PQ}}$. Inductively, $\tau^n = \tau^{n-1} \circ \tau = \tau_{(n-1)\mathbf{PQ}} \circ \tau_{\mathbf{PQ}} = \tau_{n\mathbf{PQ}}$, for each $n \in \mathbb{N}$. Furthermore, $(\tau^n)^{-1} = (\tau_{n\mathbf{PQ}})^{-1} = \tau_{-n\mathbf{PQ}} = \tau^{-n}$, so distinct integer powers of τ are distinct translations. It follows that the set $G = \{\tau^n : n \in \mathbb{Z}\}$ is infinite. Note that G is a group: inverses were discussed above and closure follows from the fact that $\tau^n \circ \tau^m = \tau^{n+m}$. Since every element of G is an integer power of τ (or τ^{-1}), G is the infinite cyclic group generated by τ (or τ^{-1}).*

Let G be a group and let K be a non-empty subset of G. The symbol $\langle K \rangle$ denotes the set of all (finite) products of powers of elements of K and their inverses. If $K = \{k_1, k_2, \ldots\}$, we abbreviate and write $\langle k_1, k_2, \ldots \rangle$ instead of $\langle \{k_1, k_2, \ldots\} \rangle$. Thus $\langle K \rangle$ is automatically a subgroup of G since it is non-empty, the group operation is closed and contains the inverse of each element in $\langle K \rangle$.

Definition 145 *Let G be a group and let K be a non-empty subset of G. The subgroup $\langle K \rangle$ is referred to as the <u>subgroup of G generated by K</u>. A subset $K \subseteq G$ is said to be a <u>generating set</u> for G if and only if $G = \langle K \rangle$. A group G is <u>finitely generated</u> if and only if there exists a finite set K such that $G = \langle K \rangle$.*

Example 146 *A cyclic group G with generator $g \in G$ has the property that $G = \langle g \rangle$. So $\{g\}$ is a generating set for G.*

Example 147 *Let τ be a non-identity translation. Then $\langle \tau \rangle$ is infinite cyclic since $\tau^n \neq \iota$ for all $n \neq 0$ (see Example 144).*

Example 148 *Let K be the set of all reflections. Since every reflection is its own inverse, $\langle K \rangle$ consists of all (finite) products of reflections. By The Fundamental Theorem of Transformational Plane Geometry every isometry of the plane is a product of reflections. Therefore $\langle K \rangle = \mathcal{I}$, i.e., the group of all isometries, is infinitely generated by the set K of all reflections.*

Example 149 *Let H denote the set of all halfturns. Since the composition of two halfturns is a translation, the composition of two translations is a translation, and every translation can be written as a composition of two halfturns, $\mathcal{H} = \langle H \rangle$ is infinitely generated and is exactly the set of all translations and halfturns.*

Exercises

1. Recall that the six symmetries of an equilateral triangle form the dihedral group D_3 (see Example 136). Show that the set $K = \{\rho_{120}, \ \sigma_\ell\}$ is a generating set for D_3 by writing each of the other four elements in D_3 as a product of powers of elements of K and their inverses. Compute all powers of each element in D_3 and show that no single element alone generates D_3. Thus D_3 is not cyclic.

2. The dihedral group D_4 consists of the eight symmetries of a square. When the square is positioned with its centroid at the origin and its vertices on the axes, the origin is a point of symmetry and the lines $a : Y = 0$, $b : Y = X$, $c : X = 0$ and $d : Y = -X$ are lines of symmetry. Construct a multiplication table for $D_4 = \{\iota, \rho_{90}, \rho_{180}, \rho_{270}, \sigma_a, \sigma_b, \sigma_c, \sigma_d\}$.

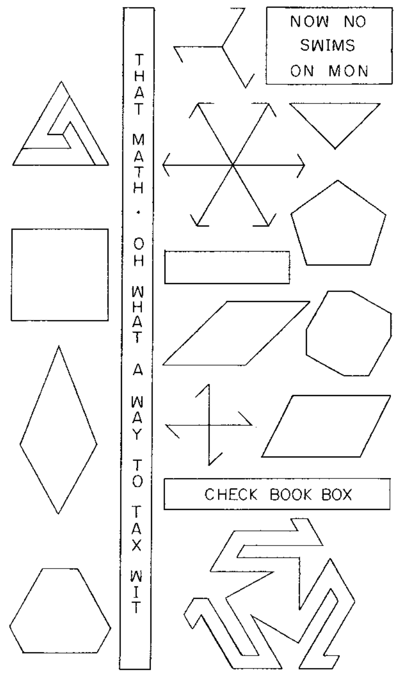
3. Find the symmetry group of

   a. A parallelogram that is neither a rectangle nor a rhombus.

   b. A rhombus that is not a square.

4. Find the symmetry group of each capital letter of the alphabet written in its most symmetric form.

5. Determine the symmetry group of each figure below:



6. The discussion following Example 143 describes how to construct a $3n$-gon whose symmetry group is C_n, where $n \geq 3$. Alter this construction to obtain a $2n$-gon whose symmetry group is C_n.

7. Let C be a point. For which rotation angles Θ is $\langle \rho_{C,\Theta} \rangle$ an infinite group?

4.3 The Rosette Groups

Definition 150 *A rosette is a plane figure R with the following properties:*

1. *There exists a non-identity rotational symmetry of R with minimal positive rotation angle Θ, i.e., if $\rho_{C,\Phi}$ is any non-identity rotational symmetry of R then $0° < \Theta° \leq \Phi°$.*
2. *All non-identity rotational symmetries of R have the same center C.*

The symmetry group of a rosette is called a rosette group.

Typically one thinks of a rosette as a pin-wheel (see Figures 4.2) or a flower with n-petals (See Figure 4.3). However, a regular polygon, a non-square rhombus, a yin-yang symbol and a pair of perpendicular lines are rosettes as well.

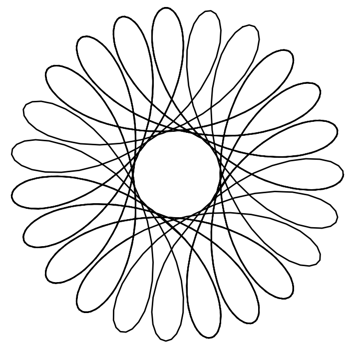


Figure 4.3: A typical rosette.

In the early sixteenth century, Leonardo da Vinci determined all possible finite groups of isometries; all but two of which are rosette groups. The two exceptions are C_1, which contains only the identity, and D_1, which contains the identity and one reflection. Note that D_1 is isomorphic to the rosette group C_2, which contains the identity and one halfturn.

