Chapter 5

Symmetry and Billiards: An Application

A billiard ball in motion on a frictionless triangular table bounces about along a path completely determined by its initial position, angle and speed. When the ball strikes a bumper at any point other than a vertex, we assume that *the angle of incidence equals the angle of reflection.* An *orbit* is any complete trajectory of the ball. If the ball returns to its initial position with its initial velocity direction, the orbit is *periodic.* A periodic orbit α has *period* n if the ball, when released from a point P not on a side of the triangle, follows α and returns to P with its initial velocity direction after n bounces. A periodic orbit with period n is *primitive* if the ball returns to its initial position for the first time after n bounces, otherwise it is an *iterate.* Orbits that are not periodic can be *singular*, with one or both endpoints at a vertex of the triangle, or *infinite,* with the ball never retracing its trajectory.

In this chapter we give a complete solution to the following billiards problem: *Find, classify and count the classes of periodic orbits of a given period on an equilateral triangle.* While periodic orbits are known to exist on all non-obtuse and certain classes of obtuse triangles, existence in general remains a long-standing open problem. The first examples of periodic orbits were discovered by Fagnano in 1745. Interestingly, his orbit of period 3 on an acute triangle, known as the "Fagnano orbit," was found not as the solution of a billiards problem, but rather as the triangle of least perimeter inscribed in a given acute triangle. This problem, known as "Fagnano's problem," is solved by the orthic triangle, whose vertices are the feet of the altitudes of the given triangle (see Figure 5.1). The orthic triangle is a billiard orbit since its angles are bisected by the altitudes of the triangle in which it is inscribed; the proof given by Coxeter and Greitzer uses exactly the unfolding technique we use below to unfold any orbit. Coxeter credits this technique to H. A. Schwarz and mentions that Frank and F. V. Morley extended Schwarz's treatment on triangles to odd-sided polygons. For a discussion of some interesting properties of the Fagnano orbit on any acute triangle.

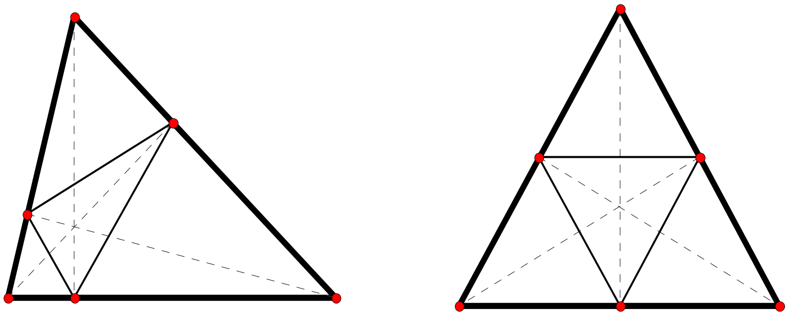

Figure 5.1. Fagnano's period 3 orbit.

Much later, in 1986, Masur proved that every *rational* polygon (one whose interior angles are rational multiples of π) admits infinitely many periodic orbits with distinct periods, but he neither constructs nor classifies them. A year later Katok proved that the number periodic orbits of a given period grows subexponentially. Existence results on various polygons were compiled by Tabachnikov in 1995.

The content of this chapter is based on a senior honors thesis written by Andrew Baxter in 2004 under my supervision.

5.1 Orbits and Tessellations

We begin by defining an equivalence relation on the set of all periodic orbits on an equilateral triangle $\triangle ABC$. But first, we make several interesting and important observations.

Proposition 169 *Every non-singular orbit strikes some side of* $\triangle ABC$ *with an angle of incidence in the range* $30^\circ \leq \theta \leq 60^\circ$.

Proof. Given a non-singular orbit α, choose a point P_1 at which α strikes $\triangle ABC$ with angle of incidence θ_1. If θ_1 lies in the desired range, set $\theta = \theta_1$. Otherwise, let α_1 be the segment of α that connects P_1 to the next strike point P_2 and label the vertices of $\triangle ABC$ so that P_1 is on side $\overline{AC}$ and P_2 is on side $\overline{BC}$. If $60^\circ < \theta_1 \leq 90^\circ$, the angle of incidence at P_2 is $\theta_2 = m\angle P_1P_2C = 120^\circ - \theta_1$ and satisfies $30^\circ \leq \theta_2 < 60^\circ$ (see Figure 5.2). So set $\theta = \theta_2$. If $0^\circ < \theta_1 < 30^\circ$, then $\theta_2 = m\angle P_1P_2B = \theta_1 + 60^\circ$ so that $60^\circ < \theta_2 < 90^\circ$. Let α_2 be the segment of α that connects P_2 to the next strike point P_3. Then as in the previous case, the angle of incidence at P_3 satisfies $30^\circ < \theta_3 < 60^\circ$; set $\theta = \theta_3$. ■

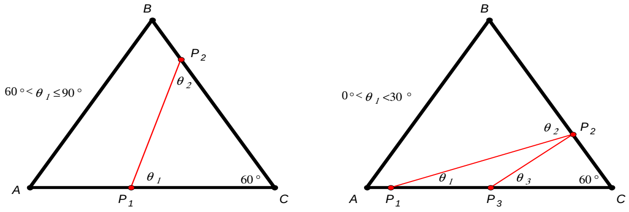

Figure 5.2. Incidence angles θ_2 (left) and θ_3 (right) in the range $30° \leq \theta \leq 60°$.

