Chapter 6

Similarity

In this chapter we consider transformations that magnify or stretch the plane. Such transformations are called "similarities" or "size transformations." One uses similarities to relate two similar triangles in much the same way one uses isometries to relate two congruent triangles. Particularly important are the "stretch" transformations, which linearly expand the plane radially outward from some fixed point. Stretch transformations are an essential component of every (non-isometric) similarity. Indeed, we shall prove that every similarity is one of the following four distinct types: an isometry, a stretch, a stretch reflection or a stretch rotation. We begin with another look at the family of dilatations, which were introduced in Section 1.6.

6.1 Similarities and Dilatations

In this section we show that every dilatation is either a translation, a stretch or a stretch followed by a halfturn.

Definition 193 *Let P and Q be points; let $r > 0$. A transformation α is a <u>similarity of ratio r</u> if and only if $P'Q' = rPQ$, where $P' = \alpha(P)$ and $Q' = \alpha(Q)$.*

Note that a similarity of ratio 1 is an isometry. Consider a similarity α with distinct fixed points P and Q. Then $P' = P$ and $Q' = Q$ so that $P'Q' = PQ$ and the ratio of similarity $r = 1$, in which case α is an isometry. Consequently, if α is a similarity with three distinct non-collinear fixed points, then $\alpha = \iota$ by Throrem 74. This proves:

Proposition 194 *Every similarity with distinct fixed points is an isometry; a similarity with three distinct non-collinear fixed points is the identity.*

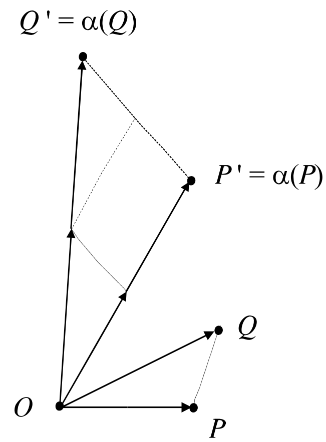

Figure 6.1: A similarity α of ratio 2.

The proof of our next result is left as an exercise.

Proposition 195 *The set of all similarities is a group under composition of functions.*

Definition 196 *Let C be a point and let $r > 0$. A stretch about C of ratio r, denoted by $\xi_{C,r}$, is the transformation with the following properties:*

1. $\xi_{C,r}(C) = C$.

2. *If P is a point, then $P' = \xi_{C,r}(P)$ is the unique point on $\overrightarrow{CP}$ such that $CP' = rCP$.*

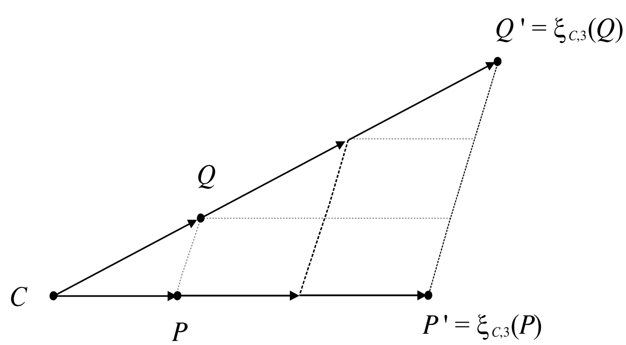

Figure 6.2: A stretch about C of ratio 3.

Of course, the identity is a stretch about every point C of ratio 1. Furthermore, the equations of a stretch about the origin are

$$\xi_{O,r} : \begin{cases} x' = rx \\ y' = ry. \end{cases}$$

To obtain the equations of a stretch about $P = \begin{bmatrix} a \\ b \end{bmatrix}$, conjugate $\xi_{O,r}$ by the translation $\tau_{\mathbf{OP}}$, i.e.,

$$\xi_{P,r} = \tau_{\mathbf{OP}} \circ \xi_{O,r} \circ \tau_{\mathbf{OP}}^{-1}.$$

Composing equations gives

$$\xi_{P,r} : \begin{cases} x' = rx + (1-r)\,a \\ y' = ry + (1-r)\,b. \end{cases}$$

Proposition 197 *A stretch preserves orientation.*

Proof. Cchoose and orientation $\left\{\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right\}$ of $\mathbb{R}^2$. Let $P = \begin{bmatrix} a \\ b \end{bmatrix}$, $U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and let $O = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Then $U' = \xi_{P,r}(U) = \begin{bmatrix} ru_1+(1-r)a \\ ru_2+(1-r)b \end{bmatrix}$, $V' = \xi_{P,r}(V) = \begin{bmatrix} rv_1+(1-r)a \\ rv_2+(1-r)b \end{bmatrix}$ and $O' = \xi_{P,r}(O) = \begin{bmatrix} (1-r)a \\ (1-r)b \end{bmatrix}$ so that

$$\mathbf{u}' = \mathbf{O}'\mathbf{U}' = \begin{bmatrix} ru_1+(1-r)a-(1-r)a \\ ru_2+(1-r)b-(1-r)b \end{bmatrix} = \begin{bmatrix} ru_1 \\ ru_2 \end{bmatrix} \text{ and similarly, } \mathbf{v}' = \begin{bmatrix} rv_1 \\ rv_2 \end{bmatrix}.$$

Therefore, $\det[\mathbf{u}'|\,\mathbf{v}'] = \det\begin{bmatrix} ru_1 & rv_1 \\ ru_2 & rv_2 \end{bmatrix} = r^2 \det\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} = r^2 \det[\mathbf{u}|\,\mathbf{v}]$, which completes the proof. ■

Proposition 198 *Let C be a point and let $r > 0$. Then $\xi_{C,r}$ is both a dilatation and a similarity of ratio r.*

Proof. If ℓ is a line passing through C, then $\xi_{C,r}(\ell) = \ell$ by definition so that $\xi_{C,r}(\ell)\|\ell$. On the other hand, if m is a line off C, choose three distinct points P, Q, and R on m (see Figure 6.3).