Theorem 151 ***(Leonardo's Theorem):*** *Every finite group of isometries is either C_n or D_n for some $n \geq 1$.*

Proof. Let G be a finite group of isometries. Then G contains only rotations and reflections since non-identity translations and glide reflections would generate infinite subgroups.

Case 1: Suppose that G contains only rotations. If $G = \{\iota\}$, then $G = C_1$ and the result holds. So assume that G contains a non-identity rotation $\rho_{C,\Theta}$. I claim that every non-identity rotation in G has center C. Suppose, on the contrary, that G contains another non-identity rotation $\rho_{B,\Phi}$ with $B \neq C$. Let $C' = \rho_{B,\Phi}(C)$; then $C' \neq C$ since C is not a fixed point. Conjugating $\rho_{C,\Theta}$ by $\rho_{B,\Phi}$ gives $\rho_{B,\Phi} \circ \rho_{C,\Theta} \circ \rho_{B,\Phi}^{-1} = \rho_{C',\Theta}$, which is an element of G by closure. Furthermore, $\rho_{C',\Theta} \circ \rho_{C,\Theta}^{-1} = \rho_{C',\Theta} \circ \rho_{C,-\Theta} \in G$, again by closure. But $\Theta + (-\Theta) \in 0^\circ$ so $\rho_{C',\Theta} \circ \rho_{C,-\Theta}$ is a non-identity translation by the Angle Addition Theorem (121), contradicting the fact that G contains no translations. So every non-identity rotation in G has center C. To prove that G is cyclic recall that every congruence class of angles Φ° has a unique class representative in the range $0 \leq \Phi < 360$. Write each element in G uniquely in the form $\rho_{C,\Phi}$ with $0 \leq \Phi < 360$. Since G is finite, there is a rotation $\rho_{C,\Phi}$ in G with the smallest positive rotation angle Φ. Thus if $\rho_{C,\Psi} \in G$, then $\Phi \leq \Psi < 360$ by the minimality of Φ, and there is a positive integer n such that $n\Phi \leq \Psi \leq (n+1)\,\Phi$. Thus $0 \leq \Psi - n\Phi \leq \Phi$. Now if both of these inequalities were strict, $\Psi - n\Phi$ would be a positive rotation angle strictly less than Φ, which violates the minimality of Φ. Therefore $\Psi = n\Phi$ or $\Psi = (n+1)\,\Phi$, i.e., $\Psi = k\Phi$ for some integer k.

Consequently, $\rho_{C,\Psi} = \rho_{C,\Phi}^k$ and G is cyclic.

Case 2: Suppose that G contains reflections. Let E be the subset of rotations in G and let $F = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$, $m \geq 1$, be the subset of reflections in G. Since E is a subgroup of the rotation group $\mathcal{R}_C$ (see Proposition 132), $E = C_n$ for some $n \geq 1$ by Case 1 above and is generated by some rotation $\rho = \rho_{C,\Theta}$, i.e., $E = \left\{\rho, \rho^2, \ldots, \rho^n\right\}$. I claim $m = n$. Choose a reflection $\sigma \in F$ and note that the subset $\sigma E = \left\{\sigma \circ \rho, \sigma \circ \rho^2, \ldots, \sigma \circ \rho^n\right\} \subset G$ contains n distinct odd isometries, which must be reflections since G has no glide reflections. Therefore $\sigma E \subseteq F$ and $n \leq m$. On the other hand, the subset $\sigma F = \{\sigma \circ \sigma_1, \sigma \circ \sigma_2, \ldots, \sigma \circ \sigma_m\} \subset G$ contains m distinct even isometries, which must be rotations since G has no translations. Therefore $\sigma F \subseteq E$ and $m \leq n$. Thus $m = n$ and $G = \langle \sigma, \rho \rangle$ contains exactly n rotations and n reflections. If $n = 1$, then $G = \langle \sigma \rangle = D_1$. But if $n > 1$,then $E = \sigma F$. So for each integer i, there is some integer k such that $\sigma\sigma_i = \rho_{C,\Theta}^k$.Hence the axis of σ_i passes through C and all lines of symmetry are concurrent at C. Therefore $G = D_n$. ■

An immediate consequence of Leonardo's Theorem is the following:

Corollary 152 *The rosette groups are either dihedral D_n or finite cyclic C_n with $n \geq 2$.*

While the notion of "symmetry type" is quite subtle for general plane figures, we can make the idea precise for rosettes. Let R_1 and R_2 be rosettes with the same minimal positive rotation angle Θ and respective centers A and B. Let $\tau = \tau_{\mathbf{AB}}$; then $\tau \circ \rho_{A,\Theta} \circ \tau^{-1} = \rho_{\tau(A),\Theta} = \rho_{B,\Theta}$ and there is an isomorphism of cyclic groups $f : \left\langle \rho_{A,\Theta} \right\rangle \rightarrow \left\langle \rho_{B,\Theta} \right\rangle$ given by $f(\alpha) = \tau \circ \alpha \circ \tau^{-1}$. If R_1 and R_2 have no lines symmetry, then f is an isomorphism of symmetry groups. On the other hand, if the respective symmetry groups G_1 and G_2 have reflections $\sigma_\ell \in G_1$ and $\sigma_m \in G_2$, the lines ℓ and m are either intersecting or parallel. If parallel, $m = \tau(\ell)$ and $\tau \circ \sigma_\ell \circ \tau^{-1} = \sigma_{\tau(\ell)} = \sigma_m$, in which case $f(\alpha) = \tau \circ \alpha \circ \tau^{-1}$ is an isomorphism of symmetry groups. If ℓ and m intersect and the directed angle measure from ℓ to m is Φ°, then $\left(\tau \circ \rho_{A,\Phi}\right) \circ \sigma_\ell \circ \left(\tau \circ \rho_{A,\Phi}\right)^{-1} = \sigma_{\left(\tau \circ \rho_{A,\Phi}\right)(\ell)} = \sigma_m$ and $f(\alpha) = \left(\tau \circ \rho_{A,\Phi}\right) \circ \alpha \circ \left(\tau \circ \rho_{A,\Phi}\right)^{-1}$ is an isomorphism of symmetry groups.

Now if G is any group and $g \in G$, the function $h : G \rightarrow G$ defined by $h(x) = gxg^{-1}$ is an isomorphism, as the reader can easily check. In particular, the map f defined above is the restriction to G_1 of an isomorphism $f : \mathcal{I} \rightarrow \mathcal{I}$, where $\mathcal{I}$ denotes the group of all plane isometries. We summarize this discussion in the definitions that follows:

Definition 153 *Let G be a group and let $g \in G$. The isomorphism $h : G \rightarrow G$ defined by $h(x) = gxg^{-1}$ is called an inner automorphism of G.*

Definition 154 *Let R_1 and R_2 be rosettes with respective symmetry groups G_1 and G_2. Rosettes R_1 and R_2 have the same symmetry type if and only if there is an an inner automorphism of $\mathcal{I}$ that restricts to an isomorphism $f : G_1 \rightarrow G_2$.*

Corollary 155 *Two rosettes have the same symmetry type if and only if their respective symmetry groups are isomorphic.*

Exercises

1. Refer to Exercise 4 in Section 1 above. Which capital letters of the alphabet written in most symmetry form are rosettes?