Now given a periodic orbit α on an equilateral triangle $\triangle ABC$, choose a point P at which α strikes the triangle with angle of incidence in the range $30° \leq \theta \leq 60°$. If necessary, relabel the vertices of the triangle so that side $\overline{BC}$ contains P. Let $\mathcal{T}$ be a regular tessellation (or tiling) of the plane by equilateral triangles each congruent to $\triangle ABC$ and positioned so that one of its families of parallel edges is horizontal. Embed $\triangle ABC$ in $\mathcal{T}$ so that its base $\overline{BC}$ is collinear with a horizontal edge of $\mathcal{T}$. Let α_1 be the segment of α in the direction of the ray from P with polar angle θ. Move from P along α_1 until it strikes a side s_1 of $\triangle ABC$ at point P_1 with angle of incidence θ_1. Let σ_1 be the reflection in the edge of $\mathcal{T}$ containing s_1 and let $\alpha_2 = \sigma_1(\alpha_1)$. Now, instead of continuing along the next segment of α in $\triangle ABC$, continue along its reflection in s_1, which is a segment of α_2 collinear with $\overrightarrow{PP_1}$ and in the basic triangle of $\mathcal{T}$ that shares sides s_1 with $\triangle ABC$. Following the directed segment from P to P_1, denoted by $\underline{PP_1}$, move along α_2 until it strikes a side s_2 of the triangle containing α_2, at point P_2 with angle of incidence θ_2. Let σ_2 be the reflection in the edge of $\mathcal{T}$ containing s_2 and let $\alpha_3 = \sigma_2(\alpha_2)$. Following the direction of $\underline{PP_2}$, move along α_3 into the triangle of $\mathcal{T}$ that contains α_3. Continue in this manner until the ball arrives at Q, the image of P, after the $(n-1)^{st}$ reflection σ_{n-1} (see Figure 5.3). Let θ_n be the angle of incidence at Q and let σ_n be the reflection in the line of $\mathcal{T}$ containing Q.

Thus we obtain a cycle of reflections $(\sigma_1, \ldots, \sigma_n)$ and a cycle of incidence angles $(\theta_1, \ldots, \theta_n)$ with $30° \leq \theta_n \leq 60°$. We refer to $\underline{PQ}$ as an *unfolding* of α, i.e., a representation of α as a directed segment with the same length as α and cutting a sequence of adjacent triangles in $\mathcal{T}$ with incidence angles $\theta_1, \ldots, \theta_n$, and to θ_n as its *representation angle.* The embedding of $\triangle ABC$ in $\mathcal{T}$ leads us to the following result for periodic orbits:

Proposition 170 *A periodic orbit strikes the sides of* $\triangle ABC$ *with at most three incidence angles, exactly one of which lies in the range* $30° \leq \theta \leq 60°$. *In fact, exactly one of the following holds:*

1. *All incidence angles measure* $60°$.

2. *There are exactly two distinct incidence angles measuring* $30°$ *and* $90°$.

3. *There are exactly three distinct incidence angles* ϕ, θ *and* ψ *such that* $0^\circ < \phi < 30^\circ < \theta < 60^\circ < \psi < 90^\circ$.

Proof. Let α be a non-singular orbit, and let $\underline{PQ}$ be an unfolding. By Proposition 169, $\underline{PQ}$ crosses each horizontal edge of $\mathcal{T}$ with angle of incidence in the range $30^\circ \leq \theta \leq 60^\circ$. Consequently, $\underline{PQ}$ crosses each left-leaning edge of $\mathcal{T}$ with angle of incidence $\phi = 120^\circ - \theta$ and crosses each right-leaning edge of $\mathcal{T}$ with angle of incidence $\psi = 60^\circ - \theta$ (see Figure 5.4). In particular, if $\theta = 60^\circ$, $\underline{PQ}$ crosses only left-leaning and horizontal edges, and all incidence angles are equal. In this case, α is either the Fagnano orbit, a primitive orbit of period 6 or some iterate these. If $\theta = 30^\circ$, then $\phi = 90^\circ$ and $\psi = 30^\circ$, and α is either a primitive orbit of period 4 or some iterate thereof (see Figure 5.3). When $30^\circ < \theta < 60^\circ$, clearly $0^\circ < \phi < 30^\circ$ and $60^\circ < \psi < 90^\circ$. ■