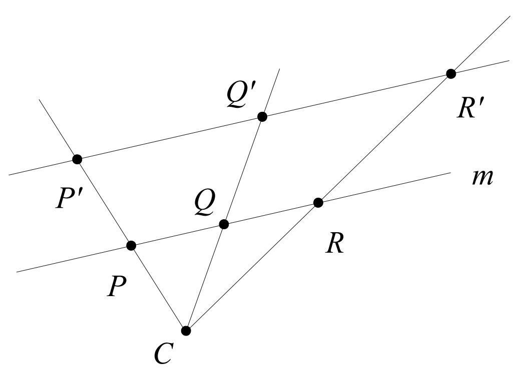

Figure 6.3.

By definition, $CP' = rCP$, $CQ' = rCQ$, and $CR' = rCR$. If $r < 1$, then $\overline{P'Q'}$ divides sides $\overline{CP}$ and $\overline{CQ}$ in $\triangle CPQ$ proportionally; on the other hand, if $r > 1$, then $\overline{PQ}$ divides sides $\overline{CP'}$ and $\overline{CQ'}$ in $\triangle CP'Q'$ proportionally. In either case $\overleftrightarrow{PQ} \| \overleftrightarrow{P'Q'}$, by a standard theorem from geometry. Similarly, $\overleftrightarrow{PR} \| \overleftrightarrow{P'R'}$. But $\overleftrightarrow{P'Q'} \| \overleftrightarrow{P'R'}$ implies that P', Q' and R' are collinear. But $m = \overleftrightarrow{PR} = \overleftrightarrow{PQ}$ and $\xi_{C,r}(m) = \overleftrightarrow{P'R'} = \overleftrightarrow{P'Q'}$; therefore $\xi_{C,r}$ is a collineation. But $\xi_{C,r}$ is also a dilatation since $m \| \xi_{C,r}(m)$. Now consider parallel lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{P'Q'}$ cut by transversal $\overleftrightarrow{CP}$. Since corresponding angles are equal, $\triangle CPQ \sim \triangle CP'Q'$ by AA. But the lengths of corresponding sides in similar triangles are proportional, hence $P'Q' = rPQ$. Therefore $\xi_{C,r}$ is a similarity of ratio r as claimed. ■

Definition 199 *Let C be a point and let $r > 0$. A* *dilation about C of ratio r,* *denoted by $\delta_{C,r}$, is either a stretch about C of ratio r or a stretch about C of ratio r followed by a halfturn about C.*

Note that the identity and all halfturns are dilations of ratio 1.

Theorem 200 *Let C be a point and let $r > 0$. Then $\delta_{C,r}$ is both a dilatation and a similarity of ratio r.*

Proof. If $\delta_{C,r} = \xi_{C,r}$, the conclusion follows from Proposition 198. So assume that $\delta_{C,r} = \varphi_C \circ \xi_{C,r}$. A halfturn is an isometry and hence a similarity; it is also a dilatation by Theorm 50. So on one hand, $\varphi_C \circ \xi_{C,r}$ is a composition of dilatations, which is a dilatation; on the other hand $\varphi_C \circ \xi_{C,r}$ is a composition of similarities, which is a similarity by Proposition 195. Proof of the fact that $\varphi_C \circ \xi_{C,r}$ is a similarity of ratio r is left to the reader. ■

Theorem 201 *If $\overleftrightarrow{AB} \| \overleftrightarrow{A'B'}$, then there is a unique dilatation α such that $A' = \alpha(A)$ and $B' = \alpha(B)$.*

Proof. First, we construct a dilatation with the required property. Given $\overleftrightarrow{AB} \| \overleftrightarrow{A'B'}$, let $C = \tau_{\mathbf{AA'}}(B)$ and consider the dilation $\delta_{A',r}$ such that $r = A'B'/A'C$ and $B' = \delta_{A',r}(C)$. Then

$$(\delta_{A',r} \circ \tau_{\mathbf{AA'}})(A) = \delta_{A',r}(A') = A' \text{ and } (\delta_{A',r} \circ \tau_{\mathbf{AA'}})(B) = \delta_{A',r}(C) = B'.$$

But $\tau_{\mathbf{AA'}}$ and $\delta_{A',r}$ are dilatations, hence $\delta_{A',r} \circ \tau_{\mathbf{AA'}}$ is a dilatation with the required property. For uniqueness, let α be any dilatation such that $\alpha(A) = A'$ and $\alpha(B) = B'$. Let P be a point off $\overleftrightarrow{AB}$ and let $P' = \alpha(P)$. Then $\overleftrightarrow{AP} \| \overleftrightarrow{A'P'}$, $\overleftrightarrow{BP} \| \overleftrightarrow{B'P'}$ and P' is the intersection of the line through A' parallel to $\overleftrightarrow{AP}$ with the line through B' parallel to $\overleftrightarrow{BP}$ (see Figure 6.4).