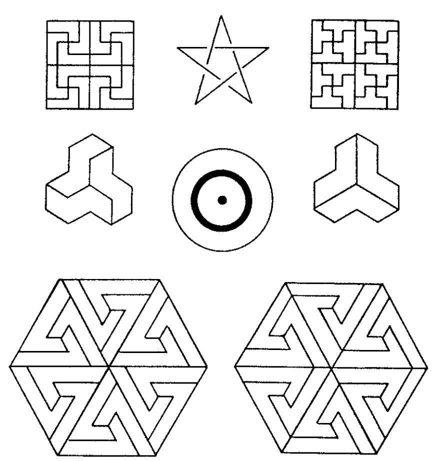
2. For $n \geq 2$, the graph of the equation $r = \cos n\theta$ in polar coordinates is a rosette.

   a. Find the rosette group of the graph for each $n \geq 2$.

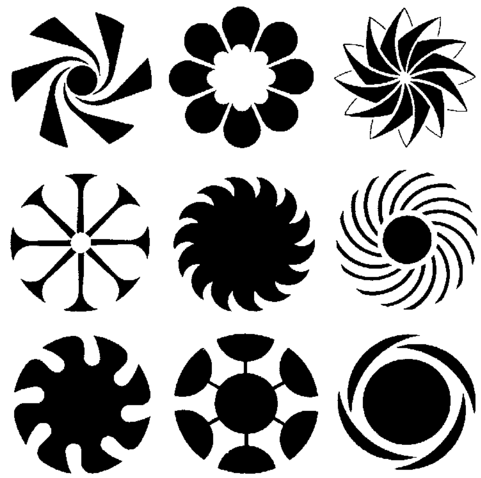
   b. Explain why the graph of the equation $r = \cos\theta$ in polar coordinates is *not* a rosette.

3. Find at least two rosettes in your campus architecture and determine their rosette groups.

4. Identify the rosette groups of the figures in the following that are rosettes:



5. Identify the rosette groups of the following rosettes:



4.4 The Frieze Groups

Frieze patterns are typically the familiar decorative borders often seen on walls or facades extended infinitely far in either direction (See Figure 4.4).

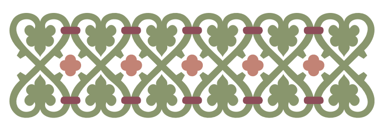


Figure 4.4: A typical frieze pattern.

In this section we identify all possible symmetries of frieze patterns and reach the startling conclusion that every frieze pattern is one of seven distinctive symmetry types.

Definition 156 *Let τ be a translation. The length of τ, denoted by $\|\tau\|$, is the length of the vector of τ.*

Definition 157 *A frieze pattern is a plane figure F with the following properties:*

1. *There exists a translational symmetry of F with minimal length, i.e., if τ' is any non-identity translational symmetry of F, then $0 < \|\tau\| \leq \|\tau'\|$.*

2. *All non-identity translational symmetries of F fix the same lines.*

The symmetry group of a frieze pattern is called a *frieze group*.

Consider an row of equally spaced letter R's extending infinitely far in either direction (see Figure 4.5)

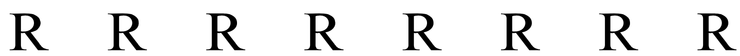


Figure 4.5: Frieze pattern F_1.

This frieze pattern, denoted by F_1, has only translational symmetry. There are

two translational symmetries of minimal length (the distance between centroid of consecutive R's)—one shifting left; the other shifting right. Let τ be a translational symmetry of shortest length; then $\tau^n \neq \iota$ for all $n \neq 0$ and the frieze group of F_1 is the infinite cyclic group $\mathcal{F}_1 = \langle \tau \rangle = \{\tau^n : n \in \mathbb{Z}\}$.

The second frieze pattern F_2 has a glide reflection symmetry (see Figure 4.6). Let γ be a glide reflection such that γ^2 is a translational symmetry of shortest length. Then $\gamma^n \neq \iota$ for all $n \neq 0$ and γ^2 generates the translation subgroup. The frieze group of F_2 is the infinite cyclic group $\mathcal{F}_2 = \langle \gamma \rangle = \{\gamma^n : n \in \mathbb{Z}\}$. Note that while the elements of $\mathcal{F}_1$ and $\mathcal{F}_2$ are very different, the two groups are isomorphic.

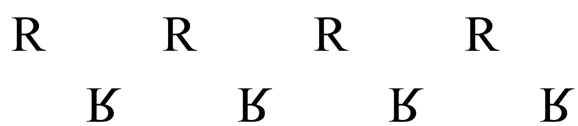


Figure 4.6: Frieze pattern F_2.

The third frieze pattern F_3 has vertical line symmetry (see Figure 4.7). Let ℓ be a line of symmetry. Choose a line m such that $\tau = \sigma_m \circ \sigma_\ell$ is a translational symmetry of minimal length. Then m is also a line of symmetry since $\sigma_m = \tau \circ \sigma_\ell$ and the composition of symmetries is a symmetry (Theorem 135). In general, the reflection $\tau^n \circ \sigma_\ell$ is a symmetry for each $n \in \mathbb{Z}$; these reflections determine all lines of symmetry. The frieze group of F_3 is $\mathcal{F}_3 = \langle \tau, \sigma_\ell \rangle = \{\tau^n \circ \sigma_\ell^m : n \in \mathbb{Z};\ m = 0, 1\}$, which is the infinite dihedral group D_∞.

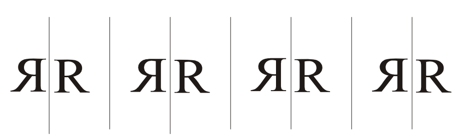


Figure 4.7: Frieze pattern F_3.

Frieze pattern F_4 has halfturn symmetry (see Figure 4.8). Let P be a point of symmetry. Choose a point Q such that $\tau = \varphi_Q \circ \varphi_P$ is a translational symmetry of shortest length. Then Q is also a point of symmetry since $\varphi_Q = \tau \circ \varphi_P$. In general, the halfturn $\tau^n \circ \varphi_P$ is a symmetry for each $n \in \mathbb{Z}$; these halfturns determine all points of symmetry. The frieze group of F_4 is $\mathcal{F}_4 = \langle \tau, \varphi_P \rangle = \{\tau^n \circ \varphi_P^m : n \in \mathbb{Z};\, m = 0, 1\}$.