Corollary 171 *Any two unfoldings of a non-singular orbit are parallel.*

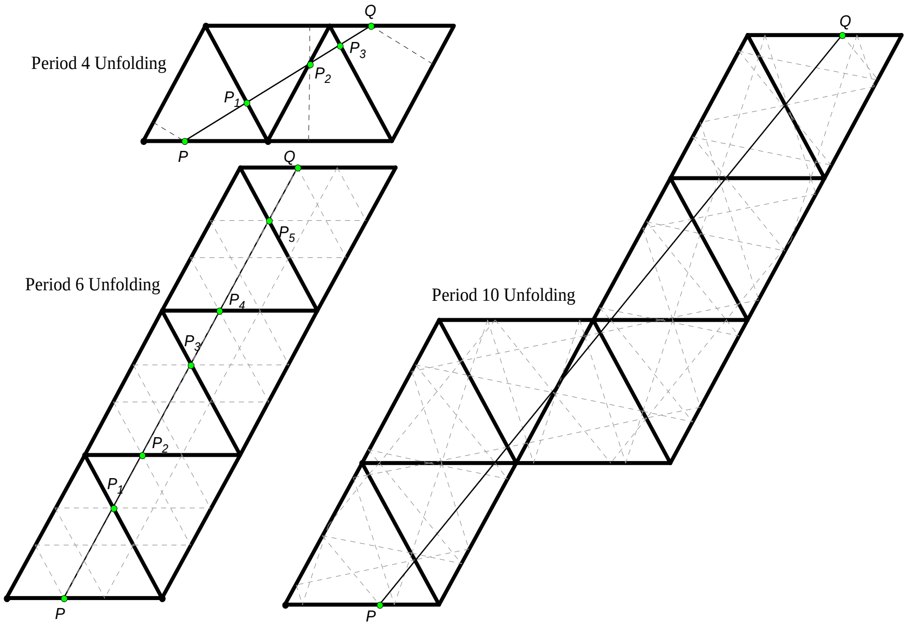

Figure 5.3. Unfolded orbits of period 4, 6, and 10.

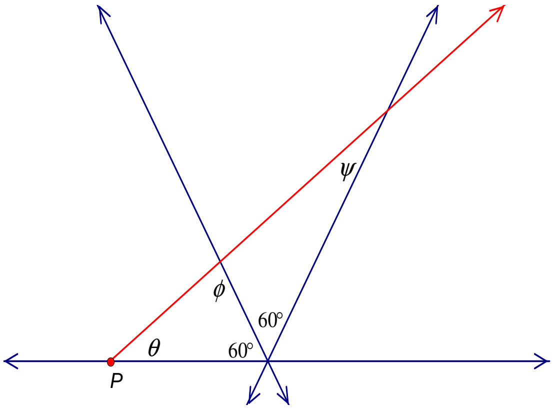

Figure 5.4. Incidence angles θ, ϕ and ψ.

Our next result is important and quite surprising:

Theorem 172 *Let $\overline{PQ}$ be an unfolding of a periodic orbit α. If Q lies on a horizontal edge of $\mathcal{T}$, then the period of α is even.*

Proof. Since both P and Q lie on horizontal edges of $\mathcal{T}$, the basic triangles of $\mathcal{T}$ cut by $\overline{PQ}$ pair up to form a polygon of rhombic tiles that contain $\overline{PQ}$ (see Figure 5.5). As the path $\overline{PQ}$ traverses this polygon, it enters each rhombic tile through an edge, crosses a diagonal of that tile (a left-leaning edge of $\mathcal{T}$), and exits through another edge. Since each exit edge is the entrace edge of the next tile and the edge containing P is identified with the edge containing Q, the number of distinct edges of $\mathcal{T}$ cut by $\overline{PQ}$ is exactly twice the number of rhombic tiles. It follows that α has even period. ■

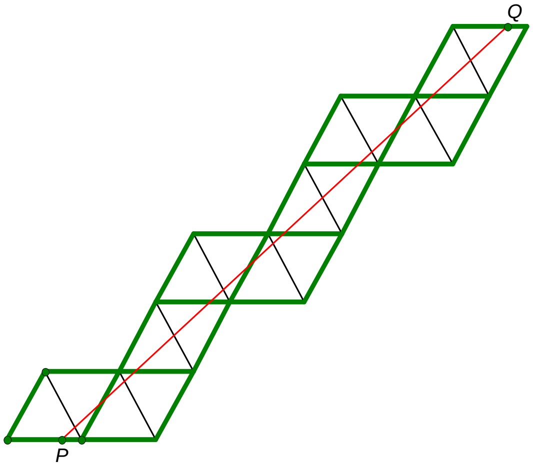

Figure 5.5. A typical rhombic tiling.

In light of Theorem 172, orbits with odd period are easy to characterize.

Theorem 173 *An orbit with odd period is either Fagnano's orbit of period* 3 *or an odd iterate thereof.*

Proof. Let $\underline{PQ}$ be an unfolding of an orbit α with odd period n. Then by Theorem 172, Q does not lie on a horizontal edge of $\mathcal{T}$. By definition of unfolding, the angle of incidence at P must equal the angle of incidence at Q, so by Proposition 170, the only possible values θ for the angle of incidence are 60° or 30°. But $\theta \neq 30^\circ$ since otherwise α has even period by Proposition 170. Thus $\theta = 60^\circ$ and Q necessarily lies on a left-leaning edge by Proposition 170. Now the composition f of reflections in the edges of $\mathcal{T}$ that sends P to Q has even parity, and since Q is on a left-leaning edge of $\mathcal{T}$, this composition is a rotation of 120° or 240°. Thus the only possible position for P is at the midpoint of side $\overline{BC}$ of $\triangle ABC$ and α is either the Fagano orbit or one of its odd iterates. ■

Let G be the group generated by all reflections in the edges of $\mathcal{T}$. Since the action of G on $\overline{BC}$ generates a regular tessellation $\mathcal{H}$ of the plane by hexagons, the point Q in any unfolding $\underline{PQ}$ of a periodic orbit must lie on some horizontal edge of $\mathcal{H}$.