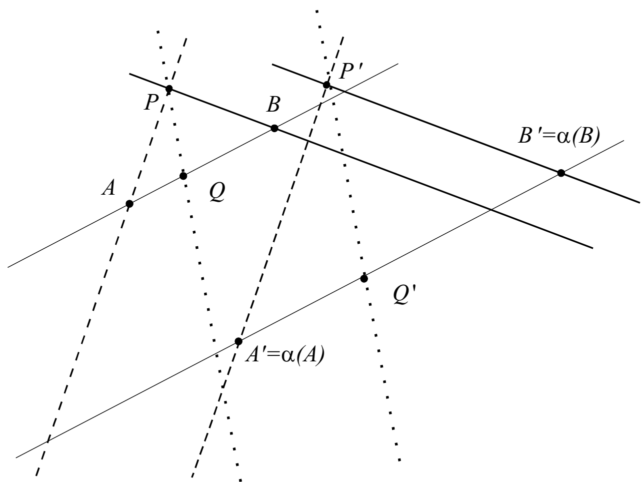

Figure 6.4.

Now if Q is a point on $\overleftrightarrow{AB}$ and $\alpha(Q) = Q'$, then $\overleftrightarrow{AQ} \| \overleftrightarrow{A'Q'}$ and $\overleftrightarrow{PQ} \| \overleftrightarrow{P'Q'}$. Since $\overleftrightarrow{AB} = \overleftrightarrow{AQ}$, the point Q' is the intersection of the line $\overleftrightarrow{A'B'}$ with the line through P' parallel to $\overleftrightarrow{PQ}$. Thus P' and Q' are uniquely determined by the given points A, A', B, and B'. Hence every similarity with the required property sends P to P' and Q to Q'; in particular,

$$(\delta_{A',r} \circ \tau_{\mathbf{AA'}})(A) = \alpha(A); \quad (\delta_{A',r} \circ \tau_{\mathbf{AA'}})(P) = \alpha(P); \text{ and } (\delta_{A',r} \circ \tau_{\mathbf{AA'}})(Q) = \alpha(Q).$$

Applying α^{-1} on the left to both sides produces the similarity $\alpha^{-1} \circ \delta_{A',r} \circ \tau_{\mathbf{AA'}}$ with non-collinear fixed points A, P, and Q. By Proposition 194,

$$\alpha^{-1} \circ \delta_{A',r} \circ \tau_{\mathbf{AA'}} = \iota$$

in which case

$$\alpha = \delta_{A',r} \circ \tau_{\mathbf{AA'}}.$$

■

Theorem 202 *If α is a dilatation and A is a point distinct from $A' = \alpha(A)$, then α fixes $\overleftrightarrow{AA'}$.*

Proof. Note that $\overleftrightarrow{AA'}$ passes through A' and is parallel to $\alpha\left(\overleftrightarrow{AA'}\right)$. On the other hand, A' also lies on $\alpha\left(\overleftrightarrow{AA'}\right)$. Therefore $\alpha\left(\overleftrightarrow{AA'}\right) = \overleftrightarrow{AA'}$. ■

We are now able to determine all of the dilatations.

Theorem 203 *Every non-identity dilatation is a translation, halfturn or dilation.*

Proof. Let α be a non-identity dilatation. If α is an isometry, it is either a translations or a halfturn by Theorems 49, 50 and 125. Let ℓ be a line distinct from $\alpha(\ell)$, its parallel image. Then if A and B are points on ℓ, their images

$$A' = \alpha(A) \text{ and } B' = \alpha(B)$$

lie on $\alpha(\ell)$. Hence

$$\overleftrightarrow{AB} \| \overleftrightarrow{A'B'}$$

and by Theorem 201, α is the unique dilatation with these properties. We consider two cases:

Case 1: Assume $\overleftrightarrow{AA'} \| \overleftrightarrow{BB'}$. Then $\square AA'B'B$ is a parallelogram so that

$$\tau_{\mathbf{AA'}}(A) = A' \text{ and } \tau_{\mathbf{AA'}}(B) = B'.$$

Since translations are dilatations by Theorem 49, $\alpha = \tau_{\mathbf{AA'}}$ by uniqueness in Theorem 201.

Case 2: Assume $\overleftrightarrow{AA'} \cap \overleftrightarrow{BB'} = C$. Then $\alpha\left(\overleftrightarrow{AA'}\right) \cap \alpha\left(\overleftrightarrow{BB'}\right) = C$, since $\alpha\left(\overleftrightarrow{AA'}\right) = \overleftrightarrow{AA'}$ and $\alpha\left(\overleftrightarrow{BB'}\right) = \overleftrightarrow{BB'}$ by Theorem 202, and it follows that C is a fixed point for α. Since $\overleftrightarrow{AB}$ is not fixed, point C lies off $\overleftrightarrow{AB}$; since A, A' and C are collinear and $\overleftrightarrow{AB} \neq \overleftrightarrow{A'B'}$, point C also lies off $\overleftrightarrow{A'B'}$ (see Figures 6.5 and 6.6). Now $\triangle ABC \sim \triangle A'B'C$ (by AA) with ratio of similarity $r = CA'/CA = CB'/CB$.