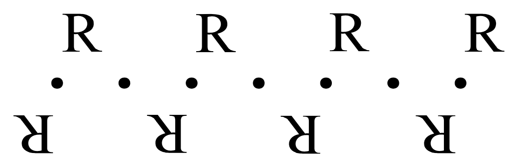


Figure 4.8: Frieze pattern F_4.

The fifth frieze pattern F_5 can be identified by its halfturn symmetry and glide reflection symmetry (see Figure 4.9). In addition, F_5 has vertical line symmetry, but as we shall see, these symmetries can be obtained by composing a glide reflection with a halfturn. Let P be a point of symmetry and let γ be a glide reflection such that γ^2 is a translational symmetry of shortest length. Choose a point Q such that $\gamma^2 = \varphi_Q \circ \varphi_P$. Then Q is also a point of symmetry since $\varphi_Q = \gamma^2 \circ \varphi_P$. In general, the halfturn $\gamma^{2n} \circ \varphi_P$ is a symmetry for each $n \in \mathbb{Z}$; these halfturns determine all points of symmetry. Now the line symmetries can be obtained from γ and φ_P as follows: Let c be the horizontal axis of γ, let a be the vertical line through P, and let ℓ be the vertical line such that $\gamma = \sigma_\ell \circ \sigma_a \circ \sigma_c$. Then $\varphi_P = \sigma_c \circ \sigma_a$ so that $\gamma \circ \varphi_P = \sigma_\ell$ and the line symmetries are the reflections $\gamma^{2n} \circ \sigma_\ell$ with $n \in \mathbb{Z}$. The frieze group of F_5 is $\mathcal{F}_5 = \langle \gamma, \varphi_P \rangle = \{\gamma^n \circ \varphi_P^m : n \in \mathbb{Z};\, m = 0, 1\}$. Note that $\mathcal{F}_3$, $\mathcal{F}_4$, and $\mathcal{F}_5$ are isomorphic groups.

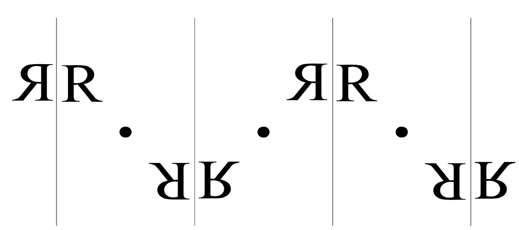


Figure 4.9: Frieze pattern F_5.

In Figure 4.10 we picture the frieze pattern F_6, which has a unique horizontal line of symmetry c. Thus the frieze group of F_6 is $\mathcal{F}_6 = \langle \tau, \sigma_c \rangle = \{\tau^n \circ \sigma_c^m : n \in \mathbb{Z};\, m = 0, 1\}$. The reader should check that $\mathcal{F}_6$ is abelian (see Exercise 5). Consequently $\mathcal{F}_6$ is not isomorphic to groups $\mathcal{F}_3$, $\mathcal{F}_4$ and $\mathcal{F}_5$.

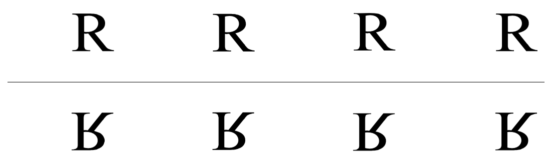


Figure 4.10: Frieze pattern F_6.

The final frieze group F_7 has vertical line symmetry and a unique horizontal line of symmetry c (see Figure 4.11). Let ℓ be a vertical line of symmetry and let τ be a translational symmetry of shortest length. Then the vertical line symmetries are the reflections $\tau^n \circ \sigma_\ell$ with $n \in \mathbb{Z}$ and the point $P = c \cap \ell$ is a point of symmetry since $\varphi_P = \sigma_c \circ \sigma_\ell$. Thus the halfturn symmetries are the halfturns $\tau^n \circ \varphi_P$ with $n \in \mathbb{Z}$. The frieze group of F_7 is $\mathcal{F}_7 = \langle \tau, \sigma_c, \sigma_\ell \rangle = \left\{ \tau^n \circ \sigma_c^m \circ \sigma_\ell^k : n \in \mathbb{Z};\, m, n = 0, 1 \right\}$.

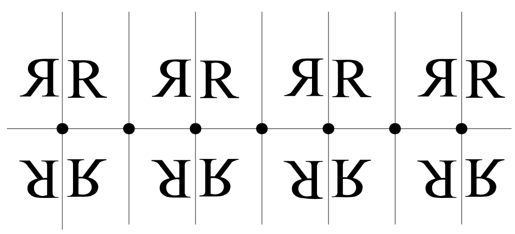


Figure 4.11: Frieze pattern F_7.

We collect the observations above as a theorem, however the proof that this list exhausts all possibilities is omitted:

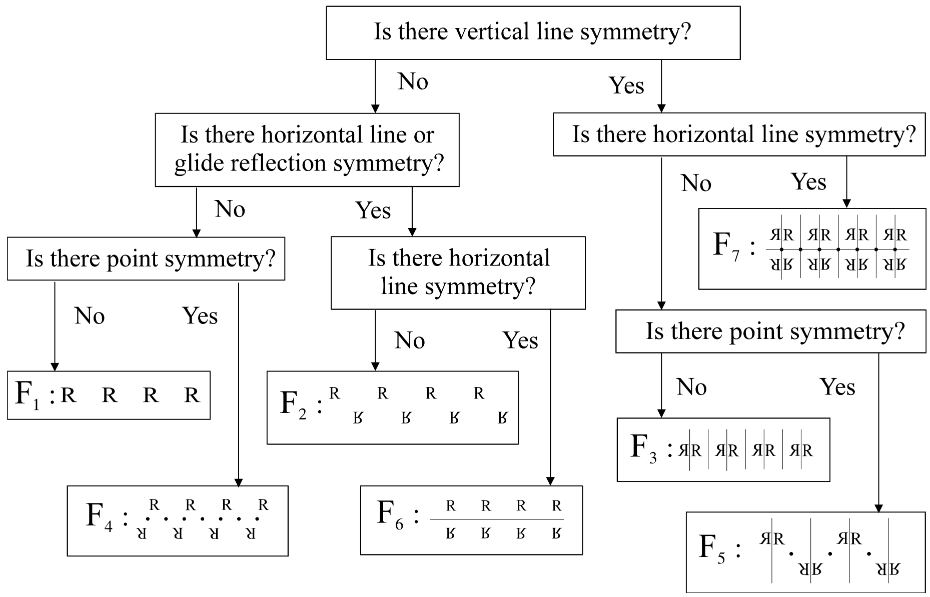
Theorem 158 *Every frieze group is one of the following:*

$$\begin{array}{lll}
\mathcal{F}_1 = \langle \tau \rangle & \mathcal{F}_2 = \langle \gamma \rangle & \\
\mathcal{F}_3 = \langle \tau, \sigma_\ell \rangle & \mathcal{F}_4 = \langle \tau, \varphi_P \rangle & \mathcal{F}_5 = \langle \gamma, \varphi_P \rangle \\
\mathcal{F}_6 = \langle \tau, \sigma_c \rangle & & \\
\mathcal{F}_7 = \langle \tau, \sigma_c, \sigma_\ell \rangle & &
\end{array}$$

where τ is a translation of shortest length, γ is a glide reflection such that $\gamma^2 = \tau$, ℓ is a vertical line of symmetry, P is a point of symmetry and c is the unique horizontal line of symmetry.