Proposition 174 *Unfoldings of orbits with even period terminate on a horizontal edge of* $\mathcal{H}$.

Proof. Let $\underline{PQ}$ be an unfolding of an orbit α with even period. If Q lies on a non-horizontal edge of $\mathcal{H}$, then, arguing as in the proof of Theorem 173, the incidence angle θ at both P and Q is either 30° or 60°. In the first case, α is either an orbit with period 4 or some iterate thereof, and all unfoldings of these terminate on a horizontal edge of $\mathcal{H}$ (see Figure 5.3). If $\theta = 60^\circ$, there are two types of primitive orbits: the Fagano orbit, which has period 3, and orbits of period 6 where P is not a midpoint of edge $\overline{BC}$ of $\triangle ABC$ (see Figures 5.3 and 5.7). The latter orbits and their iterates all terminate on horizontal edges of $\mathcal{H}$, as do all even iterates of the Fagano orbit. Thus Q must lie on a horizontal edge of $\mathcal{H}$. ■

Periodic orbits represented by horizontal translations of an unfolding $\underline{PQ}$ are typically distinct, but have the same length and incidence angles (up to permutation) as α. So it is natural to think of them as equivalent.

Definition 175 *Periodic orbits* α *and* β *are* *<u>equivalent</u>* *if there exist respective unfoldings* $\underline{PQ}$ *and* $\underline{RS}$ *and a horizontal translation* τ *such that* $\underline{RS} = \tau\left(\underline{PQ}\right)$. *The symbol* $[\alpha]$ *denotes the equivalence class of* α.

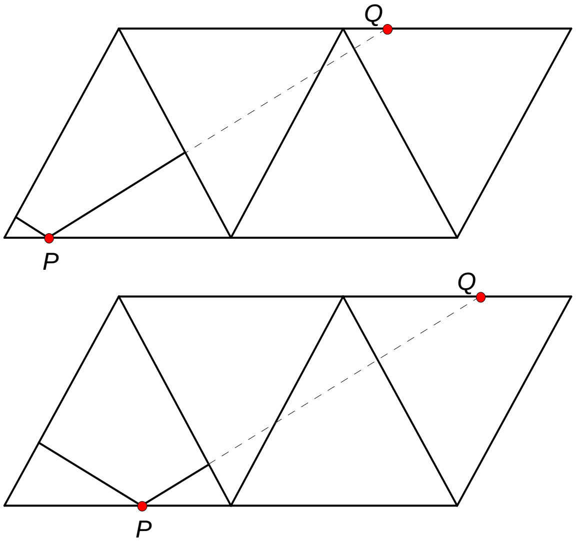

Figure 5.6. Unfoldings of equivalent period 4 orbits

Quite remarkably, the parity of an orbit's period determines the cardinality of its class.

Proposition 176 *Every class of orbits with even period is uncountable.*

Proof. Let α be an orbit with even period and let $\underline{PQ}$ be an unfolding. By Proposition 174, P and Q lie on horizontals of $\mathcal{T}$, and of course, some positive (real) horizontal distance d from the nearest vertex. Consequently, there are uncountably many horizontal right-translations of $\underline{PQ}$ through distances less than d that miss the vertices of $\mathcal{T}$ (the number of vertices is countable). Since such translations of $\underline{PQ}$ represent distinct but equivalent orbits, the class $[\alpha]$ is uncountable. ■

Note that the class of Fagnano's orbit γ is a singleton class since every unfolding $\underline{PQ}$ of γ lies on some midline of adjacent parallels of $\mathcal{T}$ (translations of $\underline{PQ}$ off its midline do not represent periodic orbits). This is also true for odd iterates of γ. The converse is an interesting consequence of Theorem 173 and Proposition 176. Thus we have:

Corollary 177 *A class* $[\alpha]$ *is a singleton if and only if* α *is Fagnano's orbit or one of its odd iterates.*

Having completely classified orbits with odd period, we turn our attention to orbits with even period. Our classification strategy is to represent equivalence classes as lattice points in some "fundamental region" of the plane, a construction of which follows below.

We wish to exclude even iterates of Fagnano's orbit γ from future considerations for the following reason: Let $\underline{PQ}$ be an unfolding of γ and let R be the point on ray $\overrightarrow{PQ}$ such that Q is the midpoint of $\underline{PR}$. Then $\underline{PR}$ represents a 2-fold iterate β of γ, while translations of $\underline{PR}$ off the midline represent *primitive* orbits equivalent to β (see Figure 5.7). When iterates of γ are excluded, "primitivity" becomes a property common to all orbits in the same class. Thus from now on, we use the phrase "orbit with even period" when we mean an orbit with even period that is not an even iterate of γ.