Subcase 2a: Assume $r = 1$. Then C is the midpoint of $\overline{AA'}$ and $\overline{BB'}$, in which case $\varphi_C(A) = A'$ and $\varphi_C(B) = B'$. Since halfturns are dilatations by Theorem 50, $\alpha = \varphi_C$ by uniqueness in Theorem 201 (see Figure 6.5).

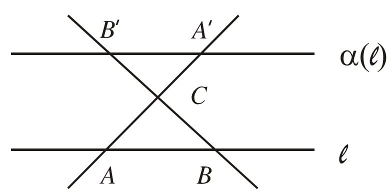

Figure 6.5.

Subcase 2b: Assume $r \neq 1$. Let $D = \xi_{C,r}(A)$ and $E = \xi_{C,r}(B)$; then D is the unique point on $\overrightarrow{CA}$ such that $CD = rCA$ and E is the unique point on $\overrightarrow{CB}$ such that $CE = rCB$. If C is between A and A', then C is the midpoint of $\overline{DA'}$ and $\overline{EB'}$. So put $\beta = \varphi_C \circ \xi_{C,r}$. Otherwise, $D = A'$ and $E = B'$, so put $\beta = \xi_{C,r}$ (see Figure 6.6). In either case, β is a dilation such that

$$\beta(A) = A' \text{ and } \beta(B) = B'.$$

Since dilations are dilatations, $\alpha = \beta$ by uniqueness in Theorem 201. ■

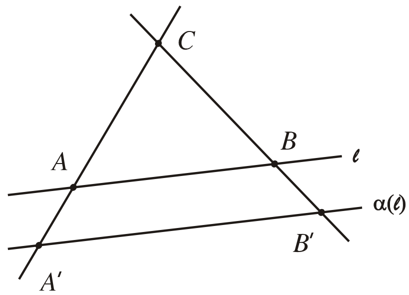

Figure 6.6.

Exercises

1. One can use the following procedure to determine the height of an object: Place a mirror flat on the ground and move back until you can see the top of the object in the mirror. Explain how this works.

2. Find the ratio of similarity r for a similarity α such that $\alpha\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}$ and $\alpha\left(\begin{bmatrix}3\\4\end{bmatrix}\right) = \begin{bmatrix}3\\4\end{bmatrix}$.

3. Find the point P such that $\xi_{P,3}\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}3x+7\\3y-5\end{bmatrix}$.

4. Let $A = \begin{bmatrix}0\\0\end{bmatrix}$, $B = \begin{bmatrix}1\\1\end{bmatrix}$, $A' = \begin{bmatrix}0\\2\end{bmatrix}$, $B' = \begin{bmatrix}2\\4\end{bmatrix}$. Identify the (unique) dilatation α such that $\alpha(A) = A'$ and $\alpha(B) = B'$ as a translation, stretch or dilation. Determine the ratio of similarity and any fixed points.

5. If α is a similarity of ratio r and β is a similarity of ratio s, prove that $\alpha \circ \beta$ is a similarity of ratio rs.

6. If α is a similarity of ratio r, prove that α^{-1} is a similarity of ratio $\frac{1}{r}$.

7. Prove that the set of all similarities is a group.

8. Let C be a point and let $r > 0$. Prove that the similarity $\varphi_C \circ \xi_C, r$ has ratio r.

9. Prove that similarities preserve angle.

10. Prove that similarities preserve betweeness.

6.2 Similarities as an Isometry and a Stretch

Given two congruent triangles, the Classificaton Theorem for Plane Isometries (Theorem 93) tells us there is exactly one isometry that maps one of two congruent triangles onto the other. A similar statement, which appears as Theorem 204 below, can be made for a pair of similar triangles, namely, there is exactly one similarity that maps one of two similar triangles onto the other. In this section we also observe that every similarity is a stretch followed by an isometry. This important fact will lead to the complete classification of all similarities in the next section.

Theorem 204 *$\triangle ABC \sim \triangle A'B'C'$ if and only if there is a unique similarity α such that $A' = \alpha(A)$, $B' = \alpha(B)$, and $C' = \alpha(C)$.*

Proof. Given similar triangles $\triangle ABC$ and $\triangle A'B'C'$, we first define a similarity that sends A to A', B to B' and C to C' then prove its uniqueness. The ratio of similarity $r = A'B'/AB$; let $D = \xi_{A,r}(B)$ and $E = \xi_{A,r}(C)$. Then $AD = rAB = A'B'$ and $AE = rAC = A'C'$. Furthermore, $\angle DAE = \angle BAC \cong \angle B'A'C'$ since corresponding angles are congruent and $\triangle ADE \cong \triangle A'B'C'$ by SAS. Let β be the isometry that maps $\triangle ADE$ congruently onto $\triangle A'B'C'$. Then $\beta \circ \xi_{A,r}$ is a similarity such that

$$(\beta \circ \xi_{A,r})(A) = \beta(A) = A',$$
$$(\beta \circ \xi_{A,r})(B) = \beta(D) = B',$$
$$(\beta \circ \xi_{A,r})(C) = \beta(E) = C'.$$