The following flowchart can be used to identify the frieze group associated with a particular frieze pattern:

Recognition Flowchart for Frieze Patterns



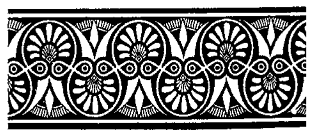
Exercises

1. Find at least two friezes in your campus architecture and identify their frieze groups.

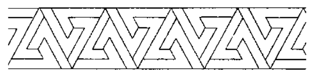
2. Find the frieze group for the pattern in Figure 4.4.

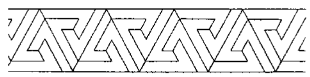
3. Prove that frieze group $\mathcal{F}_6$ is abelian.

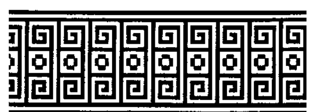
4. Identify the frieze groups for the following:

(a) 

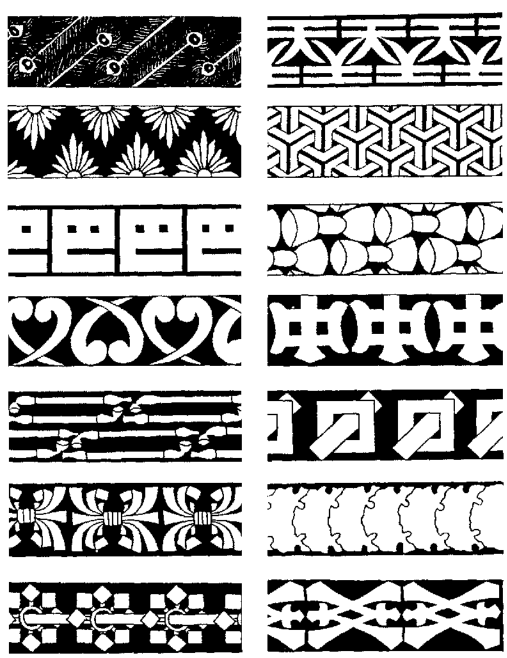
(b) 

(c) 

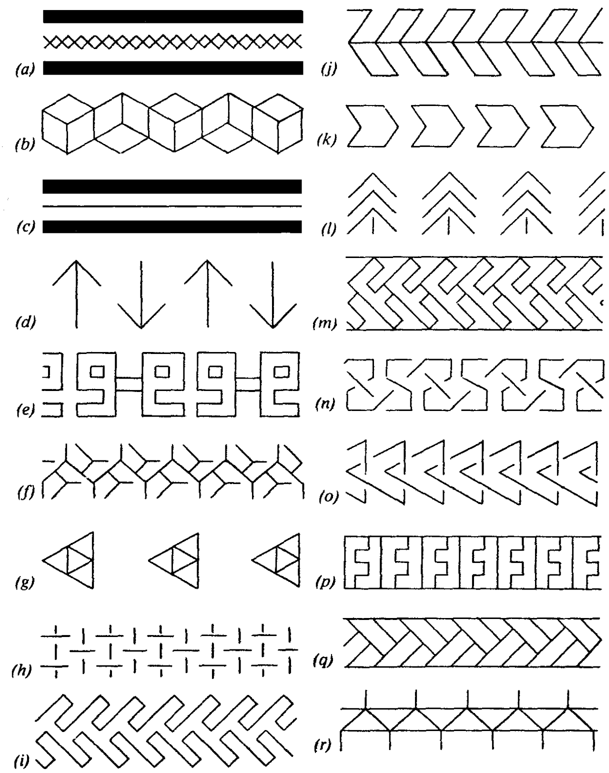
(d) 

(e) 

5. Identify the frieze groups of the following friezes taken from Theodore Menten's *Japanese Border Designs* in the Dover Pictorial Archive Series:



6. Identify the frieze groups for the following figures that are friezes:



4.5 The Wallpaper Groups

This section introduces the symmetry groups of wallpaper patterns and provides the vocabulary and techniques necessary to identify them. We omit much of the theoretical development and state the classification theorem without proof.

Definition 159 *Two translations are independent if and only if their respective glide vectors are linearly independent.*

Definition 160 *A wallpaper pattern is a plane figure W with independent translational symmetries τ_1 and τ_2 satisfying the following property: Given any translational symmetry τ, there exist integers i and j such that $\tau = \tau_2^j \circ \tau_1^i$. Translations τ_1 and τ_2 are called basic translations. The symmetry group of a wallpaper pattern is called a wallpaper group.*

Thus $\langle \tau_1, \tau_2 \rangle$ is the subgroup of translational symmetries in the wallpaper group of W.

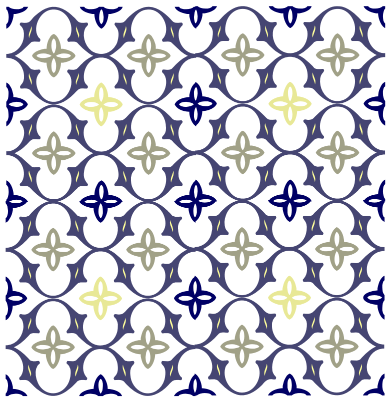


Figure 4.12: A typical wallpaper pattern.

Definition 161 *Let W be a wallpaper pattern with basic translations τ_1 and τ_2. Given any point A, let $B = \tau_1(A)$, $C = \tau_2(B)$, and $D = \tau_2(A)$. The unit cell of W with respect to A, τ_1 and τ_2 is the plane region bounded by parallelogram $\square ABCD$. The translation lattice of W determined by A is the set of points $\{(\tau_2^n \circ \tau_1^m)(A) | \ m, n \in \mathbb{Z}\}$; this lattice is square, rectangular, or rhombic if and only if the unit cell of W with respect to A, τ_1 and τ_2 is square, rectangular or rhombic.*

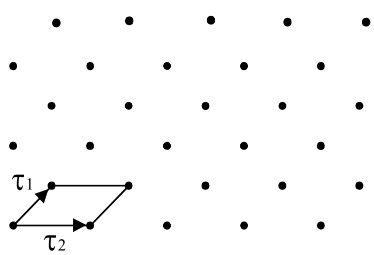


Figure 4.13: A typical translation lattice and unit cell.