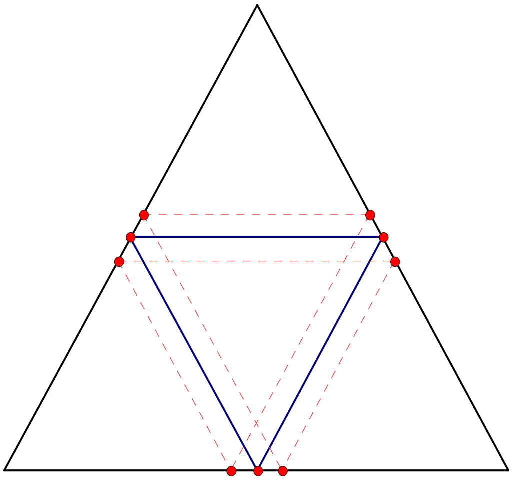

Figure 5.7. The Fagnano orbit and a related period 6 orbit (dotted).

From each class $[\alpha]$ of orbits with even period, arbitrarily choose a representative β and an unfolding $\underline{P_\beta Q_\beta}$. Then Q_β lies on a horizontal edge of $\mathcal{H}$ by Proposition 174. Choose a point R on the base $\overline{BC}$ of $\triangle ABC$ and let τ_β be the horizontal translation defined by $\tau_\beta(P_\beta) = R$. Then $\mathcal{S} = \bigcup_\beta \tau_\beta\left(\underline{P_\beta Q_\beta}\right)$ is a bouquet of directed segments with common initial point R. If necessary, horizontally translate $\mathcal{S}$ off of the vertices of $\mathcal{T}$ sending R to some point P on $\overline{BC}$ other than the midpoint and fix this point P once and for all. The polar region $30° \leq \theta \leq 60°$ centered at P is called the *fundamental region at* P and is denoted by Γ_P. By Proposition 171, each class $[\alpha]$ contains a unique representative with an unfolding $\underline{PQ}$ in Γ_P. We refer to $\underline{PQ}$ as the *fundamental unfolding of* $[\alpha]$. Note that even iterates of Fagnano's orbit have no unfolding $\underline{PQ}$ in Γ_P.

Our construction of the fundamental region moves us one step closer to classification and proves:

Proposition 178 *Each class of orbits with even period has a unique fundamental unfolding.*

To complete the classification, we must determine exactly which directed segments in Γ_P with initial point P represent orbits with even period. We address this question in the next section.

5.2 Orbits and Rhombic Coordinates

In this section we introduce the analytical structure we need to complete the classification and to count the distinct classes of orbits of a given even period. Expressing the fundamental unfolding of a class as a vector $\overrightarrow{PQ}$ allows us to exploit the natural rhombic coordinate system given by $\mathcal{T}$. Position the origin at P so that the x-axis is the horizontal edge of $\mathcal{T}$ through P. Choose the line through P with inclination 60° as the y-axis and choose BC (the length of base of $\triangle ABC$) as the unit length (see Figure 5.8).

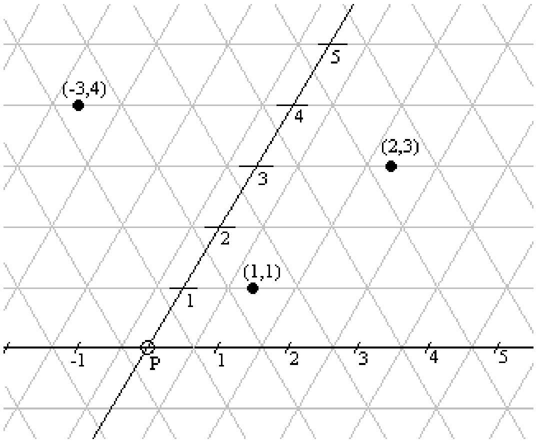

Figure 5.8. Rombic coordinates.

Let α be an orbit with even period and let $\underline{PQ}$ be the fundamental unfolding of $[\alpha]$. Note that the composition f of reflections in the edges of $\mathcal{T}$ that maps P to Q and the hexagon whose base $\overline{AB}$ contains P to the hexagon whose base $\overline{A'B'}$ contains Q has even parity. Therefore f and is either a translation or a rotation of 120° or 240°. But $\overline{AB}\|\overline{A'B'}$ so f is a translation and the position of Q on $\overline{A'B'}$ is exactly the same as the position of P on $\overline{AB}$.