For uniqueness, let α be any similarity such that $A' = \alpha(A)$, $B' = \alpha(B)$, and $C' = \alpha(C)$. Apply α^{-1} to both sides of each equation above and obtain

$$\left(\alpha^{-1} \circ \beta \circ \xi_{A,r}\right)(A) = \alpha^{-1}(A') = A,$$
$$\left(\alpha^{-1} \circ \beta \circ \xi_{A,r}\right)(B) = \alpha^{-1}(B') = B,$$
$$\left(\alpha^{-1} \circ \beta \circ \xi_{A,r}\right)(C) = \alpha^{-1}(C') = C.$$

Then $\alpha^{-1} \circ \beta \circ \xi_{A,r}$ has three non-collinear fixed points; by Proposition 194

$$\alpha^{-1} \circ \beta \circ \xi_{A,r} = \iota.$$

Therefore

$$\alpha = \beta \circ \xi_{A,r}.$$

Conversely, suppose that α is a similarity of ratio r such that $A' = \alpha(A)$, $B' = \alpha(B)$, and $C' = \alpha(C)$. Then by definition, $A'B' = rAB$, $B'C' = rBC$, and $C'A' = rCA$ so that

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}.$$

It follows that $\triangle ABC \sim \triangle A'B'C'$. ■

Definition 205 *Two plane figures s_1 and s_2 are similar if and only if there is a similarity α such that $\alpha(s_1) = s_2$.*

The proof of Theorem 204 seems to suggest that a similarity is a stretch about some point P followed by an isometry. In fact, this is true and very important.

Theorem 206 *If α is a similarity of ratio r and P is any point, there exists an isometry β such that*

$$\alpha = \beta \circ \xi_{P,r}.$$

Proof. Let α be a similarity of ratio r and let P be any arbitrarily chosen point. If $\alpha \circ \xi_{P,r}^{-1}$ is an isometry, set $\beta = \alpha \circ \xi_{P,r}^{-1}$; then $\alpha = \beta \circ \xi_{P,r}$ and our proof is complete. So we must prove that $\alpha \circ \xi_{P,r}^{-1}$ is indeed an isometry. Let Q and R be distinct points; let $Q' = \xi_{P,r}^{-1}(Q)$ and $R' = \xi_{P,r}^{-1}(R)$; let $Q'' = \alpha(Q')$ and $R'' = \alpha(R')$. Since $\xi_{P,r}^{-1}$ is a similarity of ratio $\frac{1}{r}$ we have

$$Q'R' = \frac{1}{r}QR.$$

But α is a similarity of ratio r, therefore

$$Q''R'' = rQ'R' = r\left(\frac{1}{r}QR\right) = QR$$

and $\alpha \circ \xi_{P,r}^{-1}$ is a an isometry as claimed. ■

Note that Theorem 206 follows immediately from the fact that the ratio of a composition of similarities is the product of the ratios (see Exercise 5.1.5). Thus given a similarity α of ratio r, the composition $\beta = \alpha \circ \xi_{P,r}^{-1}$ is an isometry since its ratio of similarity $r \cdot \frac{1}{r} = 1$.

Definition 207 *A stretch rotation is a non-identity stretch about some point C followed by a non-identity rotation about C.*

Definition 208 *A stretch reflection is a non-identity stretch about some point C followed by a reflection in some line through C.*

Exercises

1. Which points and lines are fixed by a stretch rotation?

2. Which points and lines are fixed by a stretch reflection?

6.3 The Classification of Similarities

In this section we prove the Classification Theorem for Similarities, which states that every similarity is either an isometry, a stretch, a stretch reflection, or a stretch rotation. We begin with a discussion of the Directed Distance Lemma, which plays an essential role in the proof.

Let $\bullet$ denote the Euclidean inner product. To each line ℓ in the plane, arbitrarily assign a unit direction vector $\mathbf{u}_\ell$.

Definition 209 *Given points A and B, the* <u>*directed distance from A to B*</u> *is defined to be*

$$\underline{AB} = \mathbf{AB} \bullet \mathbf{u}_{\overleftrightarrow{AB}}.$$

Thus $\underline{AB} = -\underline{BA}$ and $|\underline{AB}| = |\underline{BA}| = AB$.

Proposition 210 *Let A, B and Q be collinear points with B distinct from Q.*

a. *The sign of the ratio $\underline{AQ}/\underline{QB}$ is independent of the choice of $\mathbf{u}_{\overleftrightarrow{AB}}$.*

b. *The ratio $\underline{AQ}/\underline{QB} = -1$ if and only if $A = B$.*

Proof. Let A, B and Q be collinear points with B distinct from Q.

(a) $\underline{AQ}/\underline{QB} = \left(\mathbf{AQ} \bullet \mathbf{u}_{\overleftrightarrow{AB}}\right) / \left(\mathbf{QB} \bullet \mathbf{u}_{\overleftrightarrow{AB}}\right) = \left(\mathbf{AQ} \bullet -\mathbf{u}_{\overleftrightarrow{AB}}\right) / \left(\mathbf{QB} \bullet -\mathbf{u}_{\overleftrightarrow{AB}}\right)$.