Definition 162 *Let W be a wallpaper pattern. A point P is an n-center of W if and only if the group of rotational symmetries of W centered at P is C_n with $n > 1$.*

Theorem 163 *The symmetries of a wallpaper pattern fix the set of n-centers, i.e., if P is an n-center of W and α is a symmetry of W, then $\alpha(P)$ is an n-center of W.*

Proof. Let W be a wallpaper pattern with symmetry group $\mathcal{W}$ and let be P an n-center of W. Since C_n is the subgroup of rotational symmetries with center P, there is a smallest positive real number Θ such that $\rho_{P,\Theta}^n = \iota$. Now if $\alpha \in \mathcal{W}$ and $Q = \alpha(P)$, then $\alpha \circ \rho_{P,\Theta} \circ \alpha^{-1} = \rho_{Q,\pm\Theta} \in \mathcal{W}$ by closure and $\rho_{Q,\pm\Theta}^n = \left(\alpha \circ \rho_{P,\Theta} \circ \alpha^{-1}\right)^n = \alpha \circ \rho_{P,\Theta}^n \circ \alpha^{-1} = \iota$. But $\rho_{Q,\Theta} \in \mathcal{W}$ if and only if $\rho_{Q,-\Theta} \in \mathcal{W}$ so $\rho_{Q,\Theta}^n = \iota$. Thus Q is an m-center for some $m \leq n$. By the same reasoning, $\left(\alpha^{-1}\right) \circ \rho_{Q,\Theta} \circ \left(\alpha^{-1}\right)^{-1} = \rho_{P,\pm\Theta} \in \mathcal{W}$ implies that $\rho_{P,\Theta}^m = \iota$, in which case P is an n-center with $n \leq m$. Therefore $m = n$ and Q is an n-center as claimed. ■

Two n-centers in a wallpaper patterns cannot be arbitrarily close to one another.

Theorem 164 *Let W be a wallpaper pattern and let τ be a translational symmetry of shortest length. If A and B are distinct n-centers of W, then $AB \geq \frac{1}{2}\|\tau\|$.*

Proof. Let $n > 1$ and consider distinct n-centers A and B. Then $\rho_{A,360/n}$ and $\rho_{B,360/n}$ are elements of the wallpaper group $\mathcal{W}$. By closure and the Angle Addition Theorem, $\rho_{B,360/n} \circ \rho_{A,-360/n}$ is a non-identity translation in $\mathcal{W}$. Since every translation in $\mathcal{W}$ is generated by two basic translations τ_1 and τ_2, there exist integers i and j, not both zero, such that $\rho_{B,360/n} \circ \rho_{A,-360/n} = \tau_2^j \circ \tau_1^i$, or equivalently,

$$\rho_{B,360/n} = \tau_2^j \circ \tau_1^i \circ \rho_{A,360/n}.$$

Consider the point A_{ij} in the translation lattice determined by A given by

$$A_{ij} = \left(\tau_2^j \circ \tau_1^i\right)(A) = \left(\tau_2^j \circ \tau_1^i\right)\left(\rho_{A,360/n}(A)\right) = \rho_{B,360/n}(A).$$

Note that $A_{ij} \neq A$ since i and j are not both zero. Thus $AA_{ij} \geq \|\tau\|$. Now if $n = 2$, $A_{ij} = \varphi_B(A)$; and if $n > 2$, $\triangle ABA_{ij}$ is isosceles. In either case, $AB = BA_{ij}$. But $AB + BA_{ij} \geq AA_{ij}$ by the triangle inequality so it follows that $2AB \geq \|\tau\|$. ■

The next theorem, which was first proved by the Englishman W. Barlow in the late 1800's, is quite surprising. It tells us that wallpaper patterns cannot have 5-centers; consequently, crystalline structures cannot have pentagonal symmetry.

Theorem 165 *(The Crystallographic Restriction) If P is an n-center of a wallpaper pattern W, then $n \in \{2, 3, 4, 6\}$.*

Proof. Let P be an n-center of W and let τ be a translation of shortest length. We begin with an indirect argument that allows us to choose an n-center $Q \neq P$ whose distance from P is a minimum. Suppose that no such Q exists. Then there is an infinite sequence of n-centers $\{Q_k\}$ such that $PQ_1 > PQ_2 > \cdots$. By Theorem 164, $PQ_k \geq \frac{1}{2}\|\tau\|$ for all k so that $\{PQ_k\}$ is a strictly decreasing sequence of positive numbers converging to $M \geq \frac{1}{2}\|\tau\|$, i.e., given

$\epsilon > 0$, there is a positive integer N such that if $k > N$ then $M < PQ_k < M + \epsilon$. But this means that infinitely many n-centers Q_k lie within ϵ of the circle centered at P of radius M, which is impossible since $Q_iQ_j \geq \frac{1}{2}\|\tau\|$ for all i, j. So choose an n-center Q whose distance from P is a minimum and let $R = \rho_{Q,360/n}(P)$. By Theorem 163, R is an n-center and $PQ = QR$. Let $S = \rho_{R,360/n}(Q)$; then S is an n-center and $RQ = RS$. If $S = P$, then $\triangle PQR$ is equilateral in which case the rotation angle is 60° and $n = 6$. If $S \neq P$, then by the choice of Q, $SP \geq PQ = QR = RS$ in which case the rotation angle is at least 90° and $n \leq 4$. Therefore n is one of $2, 3, 4$, or 6. ∎

Corollary 166 *A wallpaper pattern with a* 4*-center has no* 3 *or* 6*-centers.*

Proof. If P is a 3-center and Q is a 4-center of a wallpaper pattern W, the corresponding wallpaper group $\mathcal{W}$ contains the rotations $\rho_{P,120}$ and $\rho_{Q,-90}$. By closure, $\mathcal{W}$ also contains the 30° rotation $\rho_{P,120} \circ \rho_{Q,-90}$, which generates C_{12}. Therefore there is an n-center of W with $n \geq 12$. But this contradicts Theorem 165. Similarly, if Q is a 4-center and R is a 6-center of W, there is also an n-center of W with $n \geq 12$ since $\rho_{R,-60} \circ \rho_{Q,90}$ is a 30° rotation. ∎

In addition to translational symmetry, wallpaper patterns can have line symmetry, glide reflection symmetry, and 180°, 120°, 90° or 60° rotational symmetry. Since the only rotational symmetries in a frieze group are halfturns, it is not surprising to find more wallpaper groups than frieze groups. We shall identify seventeen distinct wallpaper groups but we omit the proof that every wallpaper group is one of these seventeen. Throughout this discussion, W denotes a wallpaper pattern. We use the international standard notation to denote the various wallpaper groups. Each symbol is a string of letters and integers selected from p, c, m, g and $1, 2, 3, 4, 6$. The letter p stands for *primitive translation lattice*. The points in a primitive translation lattice are the vertices of parallelograms with no interior points of symmetry. When a point of symmetry lies at the center of some unit cell, we use the letter c. The letter m stands for *mirror* and indicates lines of symmetry; the letter g indicates glide reflection symmetry. Integers indicate the maximum order of the rotational symmetries of W.