Use the matrix

$$\begin{bmatrix} 1 & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{3}} \end{bmatrix}$$

to change from the standard to rhombic coordinates. Then in rhombic coordinates, norm and polar angle are given by

$$\|(x,y)\| = \sqrt{x^2+xy+y^2} \quad \text{and} \quad \theta(x,y) = \arctan\left(\tfrac{y\sqrt{3}}{2x+y}\right). \tag{5.1}$$

Furthermore,

$$\Gamma_P = \{(x, y) \mid 0 \leq x \leq y\}$$

and the termini Q of fundamental unfoldings $\underline{PQ}$ are elements of

$$\{a(1,1) + b\,(0,3) \mid a, b \in \mathbb{N}\}$$

since the vectors $(1,1)$ and $(0,3)$ are the two linearly independent vectors of shortest length defining translations that map the hexagon with base $\overline{BC}$ to another hexagon.

Theorem 172 can be restated in these analytical terms as follows:

Corollary 179 *If $\underline{PQ}$ is the fundamental unfolding of $[\alpha]$ and $Q = (x, y) \in \Gamma_P$ lies on a horizontal edge of $\mathcal{T}$, then the period of α is $2(x + y)$.*

Combining Proposition 174 and Corollary 179 we have:

Corollary 180 *The segment $\underline{PQ}$ in Γ_P represents an orbit with even period if and only if Q lies on some horizontal edge of $\mathcal{H}$ (see Figure 5.9).*

This analytical point of view gives a nice characterization of those vectors $\overrightarrow{PQ}$ in Γ_P representing orbits with even period. If $\overrightarrow{PQ}$ represents an orbit in Γ_P and $Q = (x, y)$, then Q lies on a horizontal edge of $\mathcal{H}$ by Proposition 174, in which case $(x, y) = a(1,1) + b(0,3)$ for some $a, b \in \mathbb{Z}$, and $x \equiv y \,(\text{mod}\, 3)$. Conversely, if $\overrightarrow{PQ}$ lies in Γ_P and $Q = (x, y)$ satisfies $x \equiv y \,(\text{mod}\, 3)$, then $Q = x\,(1,1) + k\,(0,3)$ for some $k \in \mathbb{Z}$, in which case Q lies on a horizontal edge of some hexagon of $\mathcal{H}$ in the same relative position as P on $\overline{BC}$. Thus $Q = g\,(P)$ for some $g \in G$ and $\overrightarrow{PQ}$ represents an orbit with even period. This completes the classification of orbits with even period.

Theorem 181 *(Classification) There is a bijection:*

$$\{\textit{Classes of orbits with even period}\} \leftrightarrow$$
$$\{(x, y) \in \Gamma_P \mid x \in \mathbb{N} \cup \{0\} \ \textit{and}\ x \equiv y \,(\text{mod}\, 3)\}.$$

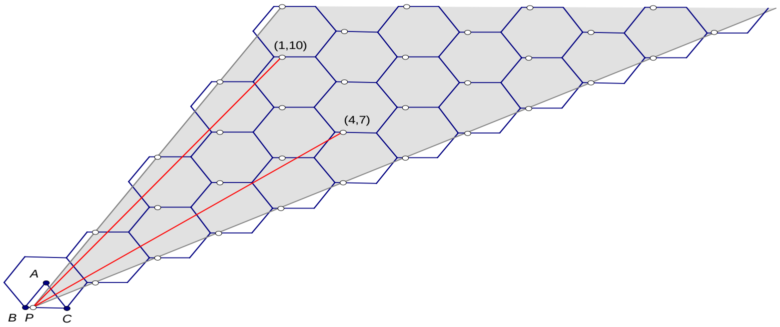

Figure 5.9. Translated images of P and orbits of period 22 in Γ_P.

In light of our Classification Theorem, we often refer to an ordered pair (x, y) as an "orbit" when we mean the class of orbits with even period to which it corresponds. Thus we may count classes of orbits of a given period $2n$ by counting integer pairs (x, y) in Γ_P such that $x \equiv y \,(\operatorname{mod} 3)$ and $x + y = n$. This is the objective of the next section.

5.3 Orbits and Partitions of n

Two questions arise: (1) Is there an orbit with period $2n$ for each $n \in \mathbb{N}$? (2) If so, exactly how many distinct classes of orbits with period $2n$ are there? If we admit k-fold iterates, question (1) has an easy answer. As noted earlier, there are no orbits of period 2. For each $n > 1$, the orbit

$$\alpha = \begin{cases} \left(\frac{n}{2}, \frac{n}{2}\right), & n \text{ even} \\ \\ \left(\frac{n-1}{2} - 1, \frac{n-1}{2} + 2\right), & n \text{ odd.} \end{cases}$$

has period $2n$. Note that the period 22 orbits $(1, 10)$ and $(4, 7)$ are not equivalent since they have different lengths and representation angles (see Figures 5.9 and 5.10). To answer question (2), we first construct a bijection between classes of orbits with period $2n$ and partitions of n with 2 and 3 as parts then count these partitions. The bijection follows from Theorem 181 and the following lemma:

Lemma 182 *For each $n \in \mathbb{N}$, let*

$$X_n = \{(x, y) \in \Gamma_P \mid x \in \mathbb{N} \cup \{0\},\ x \equiv y \,(\operatorname{mod} 3) \ \text{ and } x + y = n\}.$$

There is a bijection:

$$X_n \leftrightarrow \{\textit{Partitions of } n \textit{ with } 2 \textit{ and } 3 \textit{ as parts}\}.$$

Proof. If $(x, y) \in X_n$, then $n = x + y = 2 \cdot x + 3 \cdot \frac{y-x}{3}$ is a partition of n with 2 and 3 as parts. Conversely, given a partition of n with 2 and 3 as parts, write $n = 2a + 3b = a + (a + 3b)$ for some $a, b \geq 0$. Then $(a, a + 3b) \in X_n$. ■

Corollary 183 *For each $n \in \mathbb{N}$, there is a bijection:*

$$\{\textit{Orbits of period } 2n \textit{ in } \Gamma_P\} \leftrightarrow$$

$$\{\textit{Partitions of } n \textit{ with } 2 \textit{ and } 3 \textit{ as parts}\}.$$

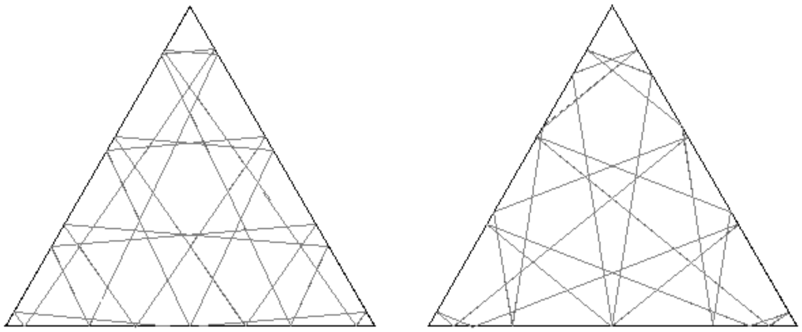

Figure 5.10. Period 22 orbits $(1,10)$ *(left) and* $(4,7)$ *(right).*

Counting partitions of n with specified parts is well understood. Let ω be a primitive cube root of unity. Using standard combinatorial techniques, we obtain the generating function

$$(1+x^2+x^4+x^6+\cdots)(1+x^3+x^6+x^9+\cdots)=\frac{1}{(1-x^2)(1-x^3)}$$

whose partial fractions decomposition

$$\frac{1}{4(1+x)}+\frac{1}{4(1-x)}+\frac{1}{6(1-x)^2}+\frac{1}{9}\left(\frac{1+2\omega}{\omega-x}+\frac{1+2\omega^2}{\omega^2-x}\right)$$

gives the following explicit formula for the coefficient of x^n :

$$\mathcal{O}(n)=\frac{(-1)^n}{4}+\frac{n}{6}+\frac{5}{12}+\frac{1}{9}\left(\omega^{2n+2}+2\omega^{2n}+\omega^{n+1}+2\omega^n\right).$$

When $n=0,1,\dots,5$ we obtain the 6 initial terms

$$\mathcal{O}(n)=\begin{cases}\lfloor\frac{n}{6}\rfloor, & n=1\\ \lfloor\frac{n}{6}\rfloor+1, & \text{otherwise,}\end{cases}$$

which together with the recurrence relation $a_{n+6}=a_n+1$ give the alternate formulation in our next theorem.

Theorem 184 *The number of distinct orbits of period* $2n$ *in* Γ_P *is exactly*

$$\mathcal{O}(n)=\left\lfloor\frac{n+2}{2}\right\rfloor-\left\lfloor\frac{n+2}{3}\right\rfloor.$$

Let us sharpen this counting formula by excluding iterates.

Definition 185 *Given an orbit* $(x,y)\in\Gamma_P$, *let* $d\in\mathbb{N}$ *be the largest value such that* $x/d\equiv y/d\,(\mathrm{mod}\,3)$. *Then* (x,y) *is* <u>*primitive*</u> *if and only if* $d=1$*; otherwise* (x,y) *is a* <u>*d-fold iterate*</u> *of a primitive orbit. All orbits in a* <u>*class of period* $2n$</u> *have period* $2n$. *All orbits in a* <u>*primitive class*</u> *are primitive; all orbits in a* <u>*non-primitive class*</u> *are d-fold iterates for some* $d>1$.

Although d is often difficult to compute, it is remarkably easy to check whether or not an orbit is primitive.

Theorem 186 *An orbit $(x,y) \in \Gamma_P$ is primitive if and only if either*

1. $\gcd(x,y) = 1$ *or*
2. $(x,y) = (3a, 3b)$, $\gcd(a,b) = 1$ *and* $a \not\equiv b \,(\mathrm{mod}\, 3)$ *for some* $a, b \in \mathbb{N} \cup \{0\}$.

Proof. If $\gcd(x,y) = 1$, the orbit (x,y) is primitive. On the other hand, if $(x,y) = (3a,3b), a \not\equiv b \,(\mathrm{mod}\, 3)$ and $\gcd(a,b) = 1$ for some a, b, let d be as in Definition 185. Then $d \neq 3$ since $a \not\equiv b \,(\mathrm{mod}\, 3)$. But $\gcd(a,b) = 1$ implies $d = 1$ so (x,y) is also primitive when either (1) or (2) holds.

Conversely, given a primitive orbit (x,y), let $c = \gcd(x,y)$. Then $cm = x \leq y = cn$ for some $m, n \in \mathbb{N} \cup \{0\}$; thus $m \leq n$, $\gcd(m,n) = 1$ and $cm \equiv cn \,(\mathrm{mod}\, 3)$. Suppose (2) fails. The reader can check that $3 \nmid c$, in which case $m \equiv n \,(\mathrm{mod}\, 3)$. But $x/c \equiv y/c \,(\mathrm{mod}\, 3)$ and the primitivity of (x,y) imply $c = 1$. On the other hand, suppose (1) fails so that $c \neq 1$. The reader can check that $3|c$, in which case $(x,y) = (3a,3b)$ is primitive and $a \not\equiv b \,(\mathrm{mod}\, 3)$. But if $e|a$ and $e|b$, then $x/e \equiv y/e \,(\mathrm{mod}\, 3)$. Therefore $e = 1$ by the primitivity of (x,y) and it follows that $\gcd(a,b) = 1$. ■

Example 187 *Using Theorem 186, the reader can verify that the following orbits of period $2n$ are primitive:*

- $n = 2k+1, k \geq 1 : (k-1, k+2)$
- $n = 2 : (1,1)$
- $n = 4k+4, k \geq 1 : (2k-1, 2k+5)$
- $n = 4k+10, k \geq 1 : (2k-1, 2k+11)$.

Corollary 190 below, asserts that there are *no* primitive orbits of period 2, 8, 12 and 20. Thus Example 187 exhibits a primitive orbit of every possible even period. To count how many classes of each there are, we make the following simple observation:

Proposition 188 *If an orbit $(x,y) \in \Gamma_P$ with period $2n$ is a d-fold iterate, then $d|n$ and the period of each iterate is $\frac{2n}{d}$.*

Proof. If $(x,y) \in \Gamma_P$ is a d-fold iterate of period $2n$, then $x/d \equiv y/d \,(\mathrm{mod}\, 3)$ so that $(\frac{x}{d}, \frac{y}{d})$ is an orbit but not necessarily primitive. Since $d|x$, $d|y$ and $x + y = n$ we have $d|n$. And furthermore, $(\frac{x}{d}, \frac{y}{d})$ has period $\frac{2n}{d}$ by Theorem 172. ■

By Theorem 184, exactly $\mathcal{O}(n/d)$ classes contain d-fold iterates of period $2n$. Hence we can account for all non-primitive classes by examining the divisors of n. Let $n = p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m}$ be the prime factorization. Then $\mathcal{O}(n/p_i)$ classes contain p_i-fold iterates of period $2n$ and the sum of these values approximates the number of non-primitive classes. Since the number of classes containing $(p_i p_j)$-fold iterates of period $2n$ have been counted twice, we must subtract

$\sum_{i<j}\mathcal{O}\left(\frac{n}{p_ip_j}\right)$. But this subtracts *too much*; we must add the number of classes containing $(p_ip_jp_k)$-fold iterates for each $i<j<k$. Continuing this process, we apply the Principle of Inclusion-Exclusion by alternately adding and subtracting until the number of classes containing $(p_1p_2\cdots p_m)$-fold iterates of period $2n$ is obtained. The final tally of these 2^m-1 summands is

$$\mathcal{I}(n)=\sum_{d>1,d|n}-\mu\left(d\right)\mathcal{O}\left(n/d\right)=\mathcal{O}\left(n\right)-\sum_{d|n}\mu\left(d\right)\mathcal{O}\left(n/d\right),$$

where

$$\mu\left(d\right)=\begin{cases}1, & d=1\\ \left(-1\right)^r, & d=p_1p_2\cdots p_r \text{ for distinct primes } p_i\\ 0, & \text{otherwise}\end{cases}$$

is the Möbius function. This proves:

Theorem 189 *For each $n\in\mathbb{N}$, exactly $\mathcal{I}\left(n\right)$ non-primitive classes have period $2n$.*

Furthermore, since a class is either primitives or non-primitive, we immediately obtain:

Corollary 190 *For each $n\in\mathbb{N}$, exactly*

$$\mathcal{P}(n)=\mathcal{O}(n)-\mathcal{I}(n)=\sum_{d|n}\mu(d)\mathcal{O}(n/d)$$

primitive classes have period $2n$.

Table 1 in the Appendix displays sample values for $\mathcal{O},\mathcal{P}$, and $\mathcal{I}$. Interesting special cases arise for certain values of n. For example:

Corollary 191 *$\mathcal{O}(n)=0$ if and only if $n=1$; $\mathcal{P}(n)=0$ if and only if $n=1,4,6,10$.*

For example, $\mathcal{O}(4)=1$ and $\mathcal{P}(4)=0$ indicates that the only class of orbits with period 8 contains 2-fold iterates of the period 4.

Corollary 192 *The following are equivalent:*

1. *The integer n is prime.*
2. *$\mathcal{P}(n)=\mathcal{O}(n)$.*
3. *All classes of period $2n$ are primitive.*