(b) $\underline{AQ}/\underline{QB} = -1$ if and only if $\underline{AQ} = -\underline{QB}$ if and only if $0 = \underline{AQ} + \underline{QB} = \mathbf{AQ} \bullet \mathbf{u}_{\overleftrightarrow{AB}} + \mathbf{QB} \bullet \mathbf{u}_{\overleftrightarrow{AB}} = (\mathbf{AQ} + \mathbf{QB}) \bullet \mathbf{u}_{\overleftrightarrow{AB}} = \mathbf{AB} \bullet \mathbf{u}_{\overleftrightarrow{AB}} = \underline{AB}$ if and only if $A = B$. ■

Lemma 211 ***(The Directed Distance Lemma):*** *Let A and B be distinct points.*

a. *If $r \neq -1$, the point P determined by $\underline{AP} = \left(\dfrac{r}{1+r}\right)\underline{AB}$ is the unique point on $\overleftrightarrow{AB}$ distinct from B such that $\underline{AP}/\underline{PB} = r$.*

b. *If Q is a point on $\overleftrightarrow{AB}$ distinct from B, then $\underline{AQ}/\underline{QB} \neq -1$.*

c. *A point Q is between A and B if and only if $\underline{AQ}/\underline{QB} > 0$.*

Proof. Let A and B be distinct points.

(a) We begin by proving uniqueness: If $r \neq -1$ and P is a point on $\overleftrightarrow{AB}$ distinct from B such that $\underline{AP}/\underline{PB} = r$, then P is unique. Define a real valued function $f : \overleftrightarrow{AB} \to \mathbb{R}$ as follows: If X is a point on line $\overleftrightarrow{AB}$, define

$$f(X) = \underline{AX}/\underline{AB}.$$

Note that $f(X) = f(Y)$ implies $\underline{AX}/\underline{AB} = \underline{AY}/\underline{AB}$, in which case $\underline{AX} = \underline{AY}$ and $X = Y$; so f is injective. Furthermore,

$$\underline{XB} = \underline{XA} + \underline{AB} = \underline{AB} - \underline{AX} = \underline{AB} - f(X)\underline{AB} = (1 - f(X))\underline{AB}$$

so that

$$1 - f(X) = \underline{XB}/\underline{AB}.$$

Consequently,

$$\underline{AX}/\underline{XB} = (\underline{AX}/\underline{AB})\,/\,(\underline{XB}/\underline{AB}) = \frac{f(X)}{1 - f(X)} \neq -1.$$

Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(u) = \frac{u}{1-u},$$

which is injective, as the reader can easily check. Since the composition of injective functions is injective, $g \circ f : \overleftrightarrow{AB} \to \mathbb{R}$ given by

$$(g \circ f)(X) = \frac{f(X)}{1 - f(X)} = \underline{AX}/\underline{XB}$$

is injective. Therefore if X and Y are points on $\overleftrightarrow{AB}$ such that $\underline{AX}/\underline{XB} = \underline{AY}/\underline{YB}$, then $X = Y$. In particular, if $r \neq -1$ and P is a point on $\overleftrightarrow{AB}$ distinct from B such that $(g \circ f)(P) = \underline{AP}/\underline{PB} = r$, then P is unique.

Now if $r \neq -1$ and P is the point on $\overleftrightarrow{AB}$ such that

$$\underline{AP} = \left(\frac{r}{1+r}\right)\underline{AB},$$

then

$$(1+r)\underline{AP} = r\underline{AB}$$

and

$$\underline{AP} = r\underline{AB} - r\underline{AP} = r\,(\underline{AB} - \underline{AP}) = r\,(\underline{AB} + \underline{PA}) = r\underline{PB}.$$

Therefore

$$\underline{AP}/\underline{PB} = r.$$

(b) This follows immediately from Proposition 210.

(c) A point Q on $\overleftrightarrow{AB}$ is between A and B if and only if $\underline{AQ}$ and $\underline{QB}$ have the same sign if and only if $\underline{AQ}/\underline{QB} > 0$. ■

Theorem 212 *Every non-isometric similarity has a fixed point.*

Proof. First note that every non-isometric dilatation has a fixed point since it is a dilation by Theorem 203; so the statement holds in this case. Let α be a similarity that is neither an isometry nor a dilatation. Choose a line ℓ that intersects its image $\ell' = \alpha(\ell)$, let $A = \ell \cap \ell'$ and let $A' = \alpha(A)$. If $A' = A$, then α has a fixed point as claimed. So assume that $A' \neq A$ and consider the line m through A' parallel to ℓ; let $m' = \alpha(m)$. Line ℓ' intersects parallels ℓ and m with equal corresponding angles. Since similarities preserve angle, line $\ell'' = \alpha(\ell')$ intersects lines ℓ' and m' with equal corresponding angles so that $m' \| \ell'$. Let $B = m \cap m'$ and let $B' = \alpha(B)$. Then B' is on m' and is distinct from A' since A' lies on ℓ'. If $B' = B$, then α has a fixed point as claimed. So assume that $B' \neq B$. Then $\ell' = \overleftrightarrow{AA'}$, $m' = \overleftrightarrow{BB'}$, and $\overleftrightarrow{AA'} \| \overleftrightarrow{BB'}$. If $\overleftrightarrow{AB} \| \overleftrightarrow{A'B'}$ then $\square ABB'A'$ is a parallelogram and $AB = A'B'$, in which case α is an isometry. Since our hypothesis excludes this possibility, lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ must intersect at some point P off parallel lines $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$ (see Figure 6.7).