There are four symmetry types of wallpaper patterns with no n-centers. These are analyzed as follows: If W has no line symmetry or glide reflection symmetry, the corresponding wallpaper group consists only of translations and is denoted by $p1$. If W has glide reflection symmetry but no lines of symmetry, the corresponding wallpaper group is denoted by pg. There are two ways that both line symmetry and glide reflection symmetry can appear in W: (1) the axis of some glide reflection symmetry is not a line of symmetry and (2) the axis of every glide reflection symmetry is a line of symmetry. The corresponding wallpaper groups are denoted by cm and pm, respectively.

There are five symmetry types whose n-centers are all 2-centers. If W has neither lines of symmetry nor glide reflection symmetries, the corresponding wallpaper group is denoted by $p2$. If W has no line symmetry but has glide reflection symmetry, the corresponding group is denoted by pgg. If W has parallel lines of symmetry, the corresponding group is denoted by *pmg*. If W has lines of symmetry in two directions, there are two ways to configure them relative to the 2-centers in W: (1) all 2-centers lie on a line of symmetry and (2) not all 2-centers lie on a line of symmetry. The corresponding wallpaper groups are denoted by *pmm* and *cmm*, respectively.

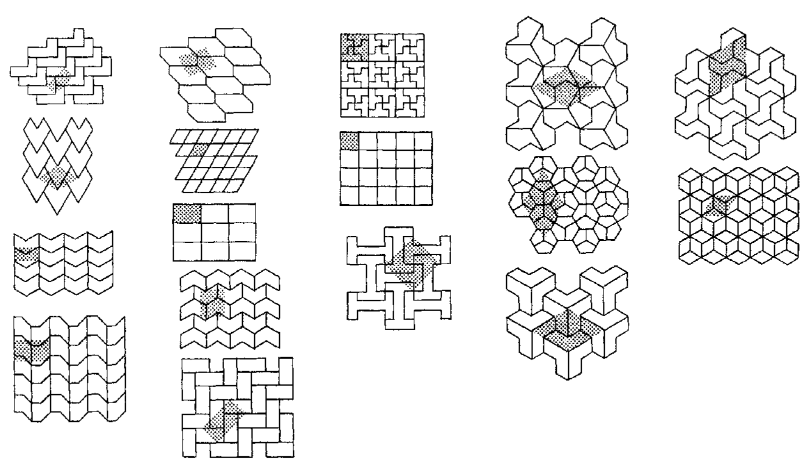
Three wallpaper patterns have n-centers whose smallest rotation angle is $90°$. Those with no lines of symmetry have wallpaper group $p4$. Those with lines of symmetry in four directions have wallpaper group $p4m$; other patterns with lines of symmetry have wallpaper group $p4g$.

Three symmetry types have n-centers whose smallest rotation angle is $120°$. Those with no lines of symmetry have wallpaper group $p3$. Those whose 3-centers lie on lines of symmetry have wallpaper group $p3m1$; those with some 3-centers off lines of symmetry have wallpaper group $p31m$.

Finally, two symmetry types have n-centers whose smallest rotation angle is $60°$. Those with line symmetry have wallpaper group $p6m$; those with no line symmetry have wallpaper group $p6$.

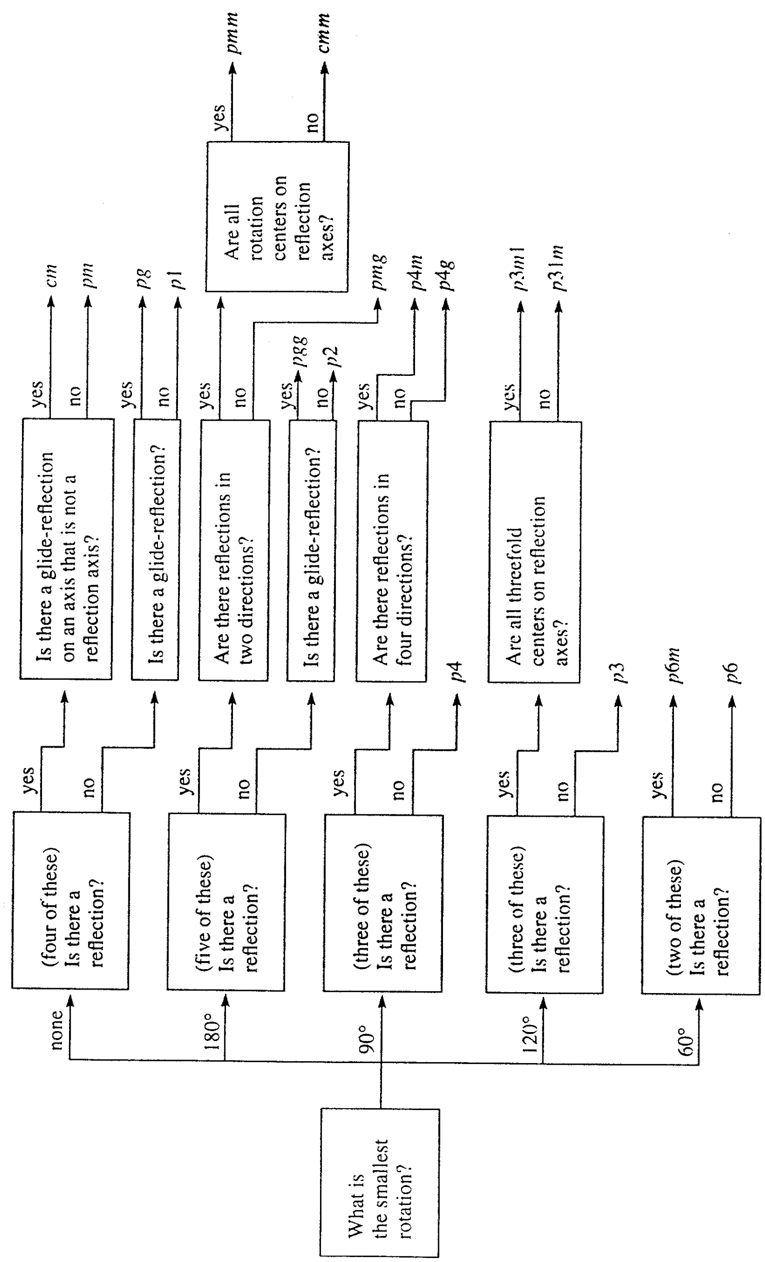
Theorem 167 *Every wallpaper group is one of the following:*