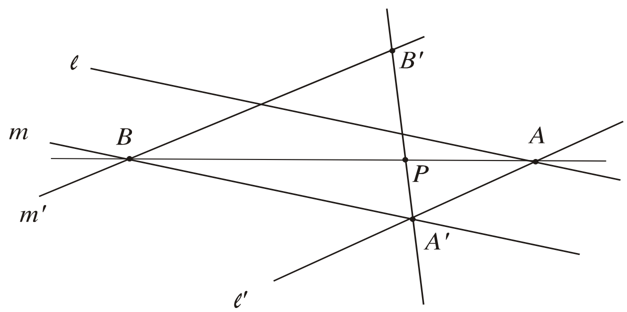

Figure 6.7.

If P is a fixed point, our proof is complete. Now $\triangle APA' \sim \triangle BPB'$ by AA, so

$$\frac{AP}{PB} = \frac{A'P}{PB'}$$

since $CPSTP$. If $P' = \alpha(P)$ and r is the ratio of α, then

$$\frac{AP}{PB} = \frac{rAP}{rPB} = \frac{A'P'}{P'B'}.$$

Therefore

$$\frac{A'P}{PB'} = \frac{A'P'}{P'B'}. \tag{6.1}$$

Since $\overleftrightarrow{AA'} \| \overleftrightarrow{BB'}$, point P is between A' and B' if and only if P is between A and B. Since similarities preserve betweenness, P is between A and B if and only if P' is between A' and B'. Hence P is between A' and B' if and only if P' is between A' and B'. By Lemma 211 part c,

$$\underline{A'P}/\underline{PB'} > 0 \text{ and } \underline{A'P'}/\underline{P'B'} > 0$$

so by (6.1)

$$\underline{A'P}/\underline{PB'} = \underline{A'P'}/\underline{P'B'}$$

or

$$\left(\underline{A'P}\right)\left(\underline{P'B'}\right) = \left(\underline{A'P'}\right)\left(\underline{PB'}\right). \tag{6.2}$$

Suppose that $P \neq P'$. Then $\underline{PP'} \neq 0$ and either $A' - P - P' - B'$ or $A' - P' - P - B'$. If $A' - P - P' - B'$, then

$$\underline{A'P'} = \underline{A'P} + \underline{PP'} \text{ and } \underline{PB'} = \underline{PP'} + \underline{P'B'},$$

in which case equation (6.2) becomes

$$\begin{aligned}(\underline{A'P})(\underline{P'B'}) &= (\underline{A'P} + \underline{PP'})(\underline{PP'} + \underline{P'B'}) \\ &= (\underline{A'P})(\underline{P'B'}) + (\underline{PP'})(\underline{A'P} + \underline{PP'} + \underline{P'B'}) \\ &= (\underline{A'P})(\underline{P'B'}) + (\underline{PP'})(\underline{A'B'}).\end{aligned}$$

Hence

$$(\underline{PP'})(\underline{A'B'}) = 0$$

so that

$$\underline{A'B'} = 0,$$

which is a contradiction. By a similar argument for $A' - P' - P - B'$ we conclude that $P' = P$ so that P is fixed point for α. ∎

We can now prove our premier result:

Theorem 213 ***(Classification of Plane Similarities)*** *A similarity is exactly one of the following: an isometry, a stretch, a stretch rotation, or a stretch reflection.*

Proof. If α is a non-isometric similarity, then α has a fixed point C by Theorem 212. By Theorem 206, there is an isometry β and a stretch ξ about C such that $\alpha = \beta \circ \xi$, or equivalently, $\alpha \circ \xi^{-1} = \beta$. But ξ^{-1} is also a stretch about C so

$$\beta(C) = (\alpha \circ \xi^{-1})(C) = \alpha(C) = C.$$

Since the isometry β has fixed point C, β is one of the following: the identity, a rotation about C or a reflection in some line passing through C. Hence α is one of the following: a stretch, a stretch rotation, or a stretch reflection. Proof of the fact that α is exactly one of these four is left to the reader. ∎

Definition 214 *A* <u>*direct*</u> *similarity preserves orientation; an* <u>*opposite*</u> *similarity reverses orientation.*

In light of Proposition 197 and Theorem 213, direct similarities are even isometries, stretches and stretch rotations and opposite similarities are odd isometries and stretch reflections. Thus the equations of a similarity are easy to obtain.

Theorem 215 *A direct similarity has equations of form*

$$\begin{cases} x' = & ax - by + c \\ y' = & bx + ay + d \end{cases}, \quad a^2 + b^2 > 0;$$

an opposite similarity has equations of form

$$\left\{\begin{array}{rr} x' = & ax - by + c \\ y' = & -bx - ay + d \end{array}\right., \ a^2 + b^2 > 0.$$

Conversely, a transformation with equations of either form is a similarity.

Proof. One can easily check that an even isometry has equations

$$\left\{\begin{array}{rr} x' = & ax - by + c \\ y' = & bx + ay + d \end{array}\right., \ a^2 + b^2 = 1;$$

and an odd isometry has equations

$$\left\{\begin{array}{rr} x' = & ax - by + c \\ y' = & -bx - ay + d \end{array}\right., \ a^2 + b^2 = 1.$$

By Theorem 213, every non-isometric similarity is a stretch, a stretch reflection or a stretch rotation. The equations of a stretch of ratio r have the form

$$\left\{\begin{array}{rr} x' = & rx + c \\ y' = & ry + d \end{array}\right., \ r > 0.$$

Composing the equations of a reflection or a rotation with those of a stretch gives the result. The converse is left to the reader. ■

We conclude with some observations about conjugation by a stretch.

Theorem 216 *If C and P are arbitrary points, $r > 0$ and $\Theta \notin 0^\circ$, then*

$$\xi_{P,r} \circ \rho_{C,\Theta} \circ \xi_{P,r}^{-1} = \rho_{\xi_{P,r}(C),\Theta}.$$

Proof. Note that $\beta = \xi_{P,r} \circ \rho_{C,\Theta} \circ \xi_{P,r}^{-1}$ is an isometry since it is a similarity of ratio 1. Let $C' = \xi_{P,r}(C)$. Then $\beta(C') = \left(\xi_{P,r} \circ \rho_{C,\Theta} \circ \xi_{P,r}^{-1}\right)(C') = \left(\xi_{P,r} \circ \rho_{C,\Theta}\right)(C) = \xi_{P,r}(C) = C'$, so β fixes C'. I claim C' is the unique fixed point of β. If Q is any fixed point of β, then $Q = \left(\xi_{P,r} \circ \rho_{C,\Theta} \circ \xi_{P,r}^{-1}\right)(Q)$ implies $\xi_{P,r}^{-1}(Q) = \rho_{C,\Theta}\left(\xi_{P,r}^{-1}(Q)\right)$. Hence $\xi_{P,r}^{-1}(Q) = C$ and $Q = \xi_{P,r}(C) = C'$, proving the claim. Therefore β is a rotation about C'. Let A' be a point distinct from C' and let $B' = \beta(A')$. Let $A = \xi_{P,r}^{-1}(A')$ and $B = \xi_{P,r}^{-1}(B')$. Then $B' = \left(\xi_{P,r} \circ \rho_{C,\Theta} \circ \xi_{P,r}^{-1}\right)(A')$ implies $\xi_{P,r}^{-1}(B') = \rho_{C,\Theta}\left(\xi_{P,r}^{-1}(A')\right)$ so that $B = \rho_{C,\Theta}(A)$. Since a stretch preserves orientation and $\xi_{P,r}(\triangle ACB) = \triangle A'C'B'$, we have $m\angle A'C'B' = m\angle ACB = \Theta$. Therefore $\beta = \rho_{C',\Theta}$. ■

Theorem 217 *If P is any point, ℓ is any line and $r > 0$, then*

$$\xi_{P,r} \circ \sigma_\ell \circ \xi_{P,r}^{-1} = \sigma_{\xi_{P,r}(\ell)}.$$

Proof. The proof is left to the reader. ■

In light of Theorems 216 and 217, we see that stretch about C commutes with rotations about C and reflections in lines through C (see Exercises 5 and 6).

Exercises

1. Consider an equilateral triangle $\triangle ABC$ and the line $\ell = \overleftrightarrow{BC}$. Find all points and lines fixed by the similarity $\sigma_\ell \circ \xi_{A,2}$.

2. A dilation with center P and ratio r has equations $x' = -2x + 3$ and $y' = -2y - 4$. Find P and r.

3. Let α be a similarity such that $\alpha\left(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $\alpha\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]$ and $\alpha\left(\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}-1\\6\end{smallmatrix}\right]$.

   a. Find the equations of α.

   b. Find $\alpha\left(\left[\begin{smallmatrix}-1\\6\end{smallmatrix}\right]\right)$.

4. Prove that if P is any point, ℓ is any line and $r > 0$, then $\xi_{P,r} \circ \sigma_\ell \circ \xi_{P,r}^{-1} = \sigma_{\xi_{P,r}(\ell)}$.

5. Let C be point. let $\Theta \in \mathbb{R}$ and let $r > 0$. Prove that $\xi_{C,r} \circ \rho_{C,\Theta} = \rho_{C,\Theta} \circ \xi_{C,r}$.

6. Let C be point, let ℓ be a line through C and let $r > 0$. Prove that $\xi_{C,r} \circ \sigma_\ell = \sigma_\ell \circ \xi_{C,r}$.

7. (*Constructing the fixed point*) Given a similarity α that is not a dilatation, choose three distinct non-collinear points A, B and C; let $A' = \alpha(A)$, $B' = \alpha(B)$ and $C' = \alpha(C)$. Then lines $m = \overleftrightarrow{AB}$ and $m' = \overleftrightarrow{A'B'}$ intersect at point P. Let n be the line through C parallel to m; let n' be the line through C' parallel to m'. Then lines n and n' intersect at point Q. Similarly, lines $\ell = \overleftrightarrow{AC}$ and $\ell' = \overleftrightarrow{A'C'}$ intersect at point R. Let k be the line through B parallel to ℓ; let k' be the line through B' parallel to ℓ'. Then lines k and k' intersect at point S. Finally, let $F = \overleftrightarrow{PQ} \cap \overleftrightarrow{RS}$ and prove that $\alpha(F) = F$.

8. Complete the proof of Theorem 213: Prove that the sets $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$, and $\mathcal{L}$ of all isometries, stretches, stretch rotations and stretch reflections, respectively, are mutually disjoint.

9. Prove that the set of all direct similarities forms a group under composition of functions.

10. Prove that similarities are injective.

11. Prove that similarities are surjective. (HINT: Generalize Theorem 14.)

12. Use Exercise 11 to prove that the set of all similarities is a group under composition of functions.

13. Prove the converse of Theorem **??**, i.e., a transformation with equations of either form indicated is a similarity.