$p1$	$p2$	$p4$	$p3$	$p6$
cm	cmm	$p4m$	$p3m1$	$p6m$
pm	pmm	$p4g$	$p31m$	
pg	pmg			
	pgg			

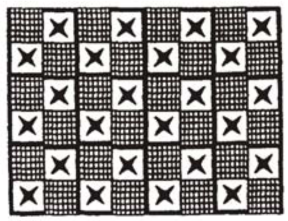


The following flowchart can be used to identify the wallpaper group associated with a particular wallpaper pattern:

Recognition Flowchart for Wallpaper Patterns



Example 168 *Here are some wallpaper patterns from around the world. Try your hand at identifying their respective wallpaper groups.*



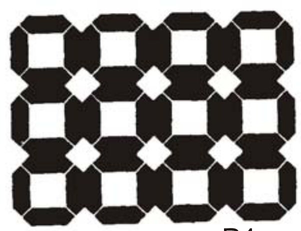
FIJI p4m

SPAIN pmm

EGYPT pmg

JAPAN cm

GERMANY cmm



ITALY P4m

EGYPT P4g

CHINA p3m1

Exercises

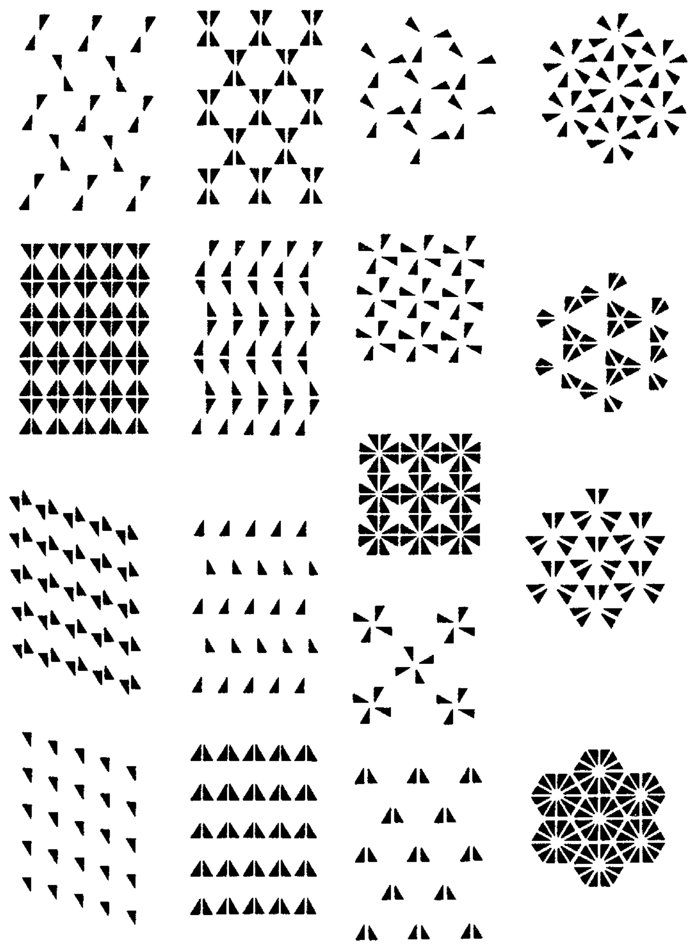
1. Identify the wallpaper group for the pattern in Figure 4.12.

2. Find at least two different wallpaper patterns on your campus and identify their wallpaper groups.

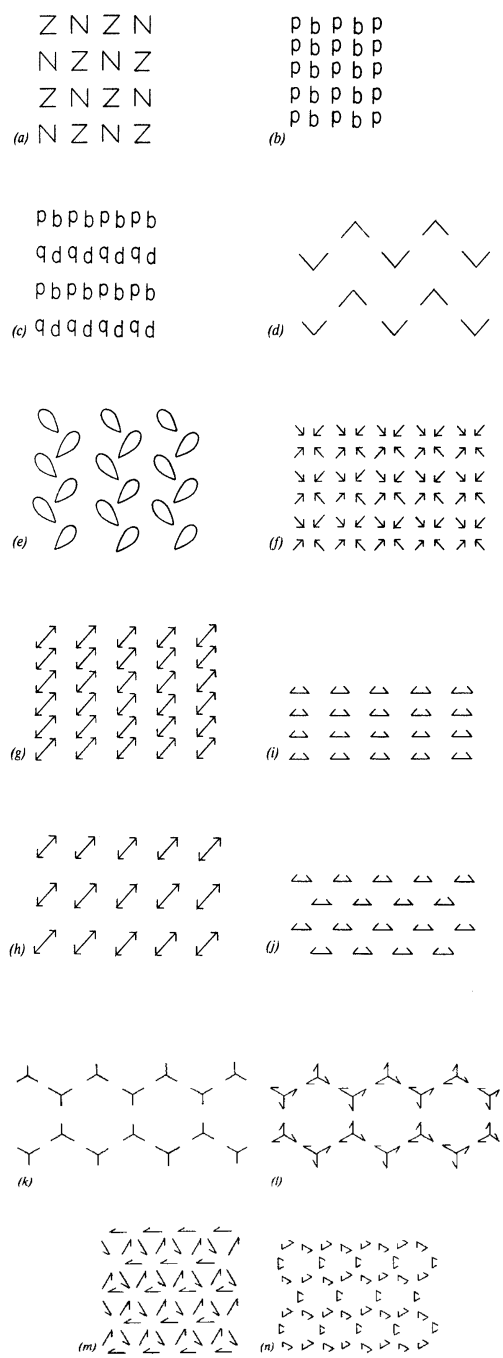
3. Identify the wallpaper groups for the following patterns.

4. Prove that if A and B are distinct points of symmetry for a plane figure F, the symmetry group of F contains a non-identity translation, and consequently has infinite order. (Hint: Consider all possible combinations of n and m such that A is an n-center and B is an m-center.)

5. Identify the wallpaper groups for the following patterns:



6. Identify the wallpaper groups for the following patterns:



7. Identify the wallpaper groups for the following patterns: