Transformational Plane Geometry

Ronald N. Umble
Department of Mathematics
Millersville University of Pennsylvania
Millersville, PA, 17551 USA

Spring 2012

Contents

Introduction

Euclidean plane geometry is the study of size and shape of objects in the plane. It is one of the oldest branches of mathematics. Indeed, by 300 BC Euclid had deductively derived the theorems of plane geometry from his five postulates. More than 2000 years later in 1628, René Descartes introduced coordinates and revolutionized the discipline by using analytical tools to attack geometrical problems. To quote Descartes, "Any problem in geometry can easily be reduced to such terms that a knowledge of the lengths of certain lines is sufficient for its construction."

About 250 years later, in 1872, Felix Klein capitalized on Descartes' analytical approach and inaugurated his so called Erlangen Program, which views plane geometry as the study of those properties of plane figures that remain unchanged under some set of transformations. Klein's startling observation that plane geometry can be completely understood from this point of view is the guiding principle of this course and provides an alternative to Eucild's axiomatic/synthetic approach. In this course, we consider two such families of transformations: (1) isometries (distance-preserving transformations), which include the translations, rotations, reflections and glide reflections and (2) plane similarities, which include the isometries, stretches, stretch rotations and stretch reflections. Our goal is to understand congruence and similarity of plane figures in terms of these particular transformations.

The classification of plane isometries and similarities solves a fundamental problem of mathematics, namely, to identify and classify the objects studied up to some equivalence. This is mathematics par excellence, and a beautiful subtext of this course. The classification of isometries goes like this:

1. Every isometry is a product of three or fewer reflections.
2. A composition of two reflections in parallel lines is a translation.
3. A composition of two reflections in intersecting lines is a rotation.
4. The identity is both a trivial translation and a trivial rotation.
5. Non-identity translations are fixed point free, but fix every line in the direction of translation.
6. A non-identity rotation, which fixes exactly one point, is not a translation.

7. A reflection, which fixes each point on its axis, is neither a translation nor a rotation.

8. A composition of three reflections in concurrent or mutually parallel lines is a reflection.

9. A composition of three reflections in non-concurrent and non-mutually parallel lines is a glide reflection.

10. A glide reflection, which has no fixed points, is neither a rotation nor a reflection.

11. A glide reflection, which only fixes its axis, is not a translation.

12. An isometry is exactly one of the following: A reflection, a rotation, a non-identity translation, or a glide reflection.

Some comments on instructional methodology are worth mentioning. Geometry is a visual science, i.e., each concept needs to be contemplated in terms of a (often mental) picture. Consequently, there will be ample opportunity throughout this course for the student to create and (quite literally) manipulate pictures that express the geometrical content of the concepts. This happens in two settings: (1) Daily homework assignments include several problems from the ancillary text Geometry: Constructions and Transformations, by Dayoub and Lott (Dale Seymour Publications, 1977 ISBN 0-86651-499-6); each construction requires a reflecting instrument such as a MIRA. (2) Biweekly laboratory assignments using the software package Geometer's Sketchpad lead the student through exploratory activities that reinforce the geometric principles presented in this text. Several of these assignments have been selected from the ancillary text Rethinking Proof with Geometer's Sketchpad by Michael de Villiers (Key Curriculum Press, 1999 ISBN 1-55953-323-4) and used by permission. This text complements the visualization skills gained using the MIRA and Geometer's Sketchpad by presenting each concept both synthetically (coordinate free) and analytically. Exercises throughout the text accommodate both points of view. The power of abstract algebra is introduced gently and slowly; prior knowledge of abstract algebra is not assumed.

Finally, I wish to thank George E. Martin, author of the text *Transformation Geometry,* UTM Springer-Verlag, NY 1982, for his encouragement and permission to reproduce some of the diagrams in his text, and my colleagues Zhigang Han and Elizabeth Sell, for carefully reading the manuscript and offering many suggestions that helped to clarify and streamline the exposition.

January 30, 2012

Chapter 1

Isometries

The first three chapters of this book are dedicated to the study of isometries and their properties. Isometries, which are distance-preserving transformations from the plane to itself, appear as reflections, translations, glide reflections, and rotations. The proof of this profound and remarkable fact will follow from our work in this and the next two chapters.

1.1 Transformations of the Plane

We denote points (respectively lines) in $\mathbb{R}^2$ by upper (respectively lower) case letters such as $A, B, C, \ldots$ (respectively $a, b, c, \ldots$). Functions are denoted by lower case Greek letters such as $\alpha, \beta, \gamma, \ldots$

Definition 1 *A transformation of the plane is a function $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ with domain $\mathbb{R}^2$.*

Example 2 *The identity transformation $\iota : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $\iota(P) = P$.*

Definition 3 *A transformation $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is injective (or one-to-one) if and only if for all $P, Q \in \mathbb{R}^2$, if $P \neq Q$, then $\alpha(P) \neq \alpha(Q)$, i.e., distinct points have distinct images.*

Example 4 *The transformation $\beta\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ y \end{bmatrix}$ fails to be injective because $\beta\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \beta\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ while $\begin{bmatrix} -1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.*

As our next example illustrates, one can establish injectivity by verifying the contrapositive of the condition in Definition 3, i.e., under the assumption that $\alpha(P) = \alpha(Q)$, prove $P = Q$.

Example 5 *To show that the transformation $\gamma\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ 2x-y \end{bmatrix}$ is injective, assume that $\gamma\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \gamma\left(\begin{bmatrix} c \\ d \end{bmatrix}\right)$. Then*

$$\begin{bmatrix} a+2b \\ 2a-b \end{bmatrix} = \begin{bmatrix} c+2d \\ 2c-d \end{bmatrix},$$

and by equating x and y coordinates we obtain

$$\begin{aligned} a + 2b &= c + 2d \\ 2a - b &= 2c - d. \end{aligned}$$

A simple calculation now shows that $b = d$ and $a = c$ so that $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = \left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$.

Definition 6 *A transformation $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is surjective (or onto) if and only if given any point $Q \in \mathbb{R}^2$, there is some point $P \in \mathbb{R}^2$ such that $\alpha(P) = Q$, i.e., $\mathbb{R}^2$ is the range of α.*

Example 7 *The transformation $\beta\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} x^2 \\ y \end{smallmatrix}\right]$ discussed in Example 4 fails to be surjective because there is no point $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathbb{R}^2$ for which $\beta\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$.*

Example 8 *Let's show that the transformation $\gamma\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} x+2y \\ 2x-y \end{smallmatrix}\right]$ discussed in Example 5 is surjective. Let $Q = \left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right] \in \mathbb{R}^2$. We must answer the following question: Are there choices for x and y such that $\gamma\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$? Equivalently, does $\left[\begin{smallmatrix} x+2y \\ 2x-y \end{smallmatrix}\right] = \left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$ for appropriate choices of x and y? The answer is yes if the system*

$$\begin{aligned} x + 2y &= c \\ 2x - y &= d \end{aligned}$$

has a solution, which indeed it does since the determinant $\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -1-4 = -5 \neq 0$. By solving simultaneously for x and y in terms of c and d we find that

$$\begin{aligned} x &= \tfrac{1}{5}c + \tfrac{2}{5}d \\ y &= \tfrac{2}{5}c - \tfrac{1}{5}d. \end{aligned}$$

Therefore $\gamma\left(\left[\begin{smallmatrix} \frac{1}{5}c+\frac{2}{5}d \\ \frac{2}{5}c-\frac{1}{5}d \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$ and γ is surjective by definition.

Definition 9 *A bijective transformation is both injective and surjective.*

Example 10 *The transformation γ discussed in Examples 5 and 8 is bijective; the transformation β discussed in Examples 4 and 7 is not.*

Definition 11 *Let α be a bijective transformation, let P be any point, and let Q be the unique point such that $\alpha(Q) = P$. The inverse of α, denoted by α^{-1}, is the function defined by $\alpha^{-1}(P) = Q$.*

Proposition 12 *Let α be a bijective transformation. Then $\beta = \alpha^{-1}$ if and only if $\alpha \circ \beta = \beta \circ \alpha = \iota$.*

Proof. Suppose $\beta = \alpha^{-1}$. If P is any point and $Q = \alpha^{-1}(P)$, then $(\alpha \circ \beta)(P) = \alpha\left(\alpha^{-1}(P)\right) = \alpha(Q) = P$ so that $\alpha \circ \beta = \iota$. Similarly, if Q is any point and $P = \alpha(Q)$, then $(\beta \circ \alpha)(Q) = \beta(\alpha(Q)) = \beta(P) = \alpha^{-1}(P) = Q$ so that $\beta \circ \alpha = \iota$. Conversely, suppose $\alpha \circ \beta = \beta \circ \alpha = \iota$. If P is any point and Q is the unique point such that $\alpha(Q) = P$, then $\beta(P) = \beta(\alpha(Q)) = (\beta \circ \alpha)(Q) = \iota(Q) = Q$ so that $\beta = \alpha^{-1}$ by Definition 11. ■

We adopt the following notation: If P and Q are distinct points, the symbols PQ, $\overline{PQ}$, $\overrightarrow{PQ}$, $\overleftrightarrow{PQ}$, and P_Q denote the distance between P and Q, the line segment connecting P and Q, the ray from P through Q, the line through P and Q, and the circle centered at P containing Q, respectively. When A, B, and C are distinct collinear points with B between A and C, we write $A - B - C$.

One of the primary goals of this course is to understand distance-preserving transformations and their properties.

Definition 13 *An isometry is a distance-preserving transformation $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$, i.e., for all $P, Q \in \mathbb{R}^2$, if $P' = \alpha(P)$ and $Q' = \alpha(Q)$, then $PQ = P'Q'$.*

Example 14 *The identity transformation ι is an isometry.*

Proposition 15 *Isometries are injective.*

Proof. Let α be an isometry and let A and B be distinct points. Then $A' = \alpha(A)$ and $B' = \alpha(B)$ are distinct since $A'B' = AB > 0$. ■

Isometries are also surjective. Our proof, which is somewhat technical, applies two useful propositions, which follow our next definition.

Definition 16 *Let P be a point. Two or more lines or circles are concurrent at P if each line or circle passes through P.*

Proposition 17 *Three concurrent circles with non-collinear centers have a unique point of concurrency.*

Proof. We prove the contrapositive. Suppose three circles centered at A, B and C are concurrent at distinct points P and Q. Since these circles share chord $\overline{PQ}$, their centers A, B and C lie on the perpendicular bisector of $\overline{PQ}$. ■

Definition 18 *Let A, B, and C be points such that $A \neq B$ and $C \neq B$, and choose a unit of measure. If A, B, and C are non-collinear, let $s \in (0, \pi)$ be the length of the unit arc centered at B and subtended by $\angle ABC$. The measure of $\angle ABC$ is the real number*

1. *$\theta = 0$ if $B - A - C$ or $A = C$.*
2. *$\theta = \pi$ if $A - B - C$.*
3. *$\theta = -s$ if the angle from $\overrightarrow{BA}$ to $\overrightarrow{BC}$ is measured clockwise.*
4. *$\theta = s$ if the angle from $\overrightarrow{BA}$ to $\overrightarrow{BC}$ is measured counter-clockwise*

The degree measure of $\angle ABC$, denoted by $m\angle ABC$, is the real number $\Theta = 180\theta/\pi$.

Note that $m\angle ABC = -m\angle CBA$ whenever A, B, and C are non-collinear.

Definition 19 *Two real numbers Θ and Φ are congruent modulo 360 if and only if $\Theta - \Phi = 360k$ for some $k \in \mathbb{Z}$, in which case we write $\Theta \equiv \Phi$. The symbol Θ° denotes the congruence class of Θ.*

Degree measure will be used exclusively throughout this text. Note that each real number Θ is congruent to exactly one real number in $(-180, 180]$. Thus, if $m\angle ABC = \Theta$ and $m\angle DEF = \Phi$, we define $m\angle ABC + m\angle DEF$ to be the unique element of $(-180, 180]$ congruent to $\Theta + \Phi$. For example, if $m\angle ABC = m\angle CBD = 120$, then $m\angle ABC + m\angle CBD = m\angle ABD = -120$.

Proposition 20 *If B, C and D are distinct points on a circle centered at A, then $m\angle BAC \equiv 2m\angle BDC$.*

Proof. Assume that $\square ABDC$ is not a crossed quadrilateral so that B and C lie on opposite sides of $\overleftrightarrow{AD}$ (the crossed quadrilateral case is left as an exercise for the reader). Label the interior angles of $\triangle ABD$ and $\triangle ACD$ as follows: $\angle 1 := \angle ABD$, $\angle 2 := \angle BDA$, $\angle 3 := \angle ADC$, $\angle 4 := \angle DCA$, $\angle 5 := \angle DAB$, and $\angle 6 := \angle CAD$. Note that the measures of these interior angles have the same sign. Thus $m\angle 1 = m\angle 2$, $m\angle 3 = m\angle 4$, and $m\angle BDC = m\angle 2 + m\angle 3 = m\angle 1 + m\angle 4$. Furthermore, the interior angle sum of $\square ABDC$ is $360 \equiv m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4 + m\angle 5 + m\angle 6 = 2m\angle 2 + 2m\angle 3 + m\angle 5 + m\angle 6$ and it follows that $m\angle BAC \equiv 360 - m\angle 5 - m\angle 6 \equiv 2m\angle 2 + 2m\angle 3 = 2m\angle BDC$.

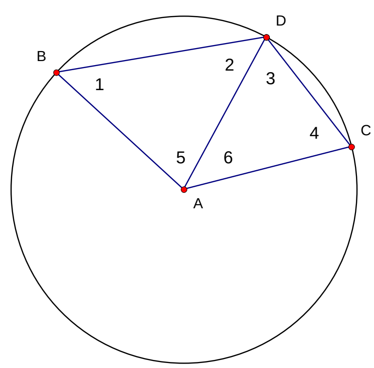


Figure 1.1.

■

To see that Proposition 20 only holds mod 360, consider $\square ABDC$ with $m\angle 2 = m\angle 3 = 60$. Then $m\angle BDC = 120$ and $m\angle BAC = -120 \equiv 240 = 2m\angle BDC$.

Definition 21 *The congruence symbol "$\cong$" has the following meanings:*

1. $\overline{AB} \cong \overline{CD}$ *if and only if* $AB = CD$.

2. $\angle ABC \cong \angle DEF$ *if and only if* $|m\angle ABC| = |m\angle DEF|$.

3. $\triangle ABC \cong \triangle DEF$ *if and only if* $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{AC} \cong \overline{DF}$, $\angle A \cong \angle D$, $\angle B \cong \angle E$, *and* $\angle C \cong \angle F$.

Since distinct elements of $(-180, 180]$ are not congruent $(\mathrm{mod}\, 360)$ we have $m\angle ABC \equiv m\angle A'B'C'$ if and only if $m\angle ABC = m\angle A'B'C'$. However, $m\angle ABC \equiv m\angle A'B'C'$ implies $\angle ABC \cong \angle A'B'C'$, but not conversely. When checking triangle congruence, it is not necessary to check congruence of all six corresponding parts:

Theorem 22 $\triangle ABC \cong \triangle DEF$ *if and only if*

1. *SAS:* $\overline{AB} \cong \overline{DE}$, $\angle A \cong \angle D$, *and* $\overline{AC} \cong \overline{DF}$.

2. *ASA:* $\angle A \cong \angle D$, $\overline{AB} \cong \overline{DE}$, *and* $\angle B \cong \angle E$.

3. *SAA:* $\overline{AB} \cong \overline{DE}$, $\angle B \cong \angle E$, *and* $\angle C \cong \angle F$.

4. *SSS:* $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, *and* $\overline{AC} \cong \overline{DF}$.

5. *HL:* $\angle C$ *and* $\angle F$ *are right angles,* $\overline{AB} \cong \overline{DE}$, *and* $\overline{BC} \cong \overline{EF}$.

Proof. The proof is left as an exercise for the reader. ■

The abbreviation "CPCTC" stands for "*Corresponding Parts of Congruent Triangles are Congruent.*"

Theorem 23 *Isometries are surjective.*

Proof. Given an isometry α and an arbitrary point A, let $A' = \alpha(A)$. If $A' = A$, we're done, so assume $A' \neq A$ and consider an equilateral triangle $\triangle ABC$ with sides of length AA'. Let $B' = \alpha(B)$ and $C' = \alpha(C)$. Again, if $B' = A$ or $C' = A$, we're done, so assume $B' \neq A$ and $C' \neq A$. Then $A'B' = AB$, $B'C' = BC$, and $C'A' = CA$ (α is an isometry), and $\triangle A'B'C' \cong \triangle ABC$ by SSS. Consider $\triangle AB'C'$, which is non-degenerate since A, B', and C' are distinct points on circle A'_A, and construct the point D on A_B such that $\angle BCD \cong \angle B'C'A$. Now $m\angle BAC = \pm 60$ and Proposition 20 implies $2m\angle BDC \equiv m\angle BAC = \pm m\angle B'A'C' \equiv \pm 2m\angle B'AC'$ so that $m\angle BDC = \pm 30 = \pm m\angle B'AC'$. Thus $\angle BDC \cong \angle B'AC'$ and $\triangle BCD \cong \triangle AB'C'$ by AAS (see Figure 1.2).

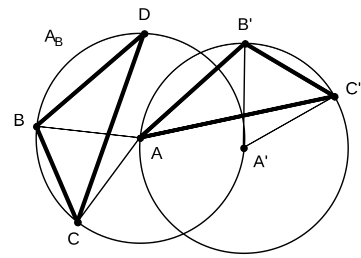


Figure 1.2.

Now $D' = \alpha(D)$ is on A'_A since $A'A = AD = A'D'$; D' is on B'_A since $B'A = BD = B'D'$ (CPCTC); and D' is on C'_A since $C'A = CD = C'D'$ (CPCTC). Thus A and D' are points of concurrency for circles A'_A, B'_A and C'_A with non-collinear centers. Therefore $D' = A$ by uniqueness in Proposition 17 (see Figure 1.3).

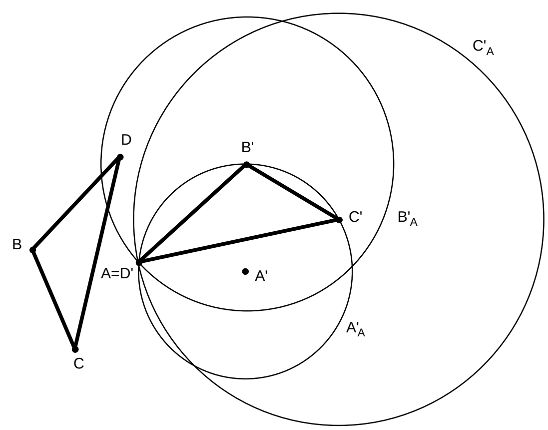


Figure 1.3.

■

Proposition 24 *Let α and β be isometries.*

1. *The composition $\alpha \circ \beta$ is an isometry.*
2. $\alpha \circ \iota = \iota \circ \alpha = \alpha$, *i.e., the identity transformation acts as an identity element.*
3. α^{-1} *is an isometry.*

Proof. The proofs are left as exercises for the reader. ■

Definition 25 *A transformation $\phi : \mathbb{R}^2 \to \mathbb{R}^2$*

- *fixes a point P if $\phi(P) = P$;*

- *is a* *<u>collineation</u>* *if* ϕ *is bijective and sends lines to lines;*
- *is* *<u>linear</u>* *if for all* $\left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right], \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in \mathbb{R}^2$ *and all* $a, b \in \mathbb{R}$,

$$\phi\left(a\begin{bmatrix} s \\ t \end{bmatrix} + b\begin{bmatrix} u \\ v \end{bmatrix}\right) = a\phi\left(\begin{bmatrix} s \\ t \end{bmatrix}\right) + b\phi\left(\begin{bmatrix} u \\ v \end{bmatrix}\right).$$

While isometries are collineations, they are not necessarily linear.

Proposition 26 *Let* α *be an isometry; let* A, B, *and* C *be distinct points; and let* $A' = \alpha(A)$, $B' = \alpha(B)$, *and* $C' = \alpha(C)$. *Then* α

1. *is a collineation.*
2. *preserves betweenness, i.e., if* $A - B - C$, *then* $A' - B' - C'$.
3. *preserves angle measure up to sign, i.e.,* $\angle A'B'C' \cong \angle ABC$.
4. *sends circles to circles.*
5. *is linear if and only if* α *fixes the origin.*

Proof. The proofs are left to the reader; item (5) is easy to prove using vector analysis. ■

Exercises

1. Which of the following transformations are injective? Which are surjective? Which are bijective?

$$\begin{array}{lll}
\alpha\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} x^3 \\ y \end{smallmatrix}\right] & \beta\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} \cos x \\ \sin y \end{smallmatrix}\right] & \gamma\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} x^3 - x \\ y \end{smallmatrix}\right] \\
\delta\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 2x \\ 3y \end{smallmatrix}\right] & \varepsilon\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} -x \\ x+3 \end{smallmatrix}\right] & \eta\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 3y \\ x+2 \end{smallmatrix}\right] \\
\rho\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} \sqrt[3]{x} \\ e^y \end{smallmatrix}\right] & \sigma\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} -x \\ -y \end{smallmatrix}\right] & \tau\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} x+2 \\ y-3 \end{smallmatrix}\right]
\end{array}$$

2. Prove that the composition of transformations is a transformation.

3. Prove that the composition of isometries is an isometry.

4. Prove that *the composition of functions is associative, i.e., if* α, β, γ are functions, *then* $\alpha \circ (\beta \circ \gamma) = (\alpha \circ \beta) \circ \gamma$. (Hint: Show that $[\alpha \circ (\beta \circ \gamma)](P) = [(\alpha \circ \beta) \circ \gamma](P)$ *for every element* P in the domain.)

5. Prove that the identity transformation ι is an identity element for the set of all transformations with respect to composition, i.e., if α is a transformation, then $\alpha \circ \iota = \iota \circ \alpha = \alpha$.

6. Prove that the inverse of a bijective transformation is a bijective transformation. (Remark: Exercises 2,4,5 and 6 show that the set of all bijective transformations is a "group" under composition.)

7. Prove that the inverse of an isometry is an isometry (Remark: Exercises 3,4,5 and 7 show that the set of all isometries is a group under composition.)

8. Let α and β be bijective transformations. Prove that $(\alpha \circ \beta)^{-1} = \beta^{-1} \circ \alpha^{-1}$, i.e., the inverse of a composition is the composition of the inverses in reverse order.

9. Find an example of a bijective transformation that is not a collineation.

10. Let ℓ be the line with equation $2X + 3Y + 4 = 0$. Using the fact that each of the following transformations is a collineation, find the equation of the image line ℓ'.

    a. $\alpha\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y \end{bmatrix}$.

    b. $\beta\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix}$.

    c. $\gamma\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ -y \end{bmatrix}$.

    d. $\delta\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5-x \\ 10-y \end{bmatrix}$.

    e. $\varepsilon\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+1 \\ y-3 \end{bmatrix}$.

11. Which of the transformations in Exercise 1 are collineations? For each collineation in Exercise 1, find the equation of the image of the line ℓ with equation $aX + bY + c = 0$.

12. Consider the collineation $\alpha\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3y \\ x-y \end{bmatrix}$ and the line ℓ' whose equation is $3X - Y + 2 = 0$. Find the equation of the line ℓ such that $\alpha(\ell) = \ell'$.

13. Prove Theorem 22.

14. Prove that an isometry is a collineation.

15. Prove that an isometry sends circles to circles.

16. Prove that an isometry α preserves betweenness.

17. Prove that an angle and its isometric image are congruent.

18. Let B, C and D be distinct points on a circle centered at A.If $\square ABDC$ is a crossed quadrilateral, prove that $m\angle BAC = 2m\angle BDC$. (Hint: Consider the diameter $\overline{DE}$, and triangles $\triangle DEB$ and $\triangle DEC$.)

19. A point P lies in the *interior* of a non-degenerate triangle if there exist points S and T on the triangle, at least one of which is not a vertex, such that $S - P - T$. Prove that two angle bisectors of a triangle meet at a point in the interior of the triangle.

20. Use the result in Problem 19 to prove that the angle bisectors of a triangle are concurrent at a point equidistant from the three sides. Thus every triangle has an inscribed circle, called the *incircle;* its center point P is called the *incenter* of the triangle.

1.2 Reflections

Definition 27 *Let ℓ be a line. The* <u>*reflection in line ℓ*</u> *is the transformation $\sigma_\ell : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

1. *Each point $P \in \ell$ is fixed by σ_ℓ.*

2. *If $P \notin \ell$ and $P' = \sigma_\ell(P)$, then ℓ is the perpendicular bisector of $\overline{PP'}$.*

Theorem 28 *Reflections are isometries.*

Proof. Let ℓ be any line, let P and Q be distinct points, and let $P' = \sigma_\ell(P)$ and $Q' = \sigma_\ell(Q)$, and consider the various configurations of P, Q, and ℓ.

<u>*Case 1:*</u> Suppose that P and Q lie on ℓ. Then $P' = P$ and $Q' = Q$ so $P'Q' = PQ$ as required.

<u>*Case 2:*</u> Suppose that P lies on ℓ and Q lies off of ℓ. Then $P = P'$ and ℓ is the $\perp$ bisector of $\overline{QQ'}$.

<u>*Subcase 2a:*</u> If $\overleftrightarrow{PQ} \perp \ell$ then Q' lies on $\overleftrightarrow{PQ}$ and P is the midpoint of $\overline{QQ'}$ so that $PQ = PQ' = P'Q'$ as required.

<u>*Subcase 2b:*</u> Otherwise, let R be the point of intersection of ℓ with $\overleftrightarrow{QQ'}$. Observe that $\triangle PQR \cong \triangle PQ'R$ by SAS, where the angles considered here are the right angles (see Figure 1.4).

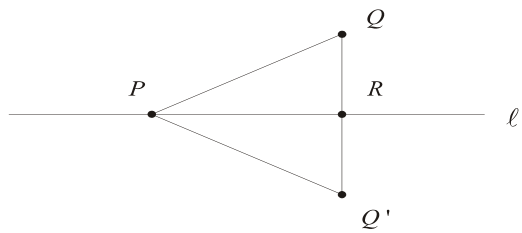


Figure 1.4.

Since corresponding parts of congruent triangles are congruent ($CPCTC$) we have $PQ = PQ' = P'Q'$ as required.

Case 3: Suppose that both P and Q lie off ℓ and on the same side of ℓ.

Subcase 3a: If $\overleftrightarrow{PQ} \perp \ell$, let R be the point of intersection of ℓ with $\overleftrightarrow{PQ}$. If $PR > QR$ then $PQ = PR - QR = P'R - Q'R = P'Q'$, and similarly if $QR > PR$.

Subcase 3b: Otherwise, let R be the point of intersection of ℓ with $\overleftrightarrow{PP'}$ and let S be the point of intersection of ℓ with $\overleftrightarrow{QQ'}$. Then $\triangle PRS \cong \triangle P'RS$ by SAS so that $PS = P'S$ by $CPCTC$ (see Figure 1.5).

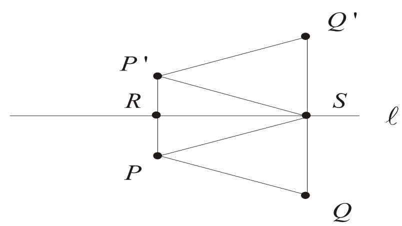


Figure 1.5.

Now $\overleftrightarrow{PP'} \perp \ell$ and $\overleftrightarrow{QQ'} \perp \ell$ so $\overleftrightarrow{PP'} \parallel \overleftrightarrow{QQ'}$. Lines $\overleftrightarrow{PS}$ and $\overleftrightarrow{P'S}$ are transversals so $\angle QSP \cong \angle SPR \cong \angle SP'R \cong \angle Q'SP'$. Since $QS = Q'S$ we have $\triangle PQS \cong \triangle P'Q'S$ from which it follows that $PQ = P'Q'$ by $CPCTC$.

Case 4: Suppose that both P and Q lie off ℓ and on opposite sides of ℓ. Proofs of the following subcases are left to the reader in Exercises 17:

Subcase 4a: If $\overleftrightarrow{PQ} \perp \ell$, let R be the point of intersection of ℓ with $\overleftrightarrow{PQ}$.

Subcase 4b: Otherwise, let R be the point of intersection of ℓ with $\overleftrightarrow{PP'}$, let S be the point of intersection of ℓ with $\overleftrightarrow{QQ'}$. ■

When $\mathbb{R}^2$ comes equipped with a Cartesian system of coordinates, one can use analytic geometry to calculate the coordinates of $P' = \sigma_\ell(P)$ from the equation of ℓ and the coordinates of P. We now derive the formulas (called equations of σ_ℓ) for doing this. Let ℓ be a line with equation $aX+bY+c = 0$, with

$a^2+b^2>0$, and consider points $P=\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]$ and $P'=\left[\begin{smallmatrix}x'\\y'\end{smallmatrix}\right]$ such that $P'=\sigma_\ell(P)$. Assume for the moment that P is off ℓ. By definition, $\overleftrightarrow{PP'}\perp\ell$. So if neither ℓ nor $\overleftrightarrow{PP'}$ is vertical, the product of their respective slopes is -1, i.e.,

$$-\frac{a}{b}\cdot\frac{y'-y}{x'-x}=-1$$

or

$$\frac{y'-y}{x'-x}=\frac{b}{a}.$$

Cross-multiplying gives

$$a(y'-y)=b(x'-x). \tag{1.1}$$

Note that equation (1.1) holds when ℓ is vertical or horizontal as well. If ℓ is vertical, its equation is $X+c=0$, in which case $a=1$ and $b=0$. But reflection in a vertical line preserves the y-coordinate so that $y=y'$. On the other hand, if ℓ is horizontal its equation is $Y+c=0$, in which case $a=0$ and $b=1$. But reflection in a horizontal line preserves the x-coordinate so that $x=x'$. But this is exactly what equation (1.1) gives in either case. Now the midpoint M of P and P' has coordinates

$$M=\begin{bmatrix}\frac{x+x'}{2}\\\frac{y+y'}{2}\end{bmatrix}.$$

Since M lies on ℓ, its coordinates satisfy $aX+bY+c=0$, which is the equation of line ℓ. Therefore

$$a\left(\frac{x+x'}{2}\right)+b\left(\frac{y+y'}{2}\right)+c=0. \tag{1.2}$$

Now rewrite equations (1.1) and (1.2) to obtain the following system of linear equations in x' and y' :

$$\begin{cases}bx'-ay'=s\\ax'+by'=t\end{cases}.$$

where $s=bx-ay$ and $t=-2c-ax-by$. Write this system in matrix form as

$$\begin{bmatrix}b&-a\\a&b\end{bmatrix}\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}s\\t\end{bmatrix}. \tag{1.3}$$

Since $a^2+b^2>0$, the coefficient matrix is invertible, we may solve for $\left[\begin{smallmatrix}x'\\y'\end{smallmatrix}\right]$ and obtain

$$\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}b&-a\\a&b\end{bmatrix}^{-1}\begin{bmatrix}s\\t\end{bmatrix}=\frac{1}{a^2+b^2}\begin{bmatrix}b&a\\-a&b\end{bmatrix}\begin{bmatrix}s\\t\end{bmatrix}=\frac{1}{a^2+b^2}\begin{bmatrix}bs+at\\bt-as\end{bmatrix}.$$

Substituting for s and t gives

$$\begin{aligned} x' &= \frac{1}{a^2+b^2}\left[b\left(bx-ay\right)+a\left(-2c-ax-by\right)\right] \qquad (1.4)\\ &= \frac{1}{a^2+b^2}\left(b^2x-bay-2ac-a^2x-aby\right)\\ &= \frac{1}{a^2+b^2}\left(b^2x+\left(a^2x-a^2x\right)-a^2x-2aby-2ac\right)\\ &= x-\frac{2a}{a^2+b^2}\left(ax+by+c\right), \end{aligned}$$

and similarly

$$y' = y - \frac{2b}{a^2+b^2}\left(ax+by+c\right). \qquad (1.5)$$

Finally, if $P = \begin{bmatrix} x \\ y \end{bmatrix}$ is on ℓ, then $ax+by+c=0$ and equations (1.4) and (1.5) reduce to

$$\begin{array}{l} x' = x \\ y' = y \end{array},$$

in which case P is a fixed point as required by the definition of σ_ℓ. We have proved:

Theorem 29 *Let ℓ be a line with equation $aX+bY+c=0$, where $a^2+b^2>0$. The equations of σ_ℓ (the reflection in line ℓ) are:*

$$\begin{array}{l} x' = x - \frac{2a}{a^2+b^2}\left(ax+by+c\right) \\ y' = y - \frac{2b}{a^2+b^2}\left(ax+by+c\right) \end{array}. \qquad (1.6)$$

Remark 30 *The equations of σ_ℓ are not to be confused with the equation of line ℓ.*

Example 31 *Let ℓ be the line given by $X-Y+5=0$. The equations for σ_ℓ are:*

$$\begin{array}{rll} x' = & x-(x-y+5) & = y-5 \\ y' = & y-(-1)(x-y+5) & = x+5. \end{array}$$

Thus $\sigma_\ell\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$ and $\sigma_\ell\left(\begin{bmatrix} 5 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$.

Definition 32 *A non-identity transformation α is an <u>involution</u> if and only if $\alpha^2 = \iota$.*

Note that an involution α has the property that $\alpha^{-1} = \alpha$.

Proposition 33 *A reflection is an involution.*

Proof. Left as an exercise for the reader. ■

Trisecting a general angle with a straight edge and compass is a classical unsolvable problem. Interestingly, this problem has a solution when the straight edge and compass are replaced with a reflecting instrument such as a MIRA. Thus the trisection algorithm presented here is an important application of reflections, and the notion of parallel lines plays an importnat role.

Definition 34 *Two lines ℓ and m are* <u>parallel</u> *if and only if $\ell = m$ or $\ell \cap m = \varnothing$.*

Algorithm 35 (Angle Trisection) *Given arbitrary rays $\overrightarrow{OX}$ and $\overrightarrow{OY}$:*

1. *Choose a point P on $\overrightarrow{OX}$.*
2. *Locate the point S on $\overrightarrow{OX}$ such that $OP = PS$.*
3. *Construct lines m and n through P such that $m \perp \overrightarrow{OY}$ and $n \perp m$.*
4. *Locate the line ℓ such that $\sigma_\ell(S)$ is on m and O is on $\sigma_\ell(n)$.*
5. *Let $R = \sigma_\ell(O)$.*

Then $m\angle SOR = 2m\angle ROY$.

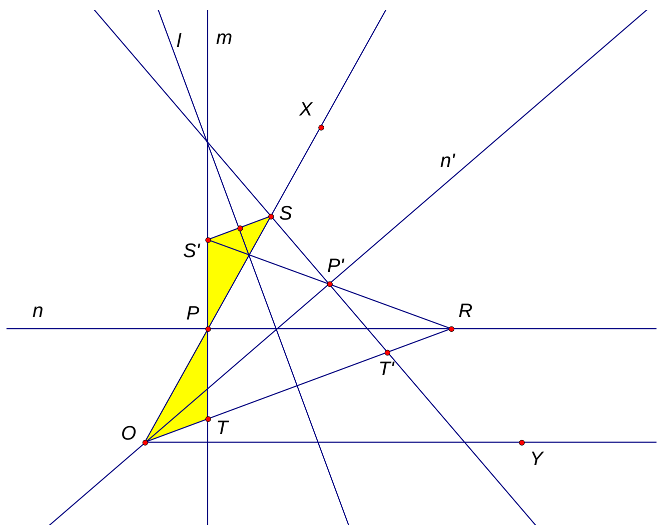


Figure 1.6. $m\angle SOR = 2m\angle ROY$

Proof. Let $S' = \sigma_\ell(S)$ and $T = m \cap \overleftrightarrow{OR}$; then $\overleftrightarrow{SS'} \parallel \overleftrightarrow{OT}$ since $\overleftrightarrow{SS'}$ and $\overleftrightarrow{OT}$ are perpendicular to ℓ, and $\angle TOP \cong \angle PSS'$ since these are alternate interior angles of parallels $\overleftrightarrow{SS'}$ and $\overleftrightarrow{OT}$ cut by transversal $\overleftrightarrow{OS}$. Furthermore, $\angle OPT \cong \angle SPS'$, since these angles are vertical, and $OP = PS$ by construction. Therefore $\triangle POT \cong \triangle PSS'$ by ASA and $S'P = PT$ by CPCTC. Let $T' = \sigma_\ell(T)$, $P' = \sigma_\ell(P)$, and $n' = \sigma_\ell(n)$; then P' is on n' since P is on n, and $SP' = P'T'$ since reflections are isometries. Furthermore, $n' \perp \overleftrightarrow{ST'}$ since $n \perp \overleftrightarrow{S'T}$. Therefore n' is the perpendicular bisector of $\overline{ST'}$. Since O is on n' by construction, n' bisects $\angle SOT'$ so that $\angle SOP' \cong \angle P'OT' = \angle P'OR$. Now $\angle P'OR \cong \angle PRO$ since these angles are the reflections of each other in line ℓ, and $\angle PRO \cong \angle ROY$ since these are alternate interior angles of parallels n and $\overleftrightarrow{OY}$ cut by transversal $\overleftrightarrow{OR}$. Therefore $\angle SOP' \cong \angle P'OR \cong \angle ROY$. ■

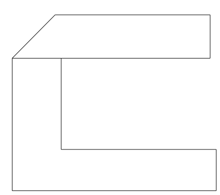
Exercises

1. Words such as `MOM` and `RADAR` that spell the same forward and backward, are called *palindromes*.

   (a) When reflected in their vertical midlines, `MOM` remains `MOM` but the `R`'s and `D` in `RADAR` appear backward. Find at least five other words like `MOM` that are preserved under reflection in their vertical midlines.

   (b) When reflected in their horizontal midlines, `MOM` becomes `WOW`, but `BOB` remains `BOB`. Find at least five other words like `BOB` that are preserved under reflection in their horizontal midlines.

2. What capital letters could be cut out of paper and given a single fold to produce the figure below?



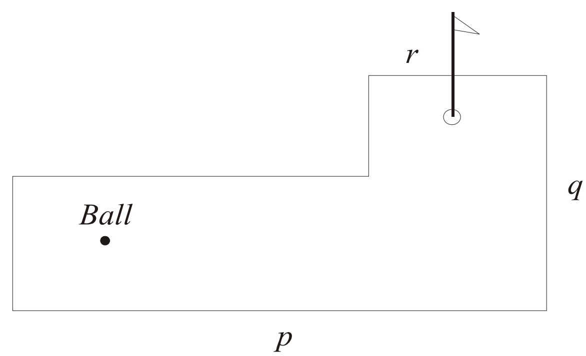
3. The diagram below shows a par 2 hole on a miniature golf course. Use a MIRA to construct the path the ball must follow to score a hole-in-one after banking the ball off

   a. wall p.

   b. wall q.

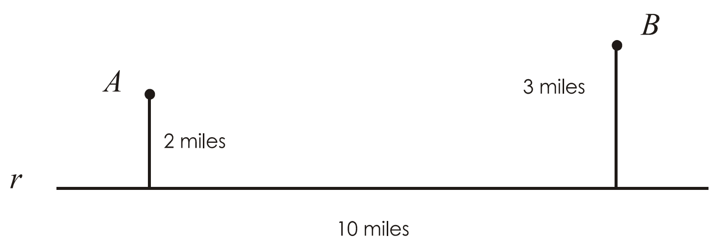
   c. walls p and q.

d. walls p, q and r.



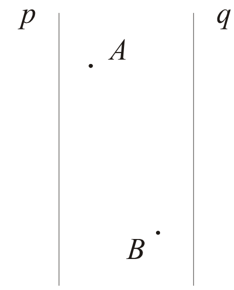
4. Two cities, located at points A and B in the diagram below, need to pipe water from the river, represented by line r. City A is 2 miles north of the river; city B is 10 miles downstream from A and 3 miles north of the river. The State will build one pumping station along the river.

   a. Use a MIRA to locate the point C along the river at which the pumping station should be built so that the minimum amount of pipe is used to connect city A to C and city B to C.



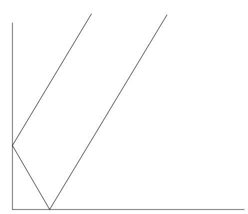
   b. Having located point C, prove that if D is any point on r distinct from C, then $AD + DB > AC + CB$.

5. Given two parallel lines p and q in the diagram below, use a MIRA to construct the path of a ray of light issuing from A and passing through B after being reflected exactly twice in p and once in q.



6. Suppose lines ℓ and m intersect at the point Q, and let $\ell' = \sigma_m(\ell)$. Let P and R be points on ℓ and ℓ' that are distinct from Q and on the same side of m. Let S and T be the feet of the perpendiculars from P and R to m. Prove that $\angle PQS \cong \angle RQT$. (Thus when a ray of light is reflected by a flat mirror, the angle of incidence equals the angle of reflection.)

7. A ray of light is reflected by two perpendicular flat mirrors. Prove that the emerging ray is parallel to the initial incoming ray as indicated in the diagram below.



8. The Smiths, who range in height from 170 cm to 182 cm, wish to purchase a flat wall mirror that allows each Smith to view the full length of his or her image. Use the fact that each Smith's eyes are 10 cm below the top of his or her head to determine the minimum length of such a mirror.

9. Prove the Prependicular Bisector Theorem: A point P lies on the perpendicular bisector of $\overline{AB}$ if and only if P is equidistant from A and B.

10. Graph the line ℓ with equation $X + 2Y - 6 = 0$ on graph paper. Plot the point $P = \left[\begin{smallmatrix}-5\\3\end{smallmatrix}\right]$ and use a MIRA to locate and mark its image P'. Visually estimate the coordinates $\left[\begin{smallmatrix}x'\\y'\end{smallmatrix}\right]$ of the image point P' and record your estimates. Using formulas (1.6), write down the equations for the reflection σ_ℓ and use them to compute the coordinates of the image point

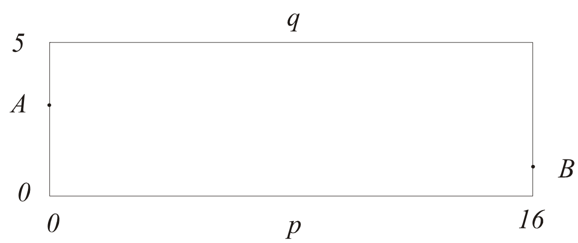
P'. Compare these analytical calculations with your visual estimates of the coordinates.

11. Fill in the missing entry in each row of the following table:

Equation of ℓ	Point P	$\sigma_\ell(P)$
$X = 0$	$\begin{bmatrix} x \\ y \end{bmatrix}$	$*$
$Y = 0$	$*$	$\begin{bmatrix} x \\ y \end{bmatrix}$
$Y = X$	$*$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$
$Y = X$	$\begin{bmatrix} x \\ y \end{bmatrix}$	$*$
$X = 2$	$\begin{bmatrix} -2 \\ 3 \end{bmatrix}$	$*$
$Y = -3$	$\begin{bmatrix} -4 \\ -1 \end{bmatrix}$	$*$

Equation of ℓ	Point P	$\sigma_\ell(P)$
$Y = -3$	$\begin{bmatrix} x \\ y \end{bmatrix}$	$*$
$*$	$\begin{bmatrix} 5 \\ 3 \end{bmatrix}$	$\begin{bmatrix} -8 \\ 3 \end{bmatrix}$
$*$	$\begin{bmatrix} 0 \\ 3 \end{bmatrix}$	$\begin{bmatrix} -3 \\ 0 \end{bmatrix}$
$*$	$\begin{bmatrix} -y \\ -x \end{bmatrix}$	$\begin{bmatrix} x \\ y \end{bmatrix}$
$Y = 2X$	$*$	$\begin{bmatrix} 4 \\ 3 \end{bmatrix}$
$2Y = 3X + 5$	$\begin{bmatrix} x \\ y \end{bmatrix}$	

12. Horizontal lines p and q in the diagram have respective equations $Y = 0$ and $Y = 5$.



a. Use a MIRA to construct the shortest path from point $A\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ to point $B\begin{bmatrix} 16 \\ 1 \end{bmatrix}$ that first touches q and then p.

b. Determine the coordinates of the point on q and the point on p touched by the path constructed in part a.

c. Find the length of the path from A to B constructed in part a.

13. For each of the following pairs of points P and P', determine the equation of the axis ℓ such that $P' = \sigma_\ell(P)$.

a. $P\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $P'\begin{bmatrix} -1 \\ -1 \end{bmatrix}$

b. $P\begin{bmatrix} 2 \\ 6 \end{bmatrix}$, $P'\begin{bmatrix} 4 \\ 8 \end{bmatrix}$

14. The equation of line ℓ is $Y = 2X - 5$. Find the coordinates of the images of $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1\\-3\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}2\\4\end{smallmatrix}\right]$ under reflection in line ℓ.

15. The equation of line m is $X - 2Y + 3 = 0$. Find the coordinates of the images of $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}4\\-1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}-3\\5\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}3\\6\end{smallmatrix}\right]$ under reflection in line m.

16. The equation of line p is $2X + 3Y + 4 = 0$; the equation of line q is $X - 2Y + 3 = 0$. Find the equation of the line $r = \sigma_q(p)$.

17. Prove subcases 4a and 4b in the proof of Theorem 28.

18. Let ℓ and m be lines such that $\sigma_m(\ell) = \ell$. Prove that either $\ell = m$ or $\ell \perp m$.

19. Find all values for a and b such that $\alpha\left(\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}ay\\x/b\end{smallmatrix}\right]$ is an involution.

20. Prove Proposition 33: A reflection is an involution.

21. Prove that an isometry is linear if and only if it fixes the origin. (Hint: Use vector algebra.)

22. Which reflections are linear? Explain.

1.3 Translations

A translation of the plane is an isometry whose effect is the same as sliding the plane in a direction parallel to some line for some finite distance.

Definition 36 *Let P and Q be points. The translation from P to Q is the transformation $\tau_{P,Q} : \mathbb{R}^2 \to \mathbb{R}^2$ with the following properties:*

1. $Q = \tau_{P,Q}(P)$.
2. If $P = Q$, then $\tau_{P,Q} = \iota$.
3. *If $P \neq Q$, let A be any point on $\overleftrightarrow{PQ}$ and let B be any point off $\overleftrightarrow{PQ}$; let $A' = \tau_{P,Q}(A)$ and let $B' = \tau_{P,Q}(B)$. Then quadrilaterals $\square PQB'B$ and $\square AA'B'B$ are parallelograms (see Figure 1.6).*

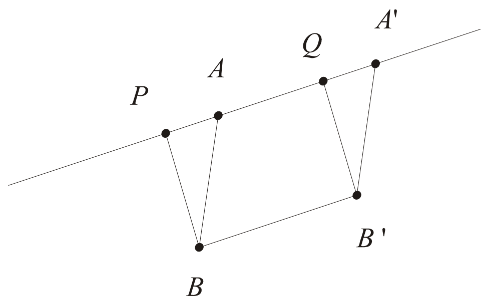


Figure 1.6.

When $P \neq Q$ one can think of a translation as a slide in the direction of $\overrightarrow{PQ}$: If A is any point and ℓ is the line through A parallel to $\overleftrightarrow{PQ}$, then $\tau_{P,Q}(A)$ is the point on ℓ whose distance from A in the direction of $\overrightarrow{PQ}$ is PQ.

Definition 37 *Let P and Q be points. The* <u>*vector* $\mathbf{PQ}$</u> *is the quantity with magnitude PQ and direction $\overrightarrow{PQ}$. If $P = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ and $Q = \left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$, the quantities $c - a$ and $d - b$ are called the* <u>*x and y components*</u> *of $\mathbf{PQ}$ and we write $\mathbf{PQ} = \left[\begin{smallmatrix} c-a \\ d-b \end{smallmatrix}\right]$. The vector $\mathbf{O} = \left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{0} \end{smallmatrix}\right]$, called the* <u>*zero vector*</u>*, has magnitude* 0 *and arbitrary direction.*

Graphically, we represent a vector $\mathbf{PQ}$ as an arrow in the plane with initial point P and terminal point Q. Thus one computes the components of $\mathbf{PQ}$ by subtracting initial coordinates from terminal coordinates. If $O = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ and P is any point, the vector $\mathbf{OP}$ is said to be in *standard position.* Since a vector is uniquely determined by its components, we identify $\mathbf{OP} = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ with the point $P = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$.

Definition 38 *Let $\mathbf{PQ} = \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ and $\mathbf{RS} = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$. The* <u>*vector sum*</u> *of $\mathbf{PQ}$ and $\mathbf{RS}$ is*

$$\mathbf{PQ} + \mathbf{RS} = \begin{bmatrix} u + x \\ v + y \end{bmatrix}.$$

To picture a vector sum, translate $\mathbf{PQ}$ so that its terminal point coincides with the initial point of $\mathbf{RS}$. Then $\mathbf{PQ}$ and $\mathbf{RS}$ determine a parallelogram whose diagonal terminating at S represents $\mathbf{PQ} + \mathbf{RS}$ (see Figure 1.8).

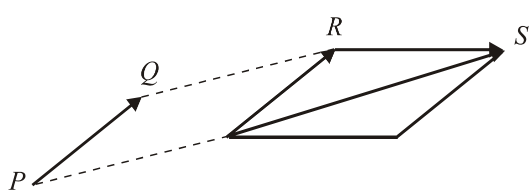


Figure 1.8.

Definition 39 *Let P and Q be points. The translation by vector $\mathbf{PQ}$ is the transformation $\tau_{\mathbf{PQ}}:\mathbb{R}^2\rightarrow\mathbb{R}^2$ defined by*

$$\tau_{\mathbf{PQ}}(R)=\mathbf{OR}+\mathbf{PQ}.$$

The vector $\mathbf{PQ}$ is called the vector of $\tau_{\mathbf{PQ}}$.

We emphasize that the the vector $\mathbf{OR}+\mathbf{PQ}$ in Definition 39 is to be thought of in standard position and is identified with its ternminal point in the plane. Definitions 36 and 39 give two ways to define the same transformation; each definition has advantages and disadvantages.

Theorem 40 *For all points P and Q, $\tau_{P,Q}=\tau_{\mathbf{PQ}}$.*

Proof. If $P=Q$, then $\tau_{P,Q}=\iota=\tau_{\mathbf{PQ}}$. So assume that $P\neq Q$. If A is any point on $\overleftrightarrow{PQ}$ and B is any point off $\overleftrightarrow{PQ}$, let $A'=\tau_{P,Q}(A)$ and $B'=\tau_{P,Q}(B)$. Then $\square PQB'B$ and $\square AA'B'B$ are parallelograms by definition, in which case $\mathbf{AA}'=\mathbf{BB}'=\mathbf{PQ}$. Therefore, $\tau_{\mathbf{PQ}}(A)=\mathbf{OA}+\mathbf{PQ}=\mathbf{OA}+\mathbf{AA}'=\mathbf{OA}'=A'=\tau_{P,Q}(A)$ and $\tau_{\mathbf{PQ}}(B)=\mathbf{OB}+\mathbf{PQ}=\mathbf{OB}+\mathbf{BB}'=\mathbf{OB}'=B'=\tau_{P,Q}(B)$. ■

Corollary 41 *If $\tau_{\mathbf{PQ}}(R)=S$, then $\tau_{\mathbf{PQ}}=\tau_{\mathbf{RS}}$.*

Proof. If $P=Q$, then $\tau_{\mathbf{PQ}}=\iota$ and $R=S$; hence $\tau_{\mathbf{RS}}=\iota$. If $P\neq Q$, then $\tau_{\mathbf{PQ}}(R)=S=\tau_{P,Q}(R)$ by Theorem 40. If R is off $\overleftrightarrow{PQ}$, then $\square PQSR$ is a parallelogram by definition of $\tau_{P,Q}$ so that $\mathbf{PQ}=\mathbf{RS}$ and $\tau_{\mathbf{PQ}}=\tau_{\mathbf{RS}}$. If R is on $\overleftrightarrow{PQ}$ and B is off $\overleftrightarrow{PQ}$, let $B'=\tau_{P,Q}(B)$. Then $\square PQB'B$ and $\square RSB'B$ are parallelograms by definition of $\tau_{P,Q}$ so that $\mathbf{PQ}=\mathbf{BB}'=\mathbf{RS}$ and $\tau_{\mathbf{PQ}}=\tau_{\mathbf{RS}}$. ■

Corollary 41 tells us that a translation is uniquely determined by any point and its image. Consequently, we shall often refer to a general translation τ without specific reference to a point P and its image Q or to a vector $\mathbf{PQ}$. When we need the vector of τ, for example, we simply evaluate τ at any point $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]$ and obtain the image point $\left[\begin{smallmatrix}x'\\y'\end{smallmatrix}\right]$; the desired components are $x'-x$ and $y'-y$. Furthermore, given $\mathbf{PQ}=\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]=\left[\begin{smallmatrix}x'-x\\y'-y\end{smallmatrix}\right]$, we immediately obtain the equations of the translation by vector $\mathbf{PQ}$:

Proposition 42 *Let $\mathbf{PQ}=\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$. The equations for the translation $\tau_{\mathbf{PQ}}$ are*

$$\begin{array}{l}x'=x+a\\y'=y+b\end{array}. \tag{1.7}$$

Example 43 *Let $P=\left[\begin{smallmatrix}4\\5\end{smallmatrix}\right]$ and $Q=\left[\begin{smallmatrix}-1\\3\end{smallmatrix}\right]$. Then $\mathbf{PQ}=\left[\begin{smallmatrix}-5\\-2\end{smallmatrix}\right]$ and the equations for $\tau_{\mathbf{PQ}}$ are*

$$\begin{array}{l}x'=x-5\\y'=y-2\end{array}.$$

In particular, $\tau_{\mathbf{PQ}}\left(\left[\begin{smallmatrix}7\\-5\end{smallmatrix}\right]\right)=\left[\begin{smallmatrix}2\\-7\end{smallmatrix}\right]$.

Theorem 44 *Translations are isometries.*

Proof. Let τ be a translation. Given points P and Q, let $P'=\tau(P)$ and $Q'=\tau(Q)$. Then $\mathbf{PP'}=\mathbf{QQ'}$ since $\tau=\tau_{\mathbf{PP'}}=\tau_{\mathbf{QQ'}}$ by Corollary 41. Therefore $\mathbf{PQ}=\mathbf{PP'}+\mathbf{P'Q}=\mathbf{QQ'}+\mathbf{P'Q}=\mathbf{P'Q}+\mathbf{QQ'}=\mathbf{P'Q'}$ and it follows that $PQ=P'Q'$. ■

Although function composition is not commutative in general, the composition of translations is commutative. Intuitively, this says that you will arrive at the same destination either by a move through directed distance d_1 parallel to line ℓ_1 followed by a move through directed distance d_2 parallel to line ℓ_2, or by a move through directed distance d_2 parallel to ℓ_2 followed by a move through directed distance d_1 parallel to ℓ_1. The paths to your destination follow the two routes along the edges a parallelogram from one vertex to its diagonal opposite. This fact is part (2) of the next proposition.

Proposition 45 *Let* P, Q, R, *and* S *be arbitrary points.*

1. The composition of translations is a translation. In fact,

$$\tau_{\mathbf{RS}}\circ\tau_{\mathbf{PQ}}=\tau_{\mathbf{PQ}+\mathbf{RS}}.$$

2. A composition of translations commutes, i.e.,

$$\tau_{\mathbf{RS}}\circ\tau_{\mathbf{PQ}}=\tau_{\mathbf{PQ}}\circ\tau_{\mathbf{RS}}.$$

3. The inverse of a translation is a translation. In fact,

$$\tau_{\mathbf{PQ}}^{-1}=\tau_{-\mathbf{PQ}}.$$

Proof. (1) Let A be any point, and identify A with the vector $\mathbf{OA}$; then $(\tau_{\mathbf{RS}}\circ\tau_{\mathbf{PQ}})(A)=\tau_{\mathbf{RS}}(\tau_{\mathbf{PQ}}(\mathbf{OA}))=\tau_{\mathbf{RS}}(\mathbf{OA}+\mathbf{PQ})=(\mathbf{OA}+\mathbf{PQ})+\mathbf{RS}=\mathbf{OA}+(\mathbf{PQ}+\mathbf{RS})=\tau_{\mathbf{PQ}+\mathbf{RS}}(A)$. Therefore $\tau_{\mathbf{RS}}\circ\tau_{\mathbf{PQ}}=\tau_{\mathbf{PQ}+\mathbf{RS}}$.

(2) By part (1) and the fact that vector addition commutes, we have $\tau_{\mathbf{PQ}}\circ\tau_{\mathbf{RS}}=\tau_{\mathbf{PQ}+\mathbf{RS}}=\tau_{\mathbf{RS}+\mathbf{PQ}}=\tau_{\mathbf{RS}}\circ\tau_{\mathbf{PQ}}$.

(3) By parts (1) and (2), $\tau_{\mathbf{PQ}}\circ\tau_{-\mathbf{PQ}}=\tau_{-\mathbf{PQ}}\circ\tau_{\mathbf{PQ}}=\tau_{\mathbf{PQ}-\mathbf{PQ}}=\iota$ so that $\tau_{\mathbf{PQ}}^{-1}=\tau_{-\mathbf{PQ}}$ by Proposition 12. ■

Definition 46 *A collineation* α *is a* <u>*dilatation*</u> *if and only if* $\alpha(\ell)\|\ell$ *for every line* ℓ *(a line is parallel to itself).*

Theorem 47 *Translations are dilatations.*

Proof. Let τ be a translation and let ℓ be a line. Let A and B be distinct points on line ℓ; let $A' = \tau(A)$ and $B' = \tau(B)$. Then $\tau(\ell) = \overleftrightarrow{A'B'}$ since τ is a collineation by Proposition 26, part 1, and $\tau = \tau_{\mathbf{AA'}} = \tau_{\mathbf{BB'}}$ by Corollary 41. If A, B, A' and B' are collinear, then $\overleftrightarrow{AB} = \overleftrightarrow{A'B'}$ so that $\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$. If A, B, A' and B' are non-collinear, then B lies off $\overleftrightarrow{AA'}$ and $\square AA'B'B$ is a parallelogram by definition of translation, and it follows that $\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$ (see Figure 1.9). ■

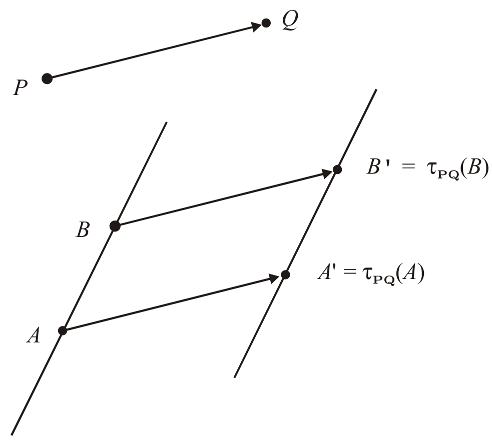


Figure 1.9.

Definition 48 *A transformation α fixes a set s if and only if $\alpha(s) = s$. A transformation α fixes a set s pointwise if and only if $\alpha(S) = S$ for each point $S \in s$.*

Example 49 *If c is a line, the reflection σ_c fixes c pointwise by definition.*

Theorem 50 *Let P and Q be distinct points. The translation $\tau_{\mathbf{PQ}}$ is fixed point free, but fixes every line parallel to $\overleftrightarrow{PQ}$.*

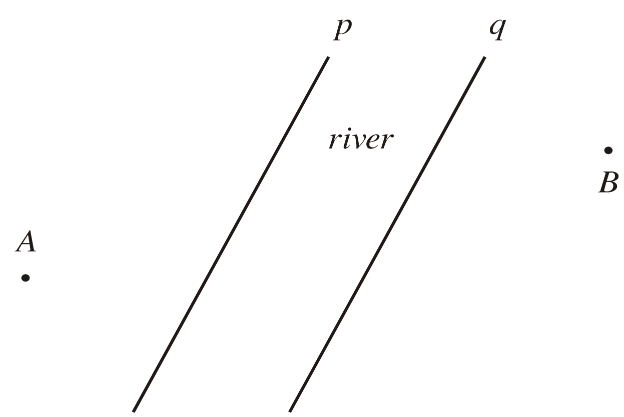
Proof. Let ℓ be a line parallel to and distinct from $\overleftrightarrow{PQ}$, let A be a point on ℓ, and let $A' = \tau_{\mathbf{PQ}}(A)$. Then A and A' are distinct. If $\ell \neq \overleftrightarrow{PQ}$, then $\square PQA'A$ is a parallelogram, by definition of translation, and $\overleftrightarrow{PQ} \| \overleftrightarrow{AA'}$. Therefore $\tau_{\mathbf{PQ}}(A) = A'$ so that $\tau_{\mathbf{PQ}}(\ell) \subseteq \ell$. Since τ is an isometry, it is a collineation by Proposition 26, part 1, and $\tau_{\mathbf{PQ}}(\ell) = \ell$. The case $\ell = \overleftrightarrow{PQ}$ is left as an exercise for the reader. ■

Since a reflection has infinitely many fixed points and a non-identity translation has none, we have:

Corollary 51 *A non-identity translation is not a reflection.*

Exercises

1. A river with parallel banks p and q is to be spanned by a bridge at right angles to p and q.



a. Using an overhead transparency to perform a translation and a MIRA, locate the bridge that minimizes the distance from city A to city B.

b. Let $\overline{PQ}$ denote the bridge at right angles to p and q constructed in part a. Prove that if $\overline{RS}$ is any other bridge spanning river r distinct from and parallel to $\overline{PQ}$, then $AR + RS + SB > AP + PQ + QB$.

2. Let τ be the translation such that $\tau\left(\begin{bmatrix}-1\\3\end{bmatrix}\right) = \begin{bmatrix}5\\2\end{bmatrix}$.

a. Find the vector of τ.

b. Find the equations of τ.

c. Find $\tau\left(\begin{bmatrix}0\\0\end{bmatrix}\right)$, $\tau\left(\begin{bmatrix}3\\-7\end{bmatrix}\right)$, and $\tau\left(\begin{bmatrix}-5\\-2\end{bmatrix}\right)$.

d. Find x and y such that $\tau\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}$.

3. Let τ be the translation such that $\tau\left(\begin{bmatrix}4\\6\end{bmatrix}\right) = \begin{bmatrix}7\\10\end{bmatrix}$.

a. Find the vector of τ.

b. Find the equations of τ.

c. Find $\tau\left(\begin{bmatrix}0\\0\end{bmatrix}\right)$, $\tau\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$, and $\tau\left(\begin{bmatrix}-3\\-4\end{bmatrix}\right)$.

d. Find x and y such that $\tau\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}$.

4. Let $P = \begin{bmatrix}4\\-1\end{bmatrix}$ and $Q = \begin{bmatrix}-3\\5\end{bmatrix}$.

a. Find the vector of $\tau_{P,Q}$.

b. Find the equations of $\tau_{P,Q}$.

c. Find $\tau_{P,Q}\left(\begin{bmatrix}3\\6\end{bmatrix}\right)$, $\tau_{P,Q}\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$, and $\tau_{P,Q}\left(\begin{bmatrix}-3\\-4\end{bmatrix}\right)$.

d. Let ℓ be the line with equation $2X+3Y+4=0$. Find the equation of the line $\ell'=\tau_{P,Q}(\ell)$.

5. Complete the proof of Theorem 50: If P and Q are distinct points and $\ell=\overleftrightarrow{PQ}$, then $\tau_{\mathbf{PQ}}(\ell)=\ell$.

6. Let A and B be points. Prove that $\tau_{\mathbf{AB}}^{-1}=\tau_{\mathbf{BA}}$.

7. Let ℓ and m be the lines with respective equations $X+Y-2=0$ and $X+Y+8=0$.

   a. Compose the equations of σ_m and σ_ℓ and show that the composition $\sigma_m\circ\sigma_\ell$ is a translation τ.

   b. Compare the norm of the vector of τ with the distance between ℓ and m.

8. Let $P=\begin{bmatrix}p_1\\p_2\end{bmatrix}$ and $Q=\begin{bmatrix}q_1\\q_2\end{bmatrix}$ be distinct points and let $c: aX+bY+d=0$ with $a^2+b^2>0$ be a line parallel to $\overleftrightarrow{PQ}$. Prove that:

   a. $a(q_1-p_1)+b(q_2-p_2)=0$.

   b. $\sigma_c\circ\tau_{\mathbf{PQ}}=\tau_{\mathbf{PQ}}\circ\sigma_c$.

9. Is a non-identity translation linear? Explain.

1.4 Halfturns

In this section we consider halfturns, which are 180° rotations of the plane about some point. Halfturns play an important role in our theoretical discussion to follow.

Definition 52 *Let C be a point. The halfturn with center C is the transformation $\varphi_C:\mathbb{R}^2\to\mathbb{R}^2$ that satisfies the following conditions:*

1. *$\varphi_C(C)=C$.*
2. *If P is a point distinct from C and $P'=\varphi_C(P)$, then C is the midpoint of $\overline{PP'}$.*

Axiom (1) says that the halfturn φ_C fixes its center C. We often refer to φ_C as the *halfturn about* C.

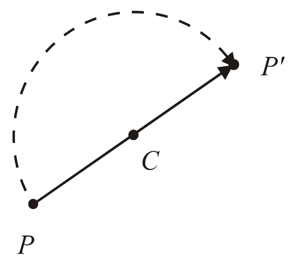


Figure 1.10: A halfturn about C.

Remark 53 *Some authors refer to the halfturn about point C as "the reflection in point C." In this course we use the "halfturn" terminology exclusively.*

Theorem 54 *Halfturns are isometries.*

Proof. Let φ_C be a halfturn about a point C, let P and Q be distinct points, let $P' = \varphi_C(P)$ and let $Q' = \varphi_C(Q)$. If $Q = C$, then $Q' = C$ by Definition 52, Axiom (1), and $PC = P'C$ by Axiom (2). Hence $PQ = PC = P'C = P'Q'$, and similarly for $P = C$. If C is distinct from both P and Q, then $PC = P'C$ and $QC = Q'C$ by Axiom (2). If C, P, and Q are non-collinear, $\angle PCQ \cong \angle P'CQ'$, since vertical angles are congruent, and it follows that $\triangle PCQ \cong \triangle P'CQ'$ by SAS (see Figure 1.11). Therefore $PQ = P'Q'$ since $CPCTC$. The case with C, P, and Q collinear is left as an exercise. ■

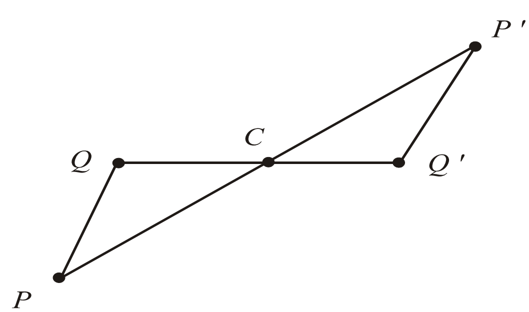


Figure 1.11.

We now derive the equations of a halfturn about the point $C = \begin{bmatrix} a \\ b \end{bmatrix}$. Let $P = \begin{bmatrix} x \\ y \end{bmatrix}$ and $P' = \begin{bmatrix} x' \\ y' \end{bmatrix}$, where $P' = \varphi_C(P)$. If $P \neq C$, then by definition C is the midpoint P and P' so that

$$a = \frac{x + x'}{2} \text{ and } b = \frac{y + y'}{2}.$$

These equations simplify to

$$\begin{array}{c} x' = 2a - x \\ y' = 2b - y \end{array}. \tag{1.8}$$

On the other hand, evaluating the equations in (1.8) at the point $C = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ gives

$$\begin{array}{c} x' = a \\ y' = b \end{array}.$$

Thus the center C is fixed by the transformation whose equations are given in (1.8). This proves:

Theorem 55 *Let $C = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ be a point in the plane. The equations of the halfturn φ_C are given by*

$$\begin{array}{c} x' = 2a - x \\ y' = 2b - y \end{array}.$$

Example 56 *Let O denote the origin; the equations for the halfturn about the origin φ_O are*

$$\begin{array}{c} x' = -x \\ y' = -y \end{array}.$$

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is odd if and only if $f(-x) = -f(x)$. An example of such a function is $f(x) = \sin(x)$. Let f be odd and consider a point $P = \left[\begin{smallmatrix} x \\ f(x) \end{smallmatrix}\right]$ on the graph of f. The image of P under the halfturn φ_O is

$$\varphi_O\left(\begin{bmatrix} x \\ f(x) \end{bmatrix}\right) = \begin{bmatrix} -x \\ -f(x) \end{bmatrix} = \begin{bmatrix} -x \\ f(-x) \end{bmatrix},$$

which is also a point on the graph of f. Thus φ_O fixes the graph of f.

Proposition 57 *A line ℓ is fixed by the halfturn φ_C if and only if C lies on ℓ.*

Proof. Let φ_C be a halfturn and let ℓ be a line. If C lies on ℓ, consider a point P on ℓ distinct from C. Let $P' = \varphi_C(P)$; by definition, C is the midpoint of $\overline{PP'}$. Hence P' is on ℓ and ℓ is fixed by φ_C. Conversely, suppose that C lies off ℓ and consider any point P on ℓ. Then $P' = \varphi_C(P)$ lies off ℓ since otherwise the midpoint of $\overline{PP'}$, which is C, would lie on ℓ. Therefore ℓ is not fixed by φ_C.
■

Proposition 58 *A halfturn is*

1. *an involution.*
2. *a dilatation.*

Proof. The proofs are left as exercises for the reader. ■

Exercises

1. People in distress on a deserted island sometimes write `SOS` in the sand.

   (a) Why is this signal particularly effective when viewed from searching aircraft?

   (b) The word `SWIMS`, like `SOS`, reads the same after performing a halfturn about its centroid. Find at least five other words that are preserved under a halfturn about their centroids.

2. Try it! Plot the graph of $y = \sin(x)$ on graph paper, pierce the graph paper at the origin with your compass point and push the compass point into your writing surface. This provides a point around which you can rotate your graph paper. Now physically rotate your graph paper 180° and observe that the graph of $y = \sin(x)$ is fixed by φ_O in the sense defined above.

3. Find the coordinates for the center of the halfturn whose equations are $x' = -x + 3$ and $y' = -y - 8$.

4. Let $P = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$; let ℓ be the line with equation $5X - Y + 7 = 0$.
   a. Find the equations of φ_P.

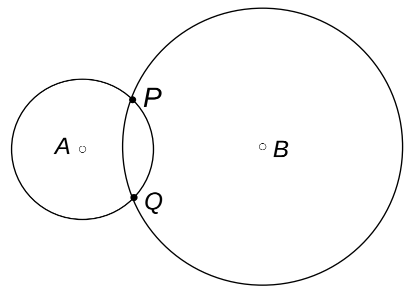
   b. Find the image of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ under φ_P.

   c. Find the equation of the line $\ell' = \varphi_P(\ell)$. On graph paper, plot point P and draw lines ℓ and ℓ'.

5. Repeat Exercise 4 with $P = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

6. $P = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $Q = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$. Find the equations of the composition $\varphi_Q \circ \varphi_P$ and inspect them carefully. These are the equations of an isometry we discussed earlier in the course. Can you identify which?

7. In the diagram below, circles A_P and B_P intersect at points P and Q. Use a MIRA to find a line through P distinct from $\overleftrightarrow{PQ}$ that intersects circles A_P and B_P in chords of equal length.



8. Let $P = \begin{bmatrix} a \\ b \end{bmatrix}$ and $Q = \begin{bmatrix} c \\ d \end{bmatrix}$ be distinct points. Find the equations of $\varphi_Q \circ \varphi_P$ and prove that the composition $\varphi_Q \circ \varphi_P$ is a translation τ. Find the vector of τ.

9. For any point P, prove that $\varphi_P^{-1} = \varphi_P$.

10. Complete the proof of Theorem 54: If C, P, and Q are distinct and collinear, $P' = \varphi_C(P)$ and $Q' = \varphi_C(Q)$, then $PQ = P'Q'$.

11. Let ℓ and m be the lines with respective equations $X + Y - 2 = 0$ and $X - Y + 8 = 0$.

    a. Compose the equations of σ_m and σ_ℓ and show that the composition $\sigma_m \circ \sigma_\ell$ is a halfturn φ_C.

    b. Find the center C of this halfturn and the coordinates of the point $P = \ell \cap m$. What do you observe?

12. Prove that $\varphi_A \circ \varphi_B = \varphi_B \circ \varphi_A$ if and only if $A = B$.

13. Prove Proposition 58, part 1: A halfturn is an involution.

14. Prove Proposition 58, part 2: A halfturn is a dilatation.

15. Which halfturns are linear? Explain.

1.5 General Rotations

In this section we define general rotations, derive their equations, and prove some fundamental properties. Recall that two real numbers Θ and Φ are *congruent* mod 360 if and only if $\Theta - \Phi = 360k$ for some $k \in \mathbb{Z}$, in which case we write

$\Theta \equiv \Phi$. The set $\Theta^\circ = \{\Theta + 360k \mid k \in \mathbb{Z}\}$ is called the *congruence class* of Θ and we define $-(\Theta^\circ) = (-\Theta)^\circ$ and $\Theta^\circ + \Phi^\circ = (\Theta + \Phi)^\circ$. Each congruence class contains exactly one real number in the interval $(-180, 180]$. For example, $370^\circ \cap (-180, 180] = 10$ and $-370^\circ \cap (-180, 180] = -10$. When $\Theta \equiv \Phi$, their congruence classes $\Theta^\circ = \Phi^\circ$ are equal as sets. For example, $300^\circ = (-60)^\circ = -(60^\circ)$. Recall that $m\angle ABC \in (-180, 180]$.

Definition 59 *Let C be a point and let $\Theta \in \mathbb{R}$. The rotation about C of Θ° is the transformation $\rho_{C,\Theta} : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

1. $\rho_{C,\Theta}(C) = C$.
2. *If $P \neq C$ and $P' = \rho_{C,\Theta}(P)$, then $CP' = CP$ and $m\angle PCP' \equiv \Theta$.*

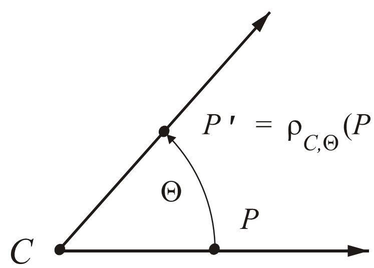


Figure 1.12.

Of course, $\rho_{C,\Theta_1} = \rho_{C,\Theta_2}$ if and only if $\Theta_1 \equiv \Theta_2$.

Example 60 *Consider a halfturn φ_C. Since $\varphi_C(C) = C$ and C is the midpoint of a point P and its image P', we have $CP = CP'$ and $m\angle PCP' = 180$. Thus φ_C is a rotation about C of 180°.*

Theorem 61 *A rotation is an isometry.*

Proof. Let $\rho_{C,\Theta}$ be a rotation. Let C, P and Q be points with P and Q distinct; let $P' = \rho_{C,\Theta}(P)$ and $Q' = \rho_{C,\Theta}(Q)$. If $P = C$, then by definition, $PQ = CQ = CQ' = P'Q'$, and similarly for $Q = C$. So assume that C, P and Q are distinct. If C, P, and Q are non-collinear, then $\triangle PCQ \cong \triangle P'CQ'$ by SAS, and $PQ = P'Q'$ since $CPCTC$. If C, P, and Q are collinear with $C - P - Q$, then $PQ = CQ - CP = CQ' - CP' = P'Q'$, since $CP = CP'$ and $CQ = CQ'$ by definition, and similarly for $C - Q - P$. But if $P - C - Q$, then $P' - C - Q'$since $m\angle PCP' = m\angle QCQ' \equiv \Theta$. Therefore $PQ = CP + CQ = CP' + CQ' = P'Q'$.

■

Before we derive the equations of a general rotation, we consider the special case of rotations $\rho_{O,\Theta}$ about the origin. Since $\rho_{O,\Theta}$ is an isometry that fixes

O, it is linear by Proposition 26, part 5. Let $E_1 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $E_2 = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$; let $E_1' = \rho_{O,\Theta}(E_1)$ and $E_2' = \rho_{O,\Theta}(E_2)$. Then

$$\begin{aligned}\rho_{O,\Theta}\left(\begin{bmatrix}x\\y\end{bmatrix}\right) &= \rho_{O,\Theta}\left(x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}\right)\\ &= x\rho_{O,\Theta}(E_1) + y\rho_{O,\Theta}(E_2)\\ &= xE_1' + yE_2'\end{aligned}$$

and $\rho_{O,\Theta}$ is completely determined by its action on E_1 and E_2. Since $m\angle E_1OE_1' = \Theta$ we have

$$E_1' = \begin{bmatrix}\cos\Theta\\ \sin\Theta\end{bmatrix}.$$

Furthermore, since $m\angle E_1OE_2 = 90$ and $m\angle E_2OE_2' = \Theta$, we have $m\angle E_1OE_2' = \Theta + 90$ and

$$E_2' = \rho_{O,\Theta+90}(E_1) = \begin{bmatrix}\cos(\Theta+90)\\ \sin(\Theta+90)\end{bmatrix} = \begin{bmatrix}-\sin\Theta\\ \cos\Theta\end{bmatrix}.$$

Consequently,

$$\begin{bmatrix}x'\\y'\end{bmatrix} = xE_1' + yE_2' = \begin{bmatrix}x\cos\Theta\\ x\sin\Theta\end{bmatrix} + \begin{bmatrix}-y\sin\Theta\\ y\cos\Theta\end{bmatrix} = \begin{bmatrix}x\cos\Theta - y\sin\Theta\\ x\sin\Theta + y\cos\Theta\end{bmatrix},$$

and we have proved:

Theorem 62 *Let $\Theta \in \mathbb{R}$. The equations for $\rho_{O,\Theta}$ are*

$$\begin{aligned}x' &= x\cos\Theta - y\sin\Theta\\ y' &= x\sin\Theta + y\cos\Theta\end{aligned}.$$

Now think of a general rotation $\rho_{C,\Theta}$ about $C = \left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$ of Θ° as the following sequence of three operations:

1. Translate by vector **CO**.
2. Perform a rotation about the origin O of Θ°.
3. Translate by vector **OC**.

This is an example of a "conjugation"; we shall return to this example in Chapter 3.

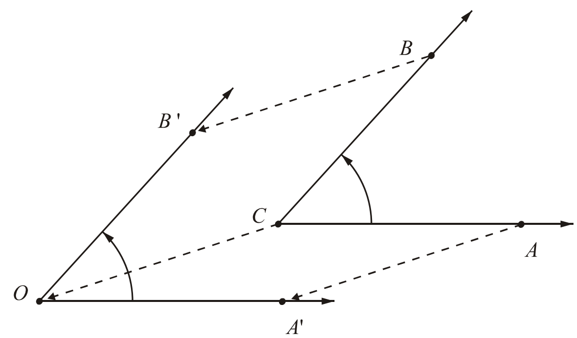


Figure 1.13.

Then

$$\rho_{C,\Theta} = \tau_{\mathbf{OC}} \circ \rho_{O,\Theta} \circ \tau_{\mathbf{CO}}$$

implies

$$\begin{aligned}
\begin{bmatrix} x' \\ y' \end{bmatrix} &= \left(\tau_{\mathbf{OC}} \circ \rho_{O,\Theta} \circ \tau_{\mathbf{CO}}\right) \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \tau_{\mathbf{OC}} \left(\rho_{O,\Theta} \left(\begin{bmatrix} x-a \\ y-b \end{bmatrix} \right) \right) \\
&= \tau_{\mathbf{OC}} \left(\begin{bmatrix} (x-a)\cos\Theta - (y-b)\sin\Theta \\ (x-a)\sin\Theta + (y-b)\cos\Theta \end{bmatrix} \right) \\
&= \begin{bmatrix} (x-a)\cos\Theta - (y-b)\sin\Theta + a \\ (x-a)\sin\Theta + (y-b)\cos\Theta + b \end{bmatrix}.
\end{aligned}$$

We have proved:

Theorem 63 *Let* $C = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ *and let* $\Theta \in \mathbb{R}$. *The equations of* $\rho_{C,\Theta}$ *are*

$$\begin{aligned}
x' &= (x-a)\cos\Theta - (y-b)\sin\Theta + a \\
y' &= (x-a)\sin\Theta + (y-b)\cos\Theta + b.
\end{aligned}$$

The proof of our next proposition is similar to the proof of Proposition 45 and is left an exercise for the reader:

Proposition 64 *Let* C *be a point and let* $\Theta, \Phi \in \mathbb{R}$.

1. A composition of rotations about C is a rotation. In fact,

$$\rho_{C,\Theta} \circ \rho_{C,\Phi} = \rho_{C,\Theta+\Phi}.$$

2. A composition of rotations about C commutes, i.e.,

$$\rho_{C,\Theta} \circ \rho_{C,\Phi} = \rho_{C,\Phi} \circ \rho_{C,\Theta}.$$

3. The inverse of a rotation about C is a rotation about C. In fact,

$$\rho_{C,\Theta}^{-1} = \rho_{C,-\Theta}.$$

Proposition 65 *A non-identity rotation $\rho_{C,\Theta}$ fixes every circle with center C and has exactly one fixed point, namely C.*

Proof. That $\rho_{C,\Theta}(C) = C$ follows by definition. From the equations of a rotation it is evident that $\rho_{C,\Theta} = \iota$ (the identity) if and only if $\Theta \equiv 0$, so $\Theta^\circ \neq 0^\circ$ by assumption. Let P be any point distinct from C, and let and $P' = \rho_{C,\Theta}(P)$; then $P \neq P'$ since $m\angle PCP' \notin 0^\circ$. Therefore C is the unique point fixed by $\rho_{C,\Theta}$. Furthermore, let Q be any point on C_P and let $Q' = \rho_{C,\Theta}(Q)$. Then by definition $CP = CQ = CQ'$ and Q' is on C_P. Hence $\rho_{C,\Theta}(C_P) \subseteq C_P$. Since $\rho_{C,\Theta}$ is an isometry, $\rho_{C,\Theta}(C_P) = C_P$ by Proposition 26, part 4. ■

Since a reflection has infinitely many fixed points, a non-identity rotation has exactly one, and a non-identity translation has none, we have:

Corollary 66 *A non-identity rotation is neither a reflection nor a translation.*

Corollary 67 *Involutory rotations are halfturns.*

Proof. An involutory rotation $\rho_{C,\Theta}$ is a non-identity rotation such that $\rho_{C,\Theta}^2 = \iota = \rho_{C,0}$. By Proposition 64, $\rho_{C,\Theta}^2 = \rho_{C,2\Theta}$ so that $\rho_{C,2\Theta} = \rho_{C,0}$ and $2\Theta \equiv 0$. Hence there exists some $k \in \mathbb{Z}$, such that $2\Theta = 360k$, or consequently $\Theta = 180k$. Now k cannot be even since $\rho_{C,\Theta} \neq \iota$ implies that Θ is not a multiple of 360. Therefore $k = 2n+1$ is odd and $\Theta = 180+360n$. It follows that $\Theta \equiv 180$ and $\rho_{C,\Theta} = \varphi_C$ as claimed. ■

Exercises

1. Find the coordinates of the point $\rho_{O,30}\left(\begin{bmatrix}3\\6\end{bmatrix}\right)$.

2. Let $Q = \begin{bmatrix}-3\\5\end{bmatrix}$. Find the coordinates of the point $\rho_{Q,45}\left(\begin{bmatrix}3\\6\end{bmatrix}\right)$.

3. Let ℓ be the line with equation $2X + 3Y + 4 = 0$.

   a. Find the equation of the line $\rho_{O,30}(\ell)$.

   b. Let $Q = \begin{bmatrix}-3\\5\end{bmatrix}$. Find the equation of the line $\rho_{Q,45}(\ell)$.

4. Let C be a point and let $\Theta, \Phi \in \mathbb{R}$. Prove that $\rho_{C,\Theta} \circ \rho_{C,\Phi} = \rho_{C,\Theta+\Phi}$.

5. Let C be a point and let $\Theta, \Phi \in \mathbb{R}$. Prove that $\rho_{C,\Theta} \circ \rho_{C,\Phi} = \rho_{C,\Phi} \circ \rho_{C,\Theta}$.

6. Let C be a point and let $\Theta \in \mathbb{R}$. Prove that $\rho_{C,\Theta}^{-1} = \rho_{C,-\Theta}$.

Chapter 2

Compositions of Isometries

As we observed in Exercise 3 of Section 1.1, the composition of isometries is an isometry. So it is natural to study the properties possessed by a composition of isometries. In this chapter we show that the composition of two reflections is a rotation or a translation, and the composition of three reflections in distinct lines is either a reflection or a "glide reflection."

2.1 Compositions of Halfturns

In this section we observe that the composition of two halfturns is a translation, and the composition of three halfturns is another halfturn.

Theorem 68 *The composition of two halfturns is a translation. In fact, given any two points A and B,*

$$\varphi_B \circ \varphi_A = \tau_{\mathbf{2AB}}.$$

Proof. Let P be a point, let $Q = \varphi_A(P)$ and let $P' = (\varphi_B \circ \varphi_A)(P)$. Then by definition, A is the midpoint of $\overline{PQ}$ and B is the midpoint of $\overline{QP'}$. Thus $\mathbf{PP'} = \mathbf{PQ} + \mathbf{QP'} = (\mathbf{PA} + \mathbf{AQ}) + (\mathbf{QB} + \mathbf{BP'}) = \mathbf{2}(\mathbf{AQ} + \mathbf{QB}) = \mathbf{2AB}$. Therefore $P' = \mathbf{OP'} = \mathbf{OP} + \mathbf{PP'} = \mathbf{OP} + 2\mathbf{AB} = \tau_{\mathbf{2AB}}(P)$. ■

When A and B are distinct, Theorem 68 tell us that $\varphi_B \circ \varphi_A$ translates a distance $2AB$ in the direction from A to B (see Figure 2.1).

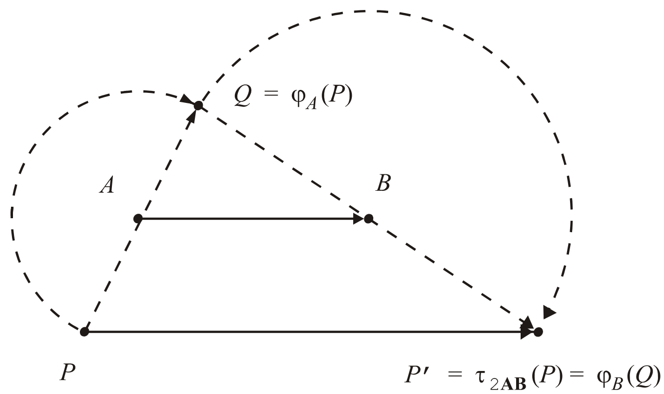


Figure 2.1: A composition of two halfturns

Theorem 69 *The composition of three halfturns is a halfturn. In fact, given any three points A, B and C,*

$$\varphi_C \circ \varphi_B \circ \varphi_A = \varphi_D, \tag{2.1}$$

where D is the unique point such that $\mathbf{AB} = \mathbf{DC}$.

Proof. Let $A = \left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right]$, $C = \left[\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right]$ and $D = \left[\begin{smallmatrix} d_1 \\ d_2 \end{smallmatrix}\right]$. Then $\mathbf{AB} = \mathbf{DC}$ implies $\left[\begin{smallmatrix} b_1 - a_1 \\ b_2 - a_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} c_1 - d_1 \\ c_2 - d_2 \end{smallmatrix}\right]$, and by equating components, $D = \left[\begin{smallmatrix} a_1 - b_1 + c_1 \\ a_2 - b_2 + c_2 \end{smallmatrix}\right]$. By Theorem 68, $\varphi_B \circ \varphi_A = \tau_{2\mathbf{AB}}$; thus

$$(\varphi_B \circ \varphi_A)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2(b_1 - a_1) \\ y + 2(b_2 - a_2) \end{bmatrix} \quad \text{and}$$

$$(\varphi_C \circ \varphi_B \circ \varphi_A)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2c_1 - (x + 2(b_1 - a_1)) \\ 2c_2 - (y + 2(b_2 - a_2)) \end{bmatrix} = \begin{bmatrix} 2(a_1 - b_1 + c_1) - x \\ 2(a_2 - b_2 + c_2) - y \end{bmatrix},$$

which is a halfturn about D. ■

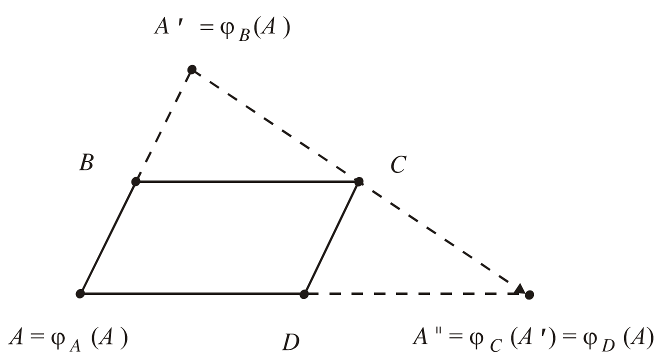


Figure 2.2.

Note that if A, B and C are non-collinear, $\square ABCD$ is a parallelogram (see Figure 2.2). This fact gives us a simple way to construct the center point D. Halfturns do not commute in general. In fact, $\varphi_A \circ \varphi_B = \varphi_B \circ \varphi_A$ if and only if $A = B$ (cf. Exercise 1.4.12). Thus *the only halfturn that commutes with φ_A is itself.* On the other hand, the fact that the product of three halfturns is a halfturn implies:

Proposition 70 *For any three points A, B and C,*

$$\varphi_C \circ \varphi_B \circ \varphi_A = \varphi_A \circ \varphi_B \circ \varphi_C.$$

Proof. By Theorem 69, there is a point D such that $\varphi_C \circ \varphi_B \circ \varphi_A = \varphi_D = \varphi_D^{-1} = (\varphi_C \circ \varphi_B \circ \varphi_A)^{-1} = \varphi_A^{-1} \circ \varphi_B^{-1} \circ \varphi_C^{-1} = \varphi_A \circ \varphi_B \circ \varphi_C$. ■

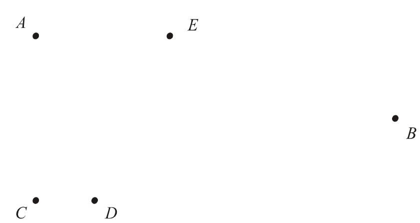
Exercises

1. In the figure below, sketch points X, Y, Z such that

   a. $\varphi_A \circ \varphi_E \circ \varphi_D = \varphi_X$

   b. $\varphi_D \circ \tau_{\mathbf{AC}} = \varphi_Y$

   c. $\tau_{\mathbf{BC}} \circ \tau_{\mathbf{AB}} \circ \tau_{\mathbf{EA}}(A) = Z.$



2. Let A, B and P be distinct points and let $P' = \tau_{\mathbf{AB}}(P)$. Prove that $\tau_{\mathbf{AB}} \circ \varphi_P \circ \tau_{\mathbf{BA}} = \varphi_{P'}$. (Hint: Use Theorem 68 to replace $\tau_{\mathbf{AB}}$ and $\tau_{\mathbf{BA}}$ with the appropriate compositions of halfturns.)

3. Use Proposition 70 to prove that any two translations commute.

4. Let τ be a translation; let C be a point. Let D be the midpoint of C and $\tau(C)$.

   (a) Prove that $\tau \circ \varphi_C = \varphi_D$.

   (b) Prove that $\varphi_C \circ \tau = \varphi_E$, where $E = \tau^{-1}(D)$.

5. If coin A in the figure below is rolled around coin B until coin A is directly under coin B, will the head on coin A right-side up or up-side down? Explain.

6. Given three points A, B, and C, construct the point D such that $\tau_{\mathbf{AB}} = \varphi_D \circ \varphi_C$.

7. Given three points A, B, and D, construct the point C such that $\tau_{\mathbf{AB}} = \varphi_D \circ \varphi_C$.

8. Given three points A, C, and D, construct the point B such that $\tau_{\mathbf{AB}} = \varphi_D \circ \varphi_C$.

9. Given three points B, C, and D, construct the point A such that $\tau_{\mathbf{AB}} = \varphi_D \circ \varphi_C$.

2.2 Compositions of Two Reflections

In this section we prove the Three Points Theorem and apply it to characterize rotations and translations in a new and important way. We shall observe that the composition of two reflections in intersecting lines is a rotation and the composition of two reflections in parallel lines is a translation.

Theorem 71 *An isometry with two distinct fixed points P and Q fixes $\overleftrightarrow{PQ}$ pointwise.*

Proof. Let P and Q be distinct fixed points of an isometry α, let R be any point on $\overleftrightarrow{PQ}$ distinct from P and Q, and let $R' = \alpha(R)$. Then $R' \in P_R$ since $PR = PR'$, and $R' \in Q_R$ since $QR = QR'$ (see Figure 2.3). If $R' \neq R$, then $\overleftrightarrow{PQ}$ is the prependicular bisector of $\overline{RR'}$, in which case P, Q, and R are non-collinear. Since this is a contradiction, $R' = R$. $\blacksquare$

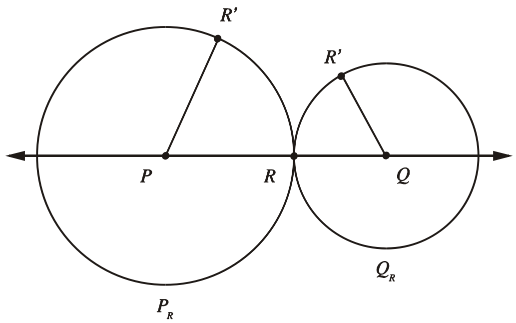


Figure 2.3.

Theorem 72 *An isometry with three non-collinear fixed points is the identity.*

Proof. Let P, Q, and R be non-collinear points of an isometry α. Then α fixes $\overleftrightarrow{PQ}$, $\overleftrightarrow{PR}$, and $\overleftrightarrow{QR}$ pointwise by Theorem 71. Let Z be any point off of these three lines and let M be any point in the interior of $\triangle PQR$ and distinct from Z (see Figure 2.4). Then $\overleftrightarrow{ZM}$ intersects $\triangle PQR$ in two distinct points A and B (one possibly a vertex). Since α fixes the points A and B, it fixes $\overleftrightarrow{ZM}$ pointwise by Theorem 71. Thus α fixes Z and $\alpha = \iota$, as claimed. ■

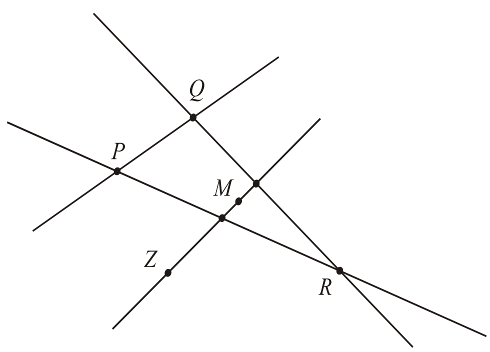


Figure 2.4.

Theorem 73 ***(Three Points Theorem)*** *Two isometries that agree on three non-collinear points are equal.*

Proof. Suppose α and β are isometries and P, Q, and R are non-collinear points such that

$$\alpha(P) = \beta(P), \quad \alpha(Q) = \beta(Q), \quad \text{and} \quad \alpha(R) = \beta(R). \tag{2.2}$$

Apply α^{-1} to both sides of each equation in (5.3) and obtain

$$P = (\alpha^{-1} \circ \beta)(P), \quad Q = (\alpha^{-1} \circ \beta)(Q), \quad \text{and} \quad R = (\alpha^{-1} \circ \beta)(R).$$

Thus $\alpha^{-1} \circ \beta$ is an isometry that fixes three non-collinear points. Hence $\alpha^{-1} \circ \beta = \iota$ by Theorem 72, and applying α to both sides gives $\alpha = \beta$. ■

Definition 74 *Let ℓ and m be distinct lines intersecting at C. Let A and B be points distinct from C on ℓ and m, respectively, and let $A' = \varphi_C(A)$. Then $\angle ACB$ and $\angle A'CB$ are the* ***angles from*** *ℓ* ***to*** *m.*

Note that $\angle ACB$ and $\angle A'CB$ are *supplementary*, i.e., their union is a straight angle, and $m\angle ACB > 0$ if and only if $m\angle A'CB < 0$. Thus $m\angle ACB - m\angle A'CB = 180$ when $m\angle ACB > 0$, and $m\angle ACB - m\angle A'CB = -180$ when $m\angle ACB < 0$. In either case, $2m\angle ACB - 2m\angle A'CB = \pm 360$ and the *double angles from ℓ to m have congruent measures.*

Theorem 75 *Given distinct lines ℓ and m intersecting at C, let Θ be the measure of an angle from ℓ to m. Then*

$$\sigma_m \circ \sigma_\ell = \rho_{C,2\Theta}.$$

Proof. First observe that

$$(\sigma_m \circ \sigma_\ell)(C) = \sigma_m(\sigma_\ell(C)) = \sigma_m(C) = C = \rho_{C,2\Theta}(C). \tag{2.3}$$

Let L be a point on ℓ distinct from C and consider the circle C_L. Let M be the point in $m \cap C_L$ such that $m\angle LCM = \Theta$ and let $L' = \sigma_m(L)$; then m is the perpendicular bisector of $\overline{LL'}$, by definition of σ_m, so that $CL = CL'$ and $m\angle LCL' = 2\Theta$. Therefore $L' = \rho_{C,2\Theta}(L)$ by definition of $\rho_{C,2\Theta}$ and

$$(\sigma_m \circ \sigma_\ell)(L) = \sigma_m(\sigma_\ell(L)) = \sigma_m(L) = L' = \rho_{C,2\Theta}(L). \tag{2.4}$$

Let $J = \sigma_\ell(M)$; then ℓ is the perpendicular bisector of $\overline{JM}$, by the definition of σ_ℓ, so that $CJ = CM$ and $m\angle JCM = 2\Theta$. Therefore $M = \rho_{C,2\Theta}(J)$ by definition of $\rho_{C,2\Theta}$ and

$$(\sigma_m \circ \sigma_\ell)(J) = \sigma_m(\sigma_\ell(J)) = \sigma_m(M) = M = \rho_{C,2\Theta}(J). \tag{2.5}$$

By equations (2.3), (2.4), and (2.5), the isometries $\sigma_m \circ \sigma_\ell$ and $\rho_{C,2\Theta}$ agree on non-collinear points C, J and L. Therefore $\sigma_m \circ \sigma_\ell = \rho_{C,2\Theta}$ by Theorem 73, as claimed. ■

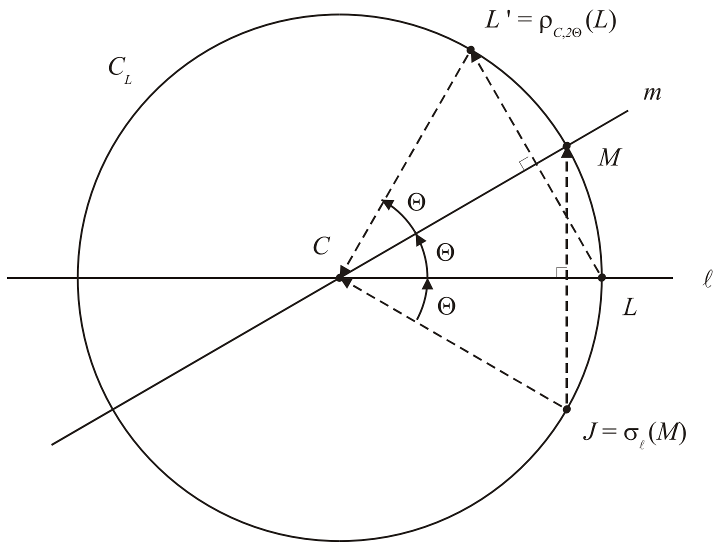


Figure 2.5.

In fact, there is the following equivalence:

Theorem 76 *A non-identity isometry α is a rotation if and only if α is the product of two reflections in distinct intersecting lines.*

Proof. The implication $\Leftarrow$ was proved in Theorem 75. For the converse, given $\rho_{C,2\Theta}$, let $\Theta' \in (-180, 180]$ such that $\Theta' \equiv \Theta$. Let ℓ be any line through C and let m be the unique line through C such that the measure of an angle from ℓ to m is Θ (see Figure 2.6). Then $\rho_{C,2\Theta} = \rho_{C,2\Theta'} = \sigma_m \circ \sigma_\ell$ by Theorem 75. ■

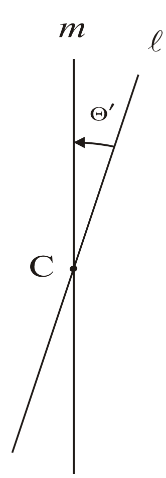


Figure 2.6.

Example 77 *Consider the lines* $\ell : X - Y = 0$ *and* $m : X = 0$. *The equations for reflections in* ℓ *and* m *are*

$$\sigma_\ell : \begin{cases} x' = y \\ y' = x \end{cases} \quad \text{and} \quad \sigma_m : \begin{cases} x' = -x \\ y' = y \end{cases}$$

and the equations for the composition $\sigma_m \circ \sigma_\ell$ *are*

$$\sigma_m \circ \sigma_\ell : \begin{cases} x' = -y \\ y' = x. \end{cases}$$

Note that the measure of the positive angle from ℓ *to* m *is* 45, *and the equations*

$$\rho_{O,90} : \begin{cases} x' = x \cos 90 - y \sin 90 = -y \\ y' = x \sin 90 + y \cos 90 = x \end{cases}$$

agree with those of equations of $\sigma_m \circ \sigma_\ell$. *Furthermore,* $-270 = 2\,(-135)$ *is twice the measure of the negative angle from* ℓ *to* m, *and of course,* $90 \equiv -270$.

Corollary 78 *Let* ℓ *and* m *be perpendicular lines intersecting at* C. *Then*

$$\varphi_C = \sigma_m \circ \sigma_\ell = \sigma_\ell \circ \sigma_m.$$

Corollary 79 *Let* ℓ *and* m *be distinct parallel lines with common perpendicular* n. *Let* $L = \ell \cap n$ *and let* $M = m \cap n$. *Then*

$$\sigma_m \circ \sigma_\ell = \tau_{\mathbf{2LM}}.$$

Proof. Refer to Figure 2.7. By Corollary 78 and Theorem 68, we have

$$\sigma_m \circ \sigma_\ell = (\sigma_m \circ \sigma_n) \circ (\sigma_n \circ \sigma_\ell) = \varphi_M \circ \varphi_L = \tau_{\mathbf{2LM}}. \tag{2.6}$$

■

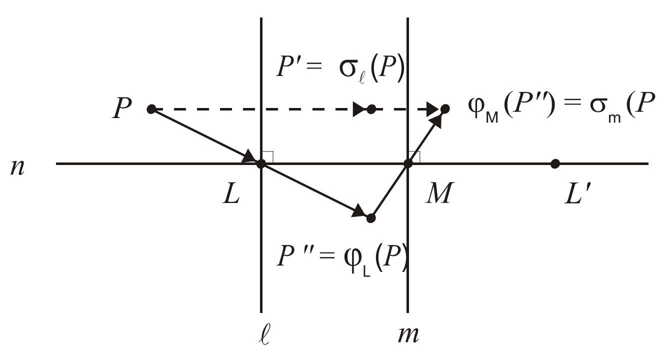


Figure 2.7.

In fact, we have the following equivalence:

Theorem 80 *A non-identity isometry α is a translation if and only if α is a product of two reflections in distinct parallel lines.*

Proof. Implication $\Leftarrow$ follows from Corollary 79. Conversely, given a non-identity translation τ and a point L, let $P = \tau(L)$; then $\tau = \tau_{\mathbf{LP}}$ by Corollary 41. Let M be the midpoint of L and P, let $n = \overleftrightarrow{LM}$, and let ℓ and m be the lines perpendicular to n at L and M, respectively. Then $\mathbf{LP} = \mathbf{LM}$, so that $\tau = \tau_{\mathbf{LP}} = \tau_{\mathbf{2LM}} = \sigma_m \circ \sigma_\ell$ by Corollary 79. ■

Theorem 81 *If lines ℓ, m, and n are concurrent at C, there exist unique lines p and q passing through C such that*

$$\sigma_m \circ \sigma_\ell = \sigma_n \circ \sigma_p = \sigma_q \circ \sigma_n.$$

Proof. If $\ell = m$, set $p = q = n$ and the conclusion follows trivially. If $\ell \neq m$, let Θ be the measure of an angle from ℓ to m; let p and q be the unique lines such that an angle from p to n and an angle from n to q have measure Θ. Then $\rho_{C,2\Theta} = \sigma_m \circ \sigma_\ell = \sigma_n \circ \sigma_p = \sigma_q \circ \sigma_n$ by Theorem 75 (see Figure 2.8). ■

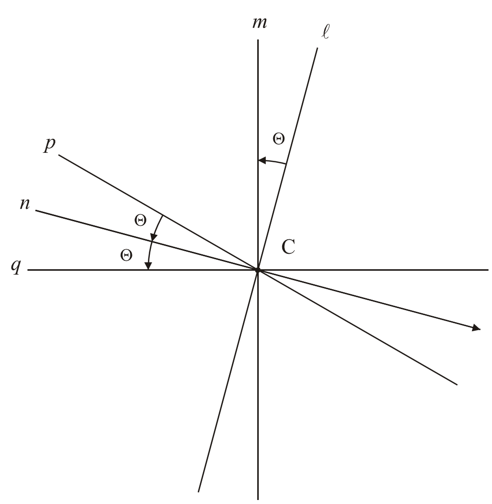


Figure 2.8.

Note that Theorem 81 does not require line n to be distinct from lines ℓ and m. If $n = m$, for example, then $p = \ell$. Also, if $\rho_{C,\Theta} = \sigma_m \circ \sigma_\ell$ and $\Theta \notin 0^\circ$, then $C = \ell \cap m$. By Theorem 81, each line n passing through C determines unique lines p and q such that $\sigma_m \circ \sigma_\ell = \sigma_n \circ \sigma_p = \sigma_q \circ \sigma_n$. Since Θ is congruent to twice the measure of an angle from p to n and an angle from n to q we have:

Corollary 82 *Let C be a point, let $\Theta \notin 0^\circ$, and let n be any line through C. Let p and q be the unique lines passing through C such that an angle from p to n and an angle from n to q is congruent to $\frac{1}{2}\Theta$. Then*

$$\rho_{C,\Theta} = \sigma_q \circ \sigma_n = \sigma_n \circ \sigma_p.$$

We know from Proposition 57 that a halfturn φ_C fixes a line ℓ if and only if C is on ℓ. Which lines are fixed by a general rotation?

Theorem 83 *Non-identity rotations that fix a line are halfturns.*

Proof. If $\Theta \notin 0^\circ$ and ℓ is a line such that $\rho_{C,\Theta}(\ell) = \ell$, let m be the line through C perpendicular to ℓ. By Corollary 82, there is a line n through C such that $\rho_{C,\Theta} = \sigma_n \circ \sigma_m$ (see Figure 2.9). Since $\ell \perp m$ we have

$$\ell = \rho_{C,\Theta}(\ell) = (\sigma_n \circ \sigma_m)(\ell) = \sigma_n(\sigma_m(\ell)) = \sigma_n(\ell).$$

Thus σ_n fixes ℓ, in which case $n = \ell$ or $n \perp \ell$, by Exercise 18 in Section 1.2. But if $n \perp \ell$, then $m \parallel n$ since ℓ is a common perpendicular. But m and n intersect at C. Therefore $n = \ell$ and $m \perp n$ so that $\rho_{C,2\Theta} = \sigma_n \circ \sigma_m = \varphi_C$ by Corollary 78. ∎

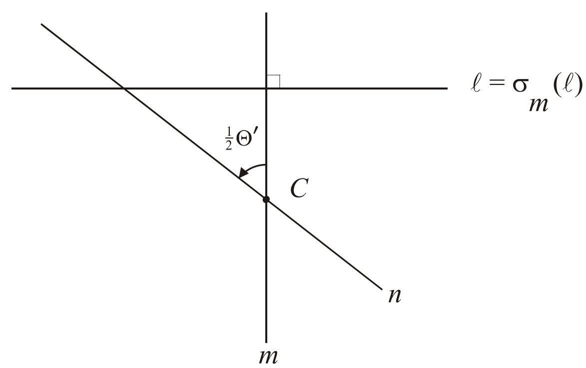


Figure 2.9. $\Theta' \equiv \Theta$

There is the following analogue of Theorem 81 for translations:

Theorem 84 *Let ℓ, m and n be parallel lines. There exist unique lines p and q parallel to ℓ such that*

$$\sigma_m \circ \sigma_\ell = \sigma_n \circ \sigma_p = \sigma_q \circ \sigma_n.$$

Proof. If $\ell = m$, set $p = q = n$ and the conclusion follows trivially. If $\ell \neq m$, choose a common perpendicular c, and let $L = \ell \cap c$, $M = m \cap c$, and $N = n \cap c$. By Theorem 69, there exist unique points P and Q on c such that $\varphi_P = \varphi_N \circ \varphi_M \circ \varphi_L$ and $\varphi_Q = \varphi_M \circ \varphi_L \circ \varphi_N$. Left-multiplying both side of the first equation by φ_N gives $\varphi_N \circ \varphi_P = \varphi_M \circ \varphi_L$; and right-multiplying both sides of the second equation by φ_N gives $\varphi_Q \circ \varphi_N = \varphi_M \circ \varphi_L$. Then by Theorem 68 we have

$$\begin{aligned} \tau_{\mathbf{2LM}} &= \varphi_M \circ \varphi_L = \varphi_N \circ \varphi_P = \tau_{\mathbf{2PN}} \\ &= \varphi_M \circ \varphi_L = \varphi_Q \circ \varphi_N = \tau_{\mathbf{2NQ}}. \end{aligned}$$

Let p and q be the lines perpendicular to c at P and Q, respectively (see Figure 2.10). Then by Corollary 79,

$$\begin{aligned}\sigma_m \circ \sigma_\ell = \tau_{\mathbf{2LM}} = \tau_{\mathbf{2PN}} = \sigma_n \circ \sigma_p \\ = \tau_{\mathbf{2LM}} = \tau_{\mathbf{2NQ}} = \sigma_q \circ \sigma_n.\end{aligned}$$

■

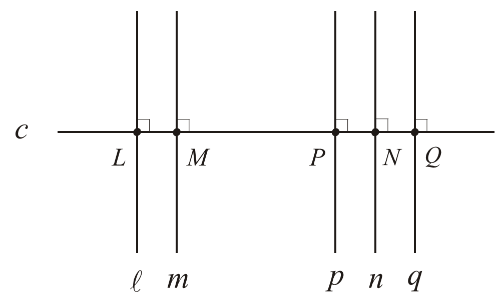


Figure 2.10.

Note that Theorem 84 does not require line n to be distinct from ℓ and m. If $n = m$, for example, then $p = \ell$. Now if $\tau_{\mathbf{PQ}} = \sigma_m \circ \sigma_\ell$, then $\ell \parallel m$. If n is also parallel to m, Theorem 84 tells us that n determines unique lines p and q parallel to m such that $\sigma_m \circ \sigma_\ell = \sigma_n \circ \sigma_p = \sigma_q \circ \sigma_n$. Since $\mathbf{PQ} = 2\mathbf{PN} = 2\mathbf{NQ}$ we have:

Corollary 85 *Let P and Q be distinct points and let n be a line perpendicular to $\overleftrightarrow{PQ}$. Then there exist unique lines p and q parallel to n such that*

$$\tau_{\mathbf{PQ}} = \sigma_q \circ \sigma_n = \sigma_n \circ \sigma_p.$$

Note that the identity $\iota = \tau_{\mathbf{PP}} = \rho_{C,0} = \sigma_\ell \circ \sigma_\ell$ for all P, C, and ℓ. Therefore

Theorem 86 *A product of two reflections is either a translation or a rotation; only the identity is both a translation and a rotation.*

Exercises

1. Consider the rotation $\rho_{C,\Theta} = \sigma_m \circ \sigma_\ell$, where $\ell : X = 3$ and $m : Y = X$.
   (a) Find the xy-coordinates of the center C and the rotation angle $\Theta' \in (-180, 180]$.
   (b) Find the equations of $\rho_{C,\Theta}$.
   (c) Compare Θ' with the (positive and negative) measures of the angles from ℓ to m.

(d) Compose the equations of σ_m with the equations of σ_ℓ and compare your result with the equations of $\rho_{C,\Theta}$.

2. Consider the rotation $\rho_{C,\Theta} = \sigma_m \circ \sigma_\ell$, where $\ell : X + Y - 2 = 0$ and $m : Y = 3$.

   (a) Find the xy-coordinates of the center C and the rotation angle $\Theta' \in (-180, 180]$.

   (b) Find the equations of $\rho_{C,\Theta}$.

   (c) Compare Θ' with the (positive and negative) measures of the angles from ℓ to m.

   (d) Compose the equations of σ_m with the equations of σ_ℓ and compare your result with the equations of $\rho_{C,\Theta}$.

3. Consider the rotation $\rho_{C,\Theta} = \sigma_m \circ \sigma_\ell$, where $\ell : Y = X$ and $m : Y = -X + 4$.

   (a) Find the xy-coordinates of the center C and the rotation angle $\Theta' \in (-180, 180]$.

   (b) Find the equations of $\rho_{C,\Theta}$.

   (c) Compare Θ' with the (positive and negative) measures of the angles from ℓ to m.

   (d) Compose the equations of σ_m with the equations of σ_ℓ and compare your result with the equations of $\rho_{C,\Theta}$.

4. Find the equations of lines ℓ and m such that $\rho_{O,90} = \sigma_m \circ \sigma_\ell$.

5. Let $C = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Find equations of lines ℓ and m such that $\rho_{C,60} = \sigma_m \circ \sigma_\ell$.

6. Lines ℓ and m have respective equations $Y = 3$ and $Y = 5$. Find the equations of the translation $\sigma_m \circ \sigma_\ell$.

7. Lines ℓ and m have respective equations $Y = X$ and $Y = X + 4$. Find the equations of the translation $\sigma_m \circ \sigma_\ell$.

8. The translation τ has vector $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$. Find the equations of lines ℓ and m such that $\tau = \sigma_m \circ \sigma_\ell$.

9. The translation τ has equations $x' = x + 6$ and $y' = y - 3$. Find equations of lines ℓ and m such that $\tau = \sigma_m \circ \sigma_\ell$.

10. Lines ℓ, m and n have respective equations $X = 0$, $Y = 2X$ and $Y = 0$.

    a. Find the equation of line p such that $\sigma_m \circ \sigma_\ell = \sigma_p \circ \sigma_n$.

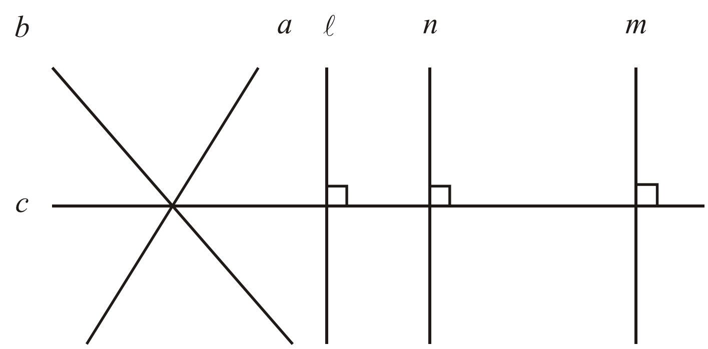
    b. Find the equation of line q such that $\sigma_m \circ \sigma_\ell = \sigma_n \circ \sigma_q$.

11. Lines ℓ, m and n have respective equations $Y = 3$, $Y = 5$ and $Y = 9$.

    a. Find the equation of line p such that $\sigma_m \circ \sigma_\ell = \sigma_p \circ \sigma_n$.

    b. Find the equation of line q such that $\sigma_m \circ \sigma_\ell = \sigma_n \circ \sigma_q$.

12. Construct the following in the figure below:



    a. Line s such that $\sigma_n \circ \sigma_s = \sigma_m \circ \sigma_\ell$.

    b. Line t such that $\sigma_c \circ \sigma_t = \sigma_b \circ \sigma_a$.

    c. The fixed point of $\sigma_t \circ \sigma_s$.

13. Given distinct points P and Q, construct the point R such that $\tau_{\mathbf{PQ}} \circ \rho_{P,45} = \rho_{R,45}$.

14. Given distinct points P, Q and R, construct the point S such that $\tau_{\mathbf{QR}} \circ \rho_{P,120} = \rho_{S,120}$.

15. Let P be a point and let a and b be lines. Prove that:

    a. There exist lines c and d with P on c such that $\sigma_b \circ \sigma_a = \sigma_d \circ \sigma_c$.

    b. There exist lines ℓ and m with P on m such that $\sigma_b \circ \sigma_a = \sigma_m \circ \sigma_\ell$.

16. Let P and Q be distinct points and let c be a line parallel to $\overleftrightarrow{PQ}$. Apply Theorem 68 and Corollary 78 to prove that $\sigma_c \circ \tau_{\mathbf{PQ}} = \tau_{\mathbf{PQ}} \circ \sigma_c$ (cf. Exercise 1.3.8).

17. Let C be a point on line ℓ and let $\Theta \in \mathbb{R}$. Prove that $\sigma_\ell \circ \rho_{C,\Theta} = \rho_{C,-\Theta} \circ \sigma_\ell$.

18. Let $\square ABCD \cong \square EFGH$ be a pair of congruent rectangles. Describe how to find a rotation $\rho_{P,\Theta}$ such that $\rho_{P,\Theta}(\square ABCD) = \square EFGH$.

19. Prove that in any non-degenerate triangle, the perpendicular bisectors of the sides are concurrent at some point P equidistant from the vertices. Thus every triangle has a circumscribed circle, called the *circumcircle.* The center P of the circumcircle is called the *circumcenter* of the triangle.

2.3 The Angle Addition Theorem

A technique similar to the one used in the proof of Theorem 84 to transform a product of halfturns into a product of reflections in parallel lines can be applied to a pair of general rotations as in the proof of our next important theorem:

Theorem 87 (The Angle Addition Theorem, part I) *Let A and B be points, and let Θ and Φ be real numbers such that $\Theta + \Phi \notin 0°$. Then there is a unique point C such that*

$$\rho_{B,\Phi} \circ \rho_{A,\Theta} = \rho_{C,\Theta+\Phi}.$$

Proof. If $A = B$, then $\rho_{B,\Phi} \circ \rho_{A,\Theta} = \rho_{B,\Phi} \circ \rho_{B,\Theta} = \rho_{B,\Theta+\Phi}$ by Proposition 64, and the conclusion holds with $C = B$. So assume that $A \neq B$ and let $\Theta', \Phi' \in (-180, 180]$ such that $\Theta' \equiv \Theta$ and $\Phi' \equiv \Phi$; then $\rho_{A,\Theta} = \rho_{A,\Theta'}$ and $\rho_{B,\Phi} = \rho_{B,\Phi'}$. If $\Theta' = 0$, then $\rho_{A,\Theta'} = \iota$ and the conclusion holds for $C = B$; similarly, if $\Phi' = 0$, the conclusion holds with $C = A$. So assume $\Theta', \Phi' \neq 0$ and let $m = \overleftrightarrow{AB}$. By Corollary 82, there exist unique lines ℓ and n passing through A and B, respectively, such that $\rho_{B,\Phi'} = \sigma_n \circ \sigma_m$ and $\rho_{A,\Theta'} = \sigma_m \circ \sigma_\ell$. Consider the angle from ℓ to m measuring $\frac{1}{2}\Theta'$ and the angle from m to n measuring $\frac{1}{2}\Phi'$. By assumption, $-360 < \Theta' + \Phi' < 360$ so that $-180 < \frac{1}{2}(\Theta' + \Phi') < 180$. Then ℓ and n are not parallel, otherwise m is a transversal and an angle from n to m has measure $\frac{1}{2}\Theta'$, in which case $\frac{1}{2}\Theta' + \frac{1}{2}\Phi' = 180$, which is a contradiction. Therefore ℓ and n intersect at some point C and we may consider $\triangle ABC$. Now $\Theta', \Phi' \in (-180, 0) \cup (0, 180]$ implies $\frac{1}{2}\Theta', \frac{1}{2}\Phi' \in (-90, 0) \cup (0, 90]$. I claim some angle from ℓ to n has measure $\frac{1}{2}(\Theta' + \Phi')$. We consider four cases.

Case 1: $\Theta', \frac{1}{2}\Phi' \in (0, 90]$. Then $m\angle CAB = \frac{1}{2}\Theta'$ and $m\angle ABC = \frac{1}{2}\Phi'$ (see Figure 2.11). By the Exterior Angle Theorem, there is an exterior angle from ℓ to n measuring $\frac{1}{2}(\Theta' + \Phi')$.

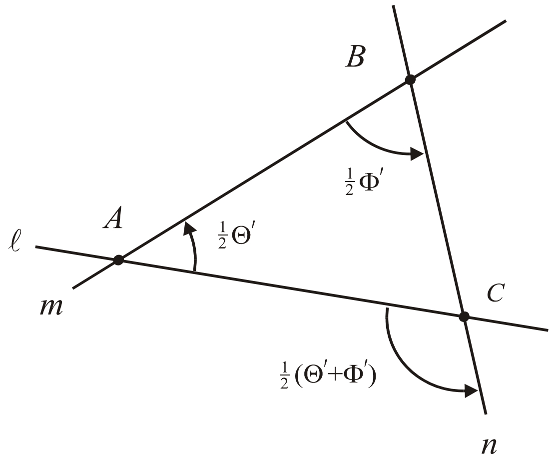


Figure 2.11.

Case 2: $\frac{1}{2}\Theta' \in (-90, 0)$ and $\frac{1}{2}\Phi' \in (0, 90]$; then $m\angle BAC = -\frac{1}{2}\Theta'$ and $\angle ACB$ is an interior angle from ℓ to n (see Figure 2.12). By the Exterior Angle Theorem, $\frac{1}{2}\Phi' = m\angle ACB - \frac{1}{2}\Theta'$; hence $m\angle ACB = \frac{1}{2}(\Theta' + \Phi')$.

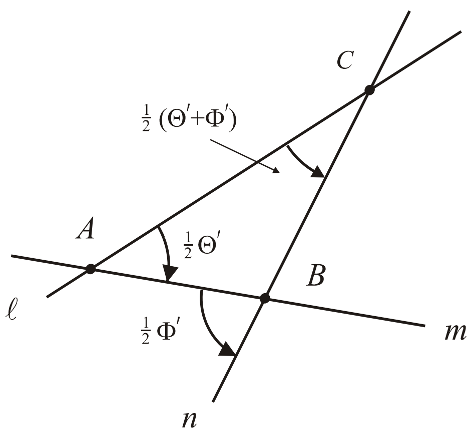


Figure 2.12.

Cases 3 and 4 are similar and left as exercises. Thus in every case, $\rho_{B,\Phi} \circ \rho_{A,\Theta} = \rho_{B,\Phi'} \circ \rho_{A,\Theta'} = (\sigma_n \circ \sigma_m) \circ (\sigma_m \circ \sigma_\ell) = \sigma_n \circ \sigma_\ell = \rho_{C,\Theta'+\Phi'} = \rho_{C,\Theta+\Phi}$. ■

Example 88 *Let $A = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. To determine $\rho_{B,180} \circ \rho_{A,90}$, let $m = \overleftrightarrow{AB} : Y = 2$, $\ell : Y = -X + 4$, and $n : X = -2$. Then ℓ and n are the unique lines such that an angle from ℓ to m measures 45 and an angle from m to n measures 90. Thus $\rho_{A,90} = \sigma_m \circ \sigma_\ell$, $\rho_{B,180} = \sigma_n \circ \sigma_m$, the center of rotation is $C = \ell \cap m = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$, and*

$$\rho_{B,180} \circ \rho_{A,90} = (\sigma_n \circ \sigma_m) \circ (\sigma_m \circ \sigma_\ell) = \sigma_n \circ \sigma_\ell = \rho_{C,-90} = \rho_{C,270}.$$

Next, we consider points A and B and a product of rotations $\rho_{B,\Phi} \circ \rho_{A,\Theta}$ such that $\Theta + \Phi \in 0°$.

Theorem 89 (The Angle Addition Theorem, part II) *Let A and B be points, and let Θ and Φ be real numbers such that $\Theta + \Phi \in 0°$. Then $\rho_{B,\Phi} \circ \rho_{A,\Theta}$ is a translation.*

Proof. If $A = B$, then Proposition 64 implies $\rho_{B,\Phi} \circ \rho_{A,\Theta} = \rho_{B,0} = \iota$, which can be thought of as a trivial translation. So assume $A \neq B$ and let $m = \overleftrightarrow{AB}$. Then by Corollary 82, there exist unique lines ℓ and n passing through A and B, respectively, such that $\rho_{A,\Theta} = \sigma_m \circ \sigma_\ell$ and $\rho_{B,\Phi} = \sigma_n \circ \sigma_m$. If $\Theta, \Phi \in 0°$, then $\rho_{B,\Phi} \circ \rho_{A,\Theta} = \iota$ can be thought of as a trivial translation. So assume that $\Theta, \Phi \notin 0°$; then $\Theta \equiv \Theta'$ and $\Phi \equiv \Phi'$ for some $\Theta', \Phi' \in (0, 360)$ so that ℓ, m, and n are distinct, some angle from ℓ to m measures $\frac{1}{2}\Theta'$, some angle from m to n measures $\frac{1}{2}\Phi'$, and $\frac{1}{2}\Theta' + \frac{1}{2}\Phi' = 180$. Thus m cuts ℓ and n with congruent corresponding angles so that $\ell \| n$ (see Figure 2.13). Therefore

$$\rho_{B,\Phi} \circ \rho_{A,\Theta} = \rho_{B,\Phi'} \circ \rho_{A,\Theta'} = (\sigma_n \circ \sigma_m) \circ (\sigma_m \circ \sigma_\ell) = \sigma_n \circ \sigma_\ell$$

is a non-identity translation by Theorem 80. ■

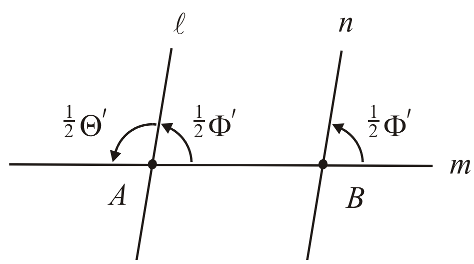


Figure 2.13. The translation $\rho_{B,\Phi} \circ \rho_{A,\Theta}$.

Finally, we consider the composition (in either order) of a translation and a non-identity rotation.

Example 90 *Let $A = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. To determine $\rho_{B,90} \circ \rho_{A,270}$, let $m = \overleftrightarrow{AB} : Y = 2$, $\ell : Y = X$, and $n : Y = X + 4$. Then ℓ and n are the unique lines such that an angle from ℓ to m measures 135 and an angle from m to n measures 45. Thus $\rho_{A,270} = \sigma_m \circ \sigma_\ell$, $\rho_{B,90} = \sigma_n \circ \sigma_m$, $\ell \parallel n$, and the vector from ℓ to n of minimal length is* **OB**. *Therefore*

$$\rho_{B,180} \circ \rho_{A,90} = (\sigma_n \circ \sigma_m) \circ (\sigma_m \circ \sigma_\ell) = \sigma_n \circ \sigma_\ell = \tau_{2\mathbf{OB}}.$$

Theorem 91 (The Angle Addition Theorem, part III) *The composition of a non-identity rotation $\rho_{C,\Theta}$ and a translation τ (in either order) is a rotation of $\Theta°$.*

Proof. If $\tau = \iota$ there is nothing to prove. So assume $\tau \neq \iota$, and let m be the line through C perpendicular to the direction of translation. Let ℓ and n be the unique lines such that $\rho_{C,\Theta} = \sigma_m \circ \sigma_\ell$ and $\tau = \sigma_n \circ \sigma_m$. Since $\rho_{C,\Theta} \neq \iota$, lines ℓ and m are distinct and intersect at C. Let $\Theta' \in (-180, 180]$ such that $\Theta \equiv \Theta'$. Since $\tau \neq \iota$, lines m and n are distinct and parallel. Hence ℓ a transversal for m and n, and the corresponding angles from ℓ to m and from ℓ to n have measure $\frac{1}{2}\Theta'$ (see Figure 2.14). Let $D = \ell \cap n$; then

$$\tau \circ \rho_{C,\Theta} = \sigma_n \circ \sigma_m \circ \sigma_m \circ \sigma_\ell = \sigma_n \circ \sigma_\ell = \rho_{D,\Theta}.$$

The product $\rho_{C,\Theta} \circ \tau$ is also a rotation of $\Theta°$ by a similar argument left to the reader. ■

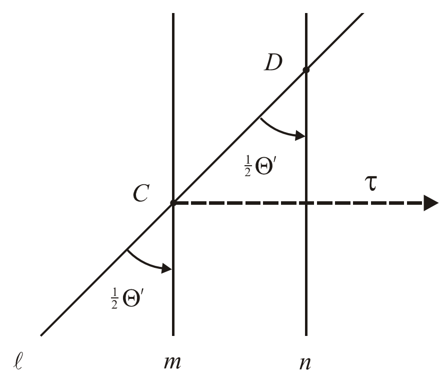


Figure 2.14.

We summarize the discussion in this section by gathering together the various parts of the Angle Addition Theorem:

Theorem 92 ***(The Angle Addition Theorem)***

a. *A rotation of* $\Theta°$ *followed by a rotation of* $\Phi°$ *is*

- *a translation if* $\Theta + \Phi \in 0°$;
- *a rotation of* $(\Theta + \Phi)°$ *otherwise.*

b. *A translation followed by a non-identity rotation of* $\Theta°$ *is a rotation of* $\Theta°$.

c. *A non-identity rotation of* $\Theta°$ *followed by a translation is a rotation of* $\Theta°$.

d. *A translation followed by a translation is a translation.*

Exercises

1. Let $O = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $C = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.
   a. Find equations of lines ℓ, m and n such that $\rho_{C,90} = \sigma_m \circ \sigma_n$ and $\rho_{O,90} = \sigma_\ell \circ \sigma_m$.

   b. Find xy-coordinates of the point D such that $\varphi_D = \rho_{O,90} \circ \rho_{C,90}$.

   c. Find xy-coordinates for the point E such that $\varphi_E = \rho_{C,90} \circ \rho_{O,90}$.

2. Let $O = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
   a. Find equations of the lines ℓ, m and n such that $\varphi_O = \sigma_m \circ \sigma_\ell$ and $\rho_{C,120} = \sigma_n \circ \sigma_m$.

   b. Find xy coordinates for the point D and the angle of rotation Θ such that $\rho_{D,\Theta} = \rho_{C,120} \circ \varphi_O$.

   c. Find xy-coordinates for the point E and the angle of rotation Φ such that $\rho_{E,\Phi} = \rho_{C,120} \circ \rho_{O,60}$.

3. Let $A = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$.
   a. Find equations of lines ℓ, m and n such that $\rho_{A,90} = \sigma_m \circ \sigma_\ell$ and $\rho_{B,120} = \sigma_n \circ \sigma_m$.

   b. Find xy coordinates for the point C and the angle of rotation Θ such that $\rho_{C,\Theta} = \rho_{B,120} \circ \rho_{A,90}$.

   c. Find xy-coordinates for the point D and the angle of rotation Φ such that $\rho_{D,\Phi} = \rho_{A,90} \circ \rho_{B,120}$.

4. Given distinct points A and B, let $\rho_{C,150} = \rho_{B,90} \circ \rho_{A,60}$ and $m = \overleftrightarrow{AB}$. Use a MIRA to construct lines ℓ and n such that $\rho_{A,60} = \sigma_m \circ \sigma_\ell$ and $\rho_{B,90} = \sigma_n \circ \sigma_m$. Label the center of rotation C.

5. Given distinct points A and B, let $\rho_{C,30} = \rho_{B,90} \circ \rho_{A,-60}$ and $m = \overleftrightarrow{AB}$. Use a MIRA to construct lines ℓ and n such that $\rho_{A,-60} = \sigma_m \circ \sigma_\ell$ and $\rho_{B,90} = \sigma_n \circ \sigma_m$. Label the center of rotation C.

6. Given distinct points A and B, let $\rho_{C,-30} = \rho_{B,-90} \circ \rho_{A,60}$ and $m = \overleftrightarrow{AB}$. Use a MIRA to construct lines ℓ and n such that $\rho_{A,60} = \sigma_m \circ \sigma_\ell$ and $\rho_{B,-90} = \sigma_n \circ \sigma_m$. Label the center of rotation C.

7. Given distinct points A and B, let $\rho_{C,-150} = \rho_{B,-90} \circ \rho_{A,-60}$ and $m = \overleftrightarrow{AB}$. Use a MIRA to construct lines ℓ and n such that $\rho_{A,-60} = \sigma_m \circ \sigma_\ell$ and $\rho_{B,-90} = \sigma_n \circ \sigma_m$. Label the center of rotation C.

8. Given distinct points A and B, let $\tau_{\mathbf{v}} = \rho_{B,120} \circ \rho_{A,240}$. Use a MIRA to construct lines ℓ and n such that $\rho_{A,240} = \sigma_m \circ \sigma_\ell$ and $\rho_{B,120} = \sigma_n \circ \sigma_m$. Construct the translation vector $\mathbf{v}$.

9. Given a point C and a non-zero vector $\mathbf{v}$, let $\rho_{D,90} = \tau_{\mathbf{v}} \circ \rho_{C,90}$. Use a MIRA to construct lines ℓ, m, and n such that $\rho_{C,90} = \sigma_m \circ \sigma_\ell$ and $\tau_{\mathbf{v}} = \sigma_n \circ \sigma_m$. Label the center of rotation D.

10. Given a point C and a non-zero vector $\mathbf{v}$, let $\rho_{D,90} = \rho_{C,90} \circ \tau_{\mathbf{v}}$. Use a MIRA to construct lines ℓ, m, and n such that $\tau_{\mathbf{v}} = \sigma_m \circ \sigma_\ell$ and $\rho_{C,90} = \sigma_n \circ \sigma_m$. Label the center of rotation D.

11. Given distinct points A and B, use a MIRA to construct a point P such that $\rho_{A,60} = \tau_{\mathbf{PB}} \circ \rho_{B,60}$.

12. Given distinct non-collinear points A, B and C, use a MIRA to construct the point D such that $\rho_{D,60} = \tau_{\mathbf{AB}} \circ \rho_{C,60}$.

13. Let C be a point and let τ be a translation. Prove there exists a point R such that $\varphi_C \circ \tau = \varphi_R$ (c.f. Section 2.1, Exercise 5).

14. Complete the proof of Theorem 87: Let A and B be points, and let Θ and Φ be real numbers such that $\Theta + \Phi \notin 0°$.

    a. Case 3. Assume $\frac{1}{2}\Theta', \frac{1}{2}\Phi' \in (-90, 0)$ and prove that $\rho_{B,\Phi} \circ \rho_{A,\Theta} = \rho_{C,\Theta+\Phi}$.

    b. Case 4. Assume $\frac{1}{2}\Theta' \in (0, 90]$ and $\Phi' \in (-90, 0)$ and prove that $\rho_{B,\Phi} \circ \rho_{A,\Theta} = \rho_{C,\Theta+\Phi}$.

15. Complete the proof of Theorem 91: Let C be a point, let $\Theta \notin 0°$, and let τ be a translation. Prove there exists a point B such that $\rho_{C,\Theta} \circ \tau = \rho_{B,\Theta}$.

2.4 Glide Reflections

If we left-multiply both sides of the equation $\sigma_m \circ \sigma_\ell = \sigma_n \circ \sigma_p$ by σ_n we obtain $\sigma_n \circ \sigma_m \circ \sigma_\ell = \sigma_p$ and the following corollary of Theorems 81 and 84:

Corollary 93 *Let ℓ, m, and n be lines (not necessarily distinct).*

a. If ℓ, m, and n are concurrent at point C, there exists a unique line p passing through C such that

$$\sigma_p = \sigma_n \circ \sigma_m \circ \sigma_\ell.$$

b. *If ℓ, m, and n are parallel, there exists a unique line p parallel to ℓ, m, and n such that*
$$\sigma_p = \sigma_n \circ \sigma_m \circ \sigma_\ell.$$

Thus the composition of three reflections in three concurrent lines (or three parallel lines) is a reflection in some unique line concurrent with (or parallel to) them. The converse is also true (see Exercise 6).

Proposition 94 *Lines ℓ, m, and n are either concurrent or mutually parallel if and only if there exists a unique line p such that $\sigma_n \circ \sigma_m \circ \sigma_\ell = \sigma_p$.*

But what do we get when we compose three reflections in distinct lines that are neither concurrent nor mutually parallel? Proposition 94 says we *don't get a reflection.* So we're left with a rotation, a translation, or perhaps something entirely new! Suppose lines ℓ, m, and n are neither concurrent nor mutually parallel. Then two of these, say ℓ and m, must intersect at a point C.

I claim that $\sigma_n \circ \sigma_m \circ \sigma_\ell$ has no fixed points and, consequently, *is not a rotation.* Let Θ be the measure of an angle from ℓ to m and let $P' = \sigma_n(P)$. Suppose there is a point P such that $P = (\sigma_n \circ \sigma_m \circ \sigma_\ell)(P)$; then composing σ_n with both sides gives $P' = \sigma_n(P) = (\sigma_m \circ \sigma_\ell)(P) = \rho_{C,2\Theta}(P)$. Hence n is the perpendicular bisector of $\overleftrightarrow{PP'}$ and passes through the center of rotation C, which is impossible since ℓ, m, and n are not concurrent.

I claim that $\sigma_n \circ \sigma_m \circ \sigma_\ell$ is not a dilatation, and consequently, *is not a translation.* Let q be the line through C parallel to n. By Theorem 81, there is a unique line p through C such that $\sigma_q \circ \sigma_p = \sigma_m \circ \sigma_\ell$. Let $\mathbf{v}$ be the vector in the orthogonal direction from q to n whose magnitude is twice the distance from q to n. Then $\sigma_n \circ \sigma_m \circ \sigma_\ell = \sigma_n \circ \sigma_q \circ \sigma_p = \tau_{\mathbf{v}} \circ \sigma_p$. Choose any line a through C distinct from q and not perpendicular to p. Let $b = \sigma_p(a)$. Then $b \nparallel a$ while $b \parallel \tau_{\mathbf{v}}(b)$ since $\tau_{\mathbf{v}}$ is a dilatation. Therefore $(\tau_{\mathbf{v}} \circ \sigma_p)(a) \nparallel a$. To summarize, we have

Proposition 95 *If distinct lines ℓ, m, and n are neither concurrent nor mutually parallel, the composition $\sigma_n \circ \sigma_m \circ \sigma_\ell$ is neither a reflection, a rotation, nor a translation.*

The fact that the composition $\sigma_n \circ \sigma_m \circ \sigma_\ell$ is a "glide reflection" is content of Theorem 100 – the main result in this section.

Definition 96 *Let $\mathbf{v}$ be a non-zero vector and let c be a line. A transformation $\gamma : \mathbb{R}^2 \to \mathbb{R}^2$ is a* ***glide reflection with glide vector*** $\mathbf{v}$ ***and axis*** c *if and only if*

1. $\gamma = \sigma_c \circ \tau_{\mathbf{v}}$;

2. $\tau_{\mathbf{v}}(c) = c$.

Imagine the pattern of footprints you make when walking in the sand on a beach, and imagine that your footprint pattern extends infinitely far in either direction. Imagine a line c positioned midway between your left and right footprints. In your mind, slide the entire pattern one-half step in a direction parallel to c then reflect it in line c. The image pattern, which exactly superimposes on the original pattern, is the result of performing a glide reflection with axis c (see Figure 2.15).

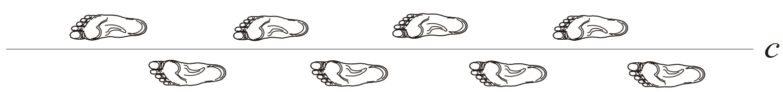


Figure 2.15: Footprints fixed by a glide reflection.

Here are some important properties of a glide reflection:

Proposition 97 *Let γ be a glide reflection with glide vector $\mathbf{v}$ and axis c.*

a. γ interchanges the halfplanes of c.

b. γ has no fixed points.

c. The midpoint of point P and its image $\gamma(P)$ lies on c.

d. γ fixes exactly one line, its axis c.

Proof. By definition, $\gamma = \sigma_c \circ \tau_{\mathbf{v}}$. Let P be any point and let $Q = \tau_{\mathbf{v}}(P)$. If P is on c, so are Q and the midpoint M of $\overline{PQ}$, since $\tau_{\mathbf{v}}(c) = c$. Furthermore, $Q \neq P$ since $\mathbf{v} \neq \mathbf{0}$; thus $\gamma(P) = \sigma_c(\tau_{\mathbf{v}}(P)) = \sigma_c(Q) = Q \neq P$ and γ has no fixed points on c. If P is off c, let $P' = \gamma(P)$. Then Q and P lie on the same side of c, in which case P and P' lie on opposite sides of c. Thus γ interchanges the halfplanes of c and has no fixed points off c. This proves (a) and (b). Let $M = \overline{PP'} \cap c$, let R be the midpoint of $\overline{QP'}$, and let S be the foot of the perpendicular from M to $\overleftrightarrow{PQ}$ (see Figure 2.16). Then R lies on c by definition of σ_c, $\overline{MS} \cong \overline{RQ} \cong \overline{RP'}$, $\angle MPS \cong \angle P'MR$ since these angles are corresponding, and the angles $\angle P'RM$ and $\angle MSP$ are right angles. Therefore $\triangle P'RM \cong \triangle MSP$ by AAS and $PM = MP'$ (CPCTC) so that M is the midpoint of $\overline{PP'}$, which proves (c). To prove (d), suppose $\gamma(\ell) = \ell$ and let P be a point on ℓ. Then $P' = \gamma(P)$ is a point on $\ell = \overleftrightarrow{PP'}$ as is the midpoint M of $\overline{PP'}$. But M is also on c by part (c) above. Hence $M' = \gamma(M)$ is on c by definition of γ and M' is on ℓ by assumption. Therefore $\ell = c$. ■

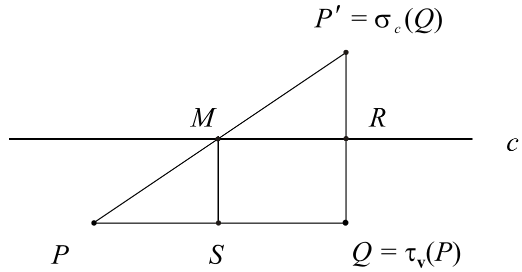


Figure 2.16.

A glide reflection can be expressed as a composition of three reflections in the following way:

Theorem 98 *A transformation γ is a glide reflection with axis c if and only if there exist distinct parallels a and b perpendicular to c such that $\gamma = \sigma_c \circ \sigma_b \circ \sigma_a$.*

Proof. Given a glide reflection γ with axis c, write $\gamma = \sigma_c \circ \tau$, where $\tau(c) = c$ and $\tau \neq \iota$. Let A be a point on c, then $A' = \tau(A)$ is also on c and is distinct from A. Let B be the midpoint of $\overline{AA'}$, and let a and b be the lines perpendicular to c at A and B, respectively. Then by Corollary 79, $\tau = \tau_{2\mathbf{AB}} = \sigma_b \circ \sigma_a \neq \iota$ and it follows that $\gamma = \sigma_c \circ \sigma_b \circ \sigma_a$. Conversely, given distinct parallel lines a and b, and a common perpendicular c, let $A = a \cap c$ and $B = b \cap c$. Then $\tau_{2\mathbf{AB}}(c) = c$ and $\tau_{2\mathbf{AB}} \neq \iota$ so that $\gamma = \sigma_c \circ \sigma_b \circ \sigma_a = \sigma_c \circ \tau_{2\mathbf{AB}}$ is a glide reflection with axis c. ■

A glide reflection can also be expressed as a reflection in some line ℓ followed by a halfturn with center off ℓ (or vice versa).

Theorem 99 *The following are equivalent:*

a. γ is a glide reflection with axis c and glide vector $\mathbf{v}$.

b. $\gamma = \sigma_c \circ \tau_{\mathbf{v}}$ and $\tau_{\mathbf{v}}(c) = c$.

c. There is a line $a \perp c$ and a point B on c and off a such that $\gamma = \varphi_B \circ \sigma_a$.

d. There is a line $b \perp c$ and a point A on c and off b such that $\gamma = \sigma_b \circ \varphi_A$.

e. $\gamma = \tau_{\mathbf{v}} \circ \sigma_c$ and $\tau_{\mathbf{v}}(c) = c$.

Proof. Statements (a) and (b) are equivalent by definition. To show the equivalence of (b), (c), (d), and (e), use Theorem 98 to choose parallels a and b perpendicular to c such that $\gamma = \sigma_c \circ \sigma_b \circ \sigma_a$. Then $\sigma_c \circ \sigma_b = \sigma_b \circ \sigma_c$ and $\sigma_c \circ \sigma_a = \sigma_a \circ \sigma_c$. Furthermore, $\tau_{\mathbf{v}} = \sigma_b \circ \sigma_a$ implies $\tau_{\mathbf{v}}(c) = c$. Let $A = a \cap c$ and $B = b \cap c$; then $\sigma_c \circ \tau_{\mathbf{v}} = \sigma_c \circ \sigma_b \circ \sigma_a = \varphi_B \circ \sigma_a$ so that (b) $\Rightarrow$ (c); $\varphi_B \circ \sigma_a = \sigma_c \circ \sigma_b \circ \sigma_a = \sigma_b \circ \sigma_c \circ \sigma_a = \sigma_b \circ \varphi_A$ so that (c) $\Rightarrow$ (d); and

$\sigma_b \circ \varphi_A = \sigma_b \circ \sigma_c \circ \sigma_a = \sigma_b \circ \sigma_a \circ \sigma_c = \tau_{\mathbf{v}} \circ \sigma_c$ so that (d) $\Rightarrow$ (e). Finally, $\tau_{\mathbf{v}} \circ \sigma_c = \sigma_b \circ \sigma_a \circ \sigma_c = \sigma_b \circ \sigma_c \circ \sigma_a = \sigma_c \circ \sigma_b \circ \sigma_a = \sigma_c \circ \tau_{\mathbf{v}}$ so that (e) $\Rightarrow$ (b). ■

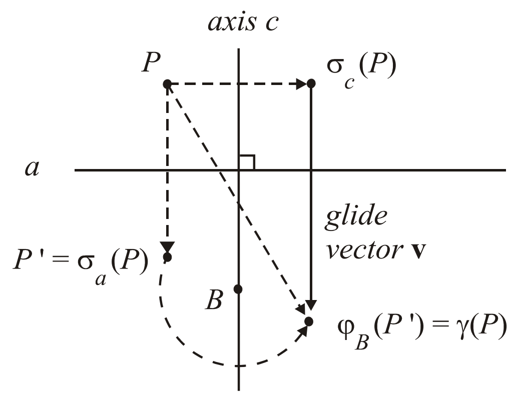


Figure 2.17. $\gamma = \varphi_B \circ \sigma_a$.

We can use Theorem 99 to construct the axis of a glide reflection. If $\gamma = \varphi_B \circ \sigma_a$ and B is off a, then γ is a glide reflection whose axis is the line through B perpendicular to a. Likewise, if $\gamma = \sigma_b \circ \varphi_A$ and A is off b, then γ is a glide reflection whose axis is the line through A perpendicular to b.

Theorem 100 *Let ℓ, m and n be distinct lines. Then $\gamma = \sigma_n \circ \sigma_m \circ \sigma_\ell$ is a glide reflection if and only if ℓ, m and n are neither concurrent nor mutually parallel.*

Proof. ($\Rightarrow$) We prove the contrapositive. Suppose distinct lines ℓ, m and n are concurrent or mutually parallel. Then $\gamma = \sigma_n \circ \sigma_m \circ \sigma_\ell$ is a reflection by Corollary 93, which is not a glide reflection (reflections have fixed points and glide reflections do not by Proposition 97, part b).

($\Leftarrow$) Assume that ℓ, m, and n are neither concurrent nor mutually parallel. Then lines ℓ and m are either intersecting or parallel, and we consider two cases:
<u>*Case 1:*</u> $\ell \cap m = P$. Since ℓ, m, and n are not concurrent, P lies off n. Let Q be the foot of the perpendicular from P to n, and let $q = \overleftrightarrow{PQ}$. By Theorem 81, there is a unique line p passing through P such that

$$\sigma_m \circ \sigma_\ell = \sigma_q \circ \sigma_p.$$

But $p \neq q$ since $\ell \neq m$, and Q is off p since Q is on q and $Q \neq P$. Thus there is a point Q off line p such that

$$\gamma = \sigma_n \circ \sigma_m \circ \sigma_\ell = \sigma_n \circ \sigma_q \circ \sigma_p = \varphi_Q \circ \sigma_p,$$

which is a glide reflection by Theorem 99 (c).
<u>*Case 2:*</u> $\ell \parallel m$. Then n is a transversal for ℓ and m and intersects m at some

point P. Consider the composition $\sigma_\ell \circ \sigma_m \circ \sigma_n$. By Case 1 above, there is a point Q off line p such that

$$\sigma_\ell \circ \sigma_m \circ \sigma_n = \varphi_Q \circ \sigma_p.$$

Hence,

$$\gamma = \sigma_n \circ \sigma_m \circ \sigma_\ell = (\sigma_\ell \circ \sigma_m \circ \sigma_n)^{-1} = \left(\varphi_Q \circ \sigma_p\right)^{-1} = \sigma_p \circ \varphi_Q,$$

which is a glide reflection by Theorem 99 (d). ■

In view of Theorem 99, the equations of a glide reflection are easily obtained.

Corollary 101 *Let γ be a glide reflection with axis $\ell : aX + bY + c = 0$ and glide vector $\mathbf{v} = \left[\begin{smallmatrix} d \\ e \end{smallmatrix}\right]$. Then $ad + be = 0$ and the equations of γ are given by*

$$\begin{array}{l} x' = x - \frac{2a}{a^2+b^2}(ax + by + c) + d \\ y' = y - \frac{2b}{a^2+b^2}(ax + by + c) + e \end{array}.$$

Proof. Using Theorem 99 (e), write $\gamma = \tau_{\mathbf{v}} \circ \sigma_\ell$. Then $\tau_{\mathbf{v}}$ with vector $\mathbf{v} = \left[\begin{smallmatrix} d \\ e \end{smallmatrix}\right]$ fixes the axis $\ell : aX + bY + c = 0$ if and only if ℓ is in the direction of $\mathbf{v}$ if and only if $ad + be = 0$. The equations of γ are the equations of the composition $\tau_{\mathbf{v}} \circ \sigma_\ell$. ■

Example 102 *Consider the line $m : 3X - 4Y + 1 = 0$ and the translation $\tau_{\mathbf{v}}$ where $\mathbf{v} = \left[\begin{smallmatrix} 4 \\ 3 \end{smallmatrix}\right]$. Since $ad + be = (3)(4) + (-4)(3) = 0$, the line m is in the direction of $\mathbf{v}$. Hence $\gamma = \sigma_m \circ \tau$ is the glide reflection with equations*

$$\begin{array}{l} x' = \frac{1}{25}(7x + 24y + 94) \\ y' = \frac{1}{25}(24x - 7y + 83). \end{array}$$

Let $P = \left[\begin{smallmatrix} 0 \\ 19 \end{smallmatrix}\right]$ and $P' = \left[\begin{smallmatrix} 22 \\ -2 \end{smallmatrix}\right] = \gamma\left(\left[\begin{smallmatrix} 0 \\ 19 \end{smallmatrix}\right]\right)$; then the midpoint $M = \left[\begin{smallmatrix} 11 \\ \frac{17}{2} \end{smallmatrix}\right]$ of $\overline{PP'}$ is on m.

Here are some useful facts about glide reflections.

Theorem 103 *Let γ be a glide reflection with axis c and glide vector $\mathbf{v}$.*

a. *γ^{-1} is a glide reflection with axis c and glide vector $-\mathbf{v}$.*

b. *If τ is any translation such that $\tau(c) = c$, then $\tau \circ \gamma = \gamma \circ \tau$.*

c. *$\gamma^2 = \tau_{2\mathbf{v}} \neq \iota$.*

Proof. The proof of statement (a) is left to the reader. To prove (b), assume that τ is a non-identity translation *such that* $\tau(c) = c$. Let A be a point on c; then $B = \tau(A) \neq A$ is a point on c and $\tau = \tau_{\mathbf{AB}}$. Thus $\sigma_c \circ \tau$ is a glide reflection and

$$\sigma_c \circ \tau = \tau \circ \sigma_c \tag{2.7}$$

by Theorem 99 (a). On the other hand, $\gamma = \sigma_c \circ \tau_{\mathbf{v}}$ by definition, and any two translations commute by Proposition 45, part (2). These facts together with equation (2.7) give

$$\gamma \circ \tau = \sigma_c \circ \tau_{\mathbf{v}} \circ \tau = \sigma_c \circ \tau \circ \tau_{\mathbf{v}} = \tau \circ \sigma_c \circ \tau_{\mathbf{v}} = \tau \circ \gamma.$$

To prove part (c), write $\gamma = \sigma_c \circ \tau_{\mathbf{v}}$ and apply Theorem 99 (a) to obtain

$$\gamma^2 = (\sigma_c \circ \tau_{\mathbf{v}})^2 = \sigma_c \circ \tau_{\mathbf{v}} \circ \sigma_c \circ \tau_{\mathbf{v}} = \sigma_c \circ \sigma_c \circ \tau_{\mathbf{v}} \circ \tau_{\mathbf{v}} = (\tau_{\mathbf{v}})^2 = \tau_{2\mathbf{v}};$$

thus γ^2 is a non-identity translation by vector $2\mathbf{v}$. ■

We conclude this section with an important construction. Let a, b and c be lines containing the sides of a non-degenerate triangle with vertices A, B, and C labeled so that A is opposite a, B is opposite b, and C is opposite c (see Figure 2.18). Let P and Q be the feet of the altitudes from C to c and from A to a, respectively. If $\angle B$ is not a right angle, $P \neq Q$ and the axis of the glide reflection $\gamma = \sigma_c \circ \sigma_b \circ \sigma_a$ is $\overleftrightarrow{PQ}$. To see this, let $m = \overleftrightarrow{CP}$; by Theorem 81 there is a unique line ℓ pasing through C such that $\sigma_b \circ \sigma_a = \sigma_m \circ \sigma_\ell$. Thus $\gamma = \sigma_c \circ \sigma_b \circ \sigma_a = \sigma_c \circ \sigma_m \circ \sigma_\ell = \varphi_P \circ \sigma_\ell$, and the line through P perpendicular to ℓ is the axis by Theorem 99 (c). Now repeat this argument at the vertex A: Let $s = \overleftrightarrow{AQ}$; by Theorem 81 there is a unique line t passing through A such that $\sigma_c \circ \sigma_b = \sigma_t \circ \sigma_s$. Thus $\gamma = \sigma_c \circ \sigma_b \circ \sigma_a = \sigma_t \circ \sigma_s \circ \sigma_a = \sigma_t \circ \varphi_Q$, and the line through Q perpendicular to t is the axis by Theorem 99 (d).

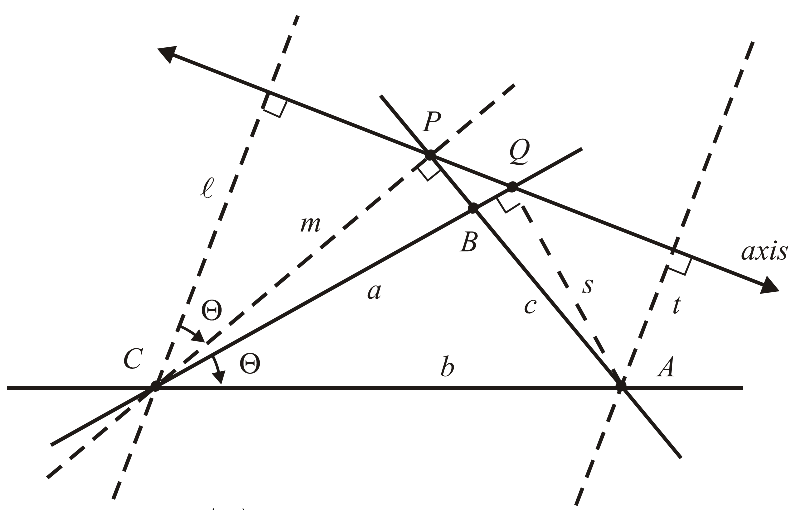


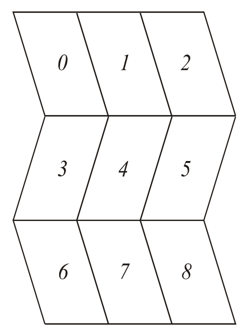
Figure 2.18. $\overleftrightarrow{PQ}$ is the axis of glide reflection $\gamma = \sigma_c \circ \sigma_b \circ \sigma_a$

In summary, we have proved:

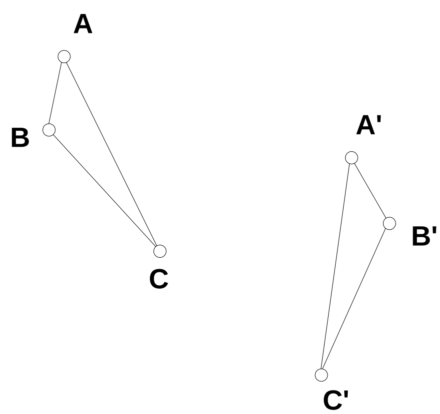
Theorem 104 *Let a, b and c be lines containing the sides of a non-degenerate triangle $\triangle ABC$ with vertices labeled so that A, B, and C are opposite a, b, and c, respectively. If $\angle B$ is not a right angle, and P and Q are the feet of* *the altitudes from C to c and from A to a, then the axis of the glide reflection* $\gamma = \sigma_c \circ \sigma_b \circ \sigma_a$ *is* $\overleftrightarrow{PQ}$.

Exercises

1. The parallelogram "0" in the figure below is mapped to each of the other eight parallelograms by a reflection, a translation, a glide reflection or a halfturn. Indicate which of these apply in each case (more than one may apply in some cases).



2. A glide reflection γ maps $\triangle ABC$ onto $\triangle A'B'C'$ in the diagram below. Use a MIRA to construct the axis and glide vector of γ.



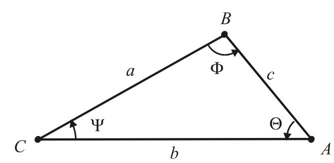
3. Consider a non-degenerate triangle $\triangle ABC$ with sides a, b and c opposite vertices A, B, and C, respectively. Assume that $\triangle ABC$ is not a right triangle, and let P, Q, and R be the feet of the respective altitudes from C to c, from A to a, and from B to b.

(a) Use a MIRA and Theorem 104 to construct the axis and glide vector of $\gamma' = \sigma_c \circ \sigma_a \circ \sigma_b$.

(b) Use a MIRA and Theorem 104 to construct the axis and glide vector of $\gamma'' = \sigma_b \circ \sigma_c \circ \sigma_a$.

(c) Use a MIRA and Theorem 104 to construct the axis and glide vector of $\gamma''' = \sigma_a \circ \sigma_b \circ \sigma_c$.

4. Consider a non-degenerate $\triangle ABC$ with sides a, b and c opposite vertices A, B, and C, respectively, let $\gamma = \sigma_a \circ \sigma_b \circ \sigma_c$, and let Θ, Φ, and Ψ be the respective measures of the interior angles of $\triangle ABC$ as indicated in the diagram below. Show that $\rho_{C,2\Psi} \circ \rho_{B,2\Phi} \circ \rho_{A,2\Theta} = \gamma^2$ but $\rho_{A,2\Theta} \circ \rho_{B,2\Phi} \circ \rho_{C,2\Psi} = \iota$.

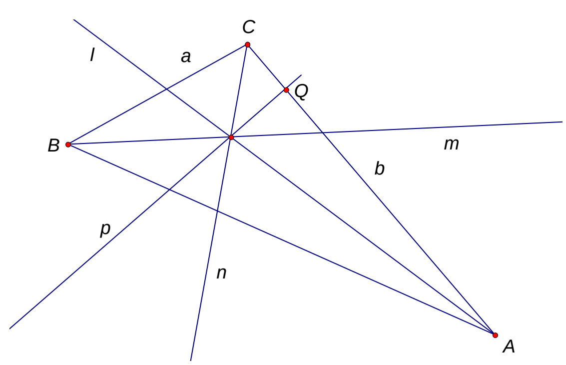


5. Let $\gamma = \sigma_c \circ \sigma_b \circ \sigma_a$ be a glide reflection such that exactly two of a, b, and c are parallel. Construct the axis and glide vector of γ in each case.

6. Prove the converse of Proposition 94: Given lines ℓ, m and n, and a line p such that $\sigma_p = \sigma_n \circ \sigma_m \circ \sigma_\ell$, prove that ℓ, m and n are either concurrent or mutually parallel.

7. If lines ℓ, m and n are either concurrent or mutually parallel, prove that $\sigma_n \circ \sigma_m \circ \sigma_\ell = \sigma_\ell \circ \sigma_m \circ \sigma_n$.

8. Given a non-degenerate triangle $\triangle ABC$, let b be the side opposite $\angle B$ and let ℓ, m and n be the respective angle bisectors of $\angle A$, $\angle B$, and $\angle C$. Then ℓ, m and n are concurrent by Exercise 20, Section 1.1, and there is a unique line p such that $\sigma_n \circ \sigma_m \circ \sigma_\ell = \sigma_p$ by Corollary 93 (a). Prove that $p \perp b$.



9. Let c be the line with equation $X - 2Y + 3 = 0$; let $P = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ and $Q = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$.
   a. Write the equations for the composite transformation $\gamma = \tau_{\mathbf{PQ}} \circ \sigma_c$.

   b. Find the image of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ -2 \end{bmatrix}$ under γ.

   c. Prove that $\gamma = \sigma_c \circ \tau_{\mathbf{PQ}}$ is a glide reflection.

10. Let γ be a glide reflection with axis c and glide vector $\mathbf{v}$. If $\gamma^2 = 2\mathbf{w}$, show that $\mathbf{w} = \mathbf{v}$.

11. Let γ be a glide reflection with axis c and glide vector $\mathbf{v}$. Given any point M on c, construct a point P off c such that M is the midpoint of P and $\gamma(P)$.

12. Given a translation τ with vector $\mathbf{v}$, find the glide vector and axis of a glide-reflection γ such that $\gamma^2 = \tau$.

13. Prove Theorem 103 (a): If γ is a glide reflection with axis c and glide vector $\mathbf{v}$, then $\gamma^{-1} = \sigma_c \circ \tau_{\mathbf{v}}^{-1}$ is a glide reflection with axis c and glide vector $-\mathbf{v}$.

14. Use Theorem 100 to prove that the perpendicular bisectors of the sides in any non-degenerate triangle, are concurrent (cf. Scetion 2, Exercise 19).

Chapter 3

Classification of Isometries

3.1 The Fundamental Theorem and Congruence

In this section we characterize isometries as products of three or fewer reflections and express the notion of congruence in terms of isometries. But first we need some additional facts about fixed points.

Theorem 105 *An isometry that fixes two distinct points is either a reflection or the identity.*

Proof. We assume that $\alpha \neq \iota$ and show that α is a reflection. Let P and Q be distinct points, let $m = \overleftrightarrow{PQ}$, and let α be an isometry that fixes P and Q. Let R be any point such that $R' = \alpha(R) \neq R$. Then P, Q, and R are non-collinear by Theorem 71 and $PR = PR'$ and $QR = QR'$ since α is an isometry. Since P and Q are equidistant from R and R', line m is the perpendicular bisector of $\overline{RR'}$. Hence

$$\alpha(R) = R' = \sigma_m(R), \quad \alpha(P) = P = \sigma_m(P), \quad \text{and} \quad \alpha(Q) = Q = \sigma_m(Q)$$

so that $\alpha = \sigma_m$ by Theorem 73. ∎

Theorem 106 *An isometry that fixes exactly one point is a non-identity rotation.*

Proof. Let α be an isometry with exactly one fixed point C, let P be a point distinct from C, and let $P' = \alpha(P)$. Since α is an isometry, $CP = CP'$ so that C is on the perpendicular bisector m of $\overline{PP'}$ (see Figure 3.1). But $\sigma_m(P') = P$ by definition of a reflection, and it follows that

$$(\sigma_m \circ \alpha)(C) = \sigma_m(\alpha(C)) = \sigma_m(C) = C$$

and

$$(\sigma_m \circ \alpha)(P) = \sigma_m(\alpha(P)) = \sigma_m(P') = P.$$

Since $\sigma_m \circ \alpha$ is an isometry and fixes the distinct points C and P, Theorem 105 tells us that either $\sigma_m \circ \alpha = \iota$ or $\sigma_m \circ \alpha = \sigma_\ell$ where $\ell = \overleftrightarrow{CP}$. However, if $\sigma_m \circ \alpha = \iota$, then $\sigma_m = \alpha$, which is impossible since σ_m has infinitely many fixed points. Therefore

$$\sigma_m \circ \alpha = \sigma_\ell$$

so that

$$\alpha = \sigma_m \circ \sigma_\ell.$$

But $\ell \neq m$ since $P \neq P'$; hence α is a non-identity rotation by Theorem 76.

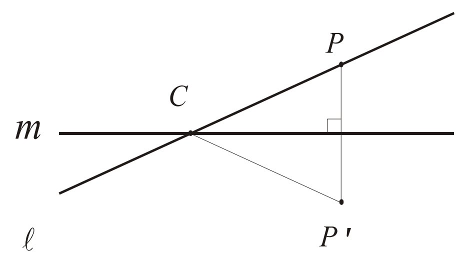


Figure 3.1.

■

Thinking of the identity as a trivial rotation, we summarize Theorems 105 and 106 as:

Theorem 107 *An isometry with a fixed point is either a reflection, or a rotation. An isometry with exactly one fixed point is a non-identity rotation.*

We have established the facts we need to prove our first major result, which characterizes an isometry as a product of three or fewer reflections:

Theorem 108 ***(The Fundamental Theorem of Transformational Plane Geometry)*** *A transformation α is an isometry if and only if α can be expressed as a composition of three or fewer reflections.*

Proof. By Exercise 1.1.3, the composition of isometries is an isometry. Since reflections are isometries, a composition of reflections is an isometry. Conversely, if $\alpha = \iota$, choose any line ℓ and write $\iota = \sigma_\ell \circ \sigma_\ell$. If $\alpha \neq \iota$, choose a point P such that $P' = \alpha(P) \neq P$ and let m be the perpendicular bisector of $\overline{PP'}$. Then

$$(\sigma_m \circ \alpha)(P) = \sigma_m(\alpha(P)) = \sigma_m(P') = P,$$

i.e., $\beta = \sigma_m \circ \alpha$ fixes the point P. By Theorem 107, β can be expressed as a composition of two or fewer reflections, which means that

$$\alpha = \sigma_m \circ \beta$$

can be expressed as a composition of three or fewer reflections. ■

Our next theorem expresses the notion of congruence in terms of isometries. Since an isometry is completely determined by its action on three distinct non-collinear points, by the Three Points Theorem, the procedure in the proof of Theorem 109 gives us a constructive way to express a given isometry as a product of three or fewer reflections.

Theorem 109 $\triangle PQR \cong \triangle ABC$ *if and only if there is a unique isometry* α *such that* $\alpha(P) = A$, $\alpha(Q) = B$ *and* $\alpha(R) = C$.

Proof. Given $\triangle PQR \cong \triangle ABC$, Theorem 73 tells us that *if* an isometry α with the required properties exists, it is unique. Our task, therefore, is to show that such an isometry α does indeed exist; we'll do this by constructing α as an explicit product of three isometries $\alpha_3 \circ \alpha_2 \circ \alpha_1$ each of which is either the identity or a reflection. Begin by noting that

$$AB = PQ, \quad AC = PR, \quad \text{and} \quad BC = QR, \tag{3.1}$$

by $CPCTC$ (see Figure 3.4).

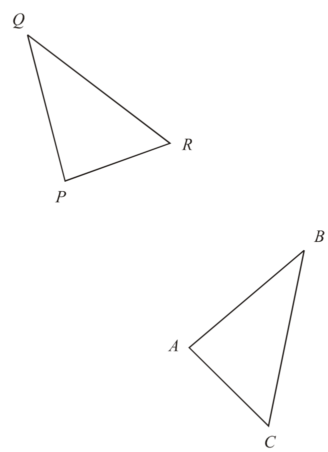


Figure 3.4.

<u>*The isometry* α_1</u>: If $P = A$, let $\alpha_1 = \iota$. Otherwise, let $\alpha_1 = \sigma_\ell$ where ℓ is the perpendicular bisector of $\overline{PA}$. In either case,

$$\alpha_1(P) = A.$$

Let

$$Q_1 = \alpha_1(Q) \quad \text{and} \quad R_1 = \alpha_1(R)$$

and note that

$$PQ = AQ_1, \quad PR = AR_1, \quad \text{and} \quad QR = Q_1R_1 \tag{3.2}$$

(see Figure 3.5).

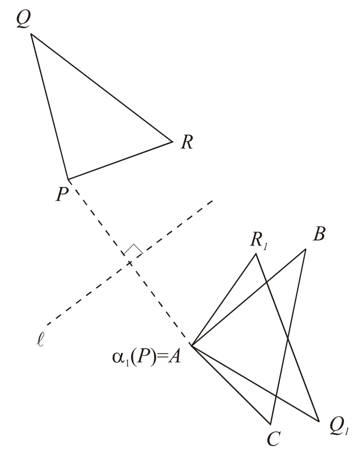


Figure 3.5.

<u>*The isometry* α_2</u>: If $Q_1 = B$, let $\alpha_2 = \iota$. Otherwise, let $\alpha_2 = \sigma_m$ where m is the perpendicular bisector of $\overline{Q_1B}$. By (5.4) and (3.2) we have

$$AB = PQ = AQ_1$$

so the point A is equidistant from points B and Q_1. Therefore A lies on m and in either case we have

$$\alpha_2(A) = A \quad \text{and} \quad \alpha_2(Q_1) = B.$$

Let

$$R_2 = \alpha_2(R_1)$$

and note that

$$AR_1 = AR_2 \quad \text{and} \quad Q_1R_1 = BR_2 \tag{3.3}$$

(see Figure 3.6).

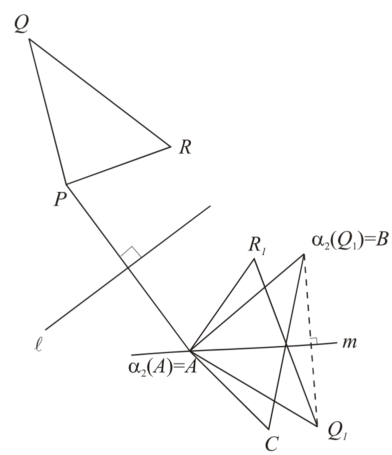


Figure 3.6.

The isometry α_3: If $R_2 = C$, let $\alpha_3 = \iota$. Otherwise, let $\alpha_3 = \sigma_n$ where n is the perpendicular bisector of $\overline{R_2C}$. By (5.4), (3.2), and (3.3) we have

$$AC = PR = AR_1 = AR_2$$

so the point A is equidistant from points C and R_2 and lies on n. On the other hand, (5.4), (3.2), and (3.3) also give

$$BC = QR = Q_1R_1 = BR_2$$

so the point B is equidistant from points C and R_2 and also lies on n. In either case we have

$$\alpha_3(A) = A, \quad \alpha_3(B) = B, \quad \text{and} \quad \alpha_3(R_2) = C.$$

Let $\alpha = \alpha_3 \circ \alpha_2 \circ \alpha_1$ and observe that

$$\alpha(P) = \alpha_3(\alpha_2(\alpha_1(P))) = \alpha_3(\alpha_2(A)) = \alpha_3(A) = A$$

$$\alpha(Q) = \alpha_3(\alpha_2(\alpha_1(Q))) = \alpha_3(\alpha_2(Q_1)) = \alpha_3(B) = B$$

$$\alpha(R) = \alpha_3(\alpha_2(\alpha_1(R)) = \alpha_3(\alpha_2(R_1)) = \alpha_3(R_2) = C.$$

Therefore α is indeed a product of three or fewer reflections. The converse follows from the fact that isometries preserve length and angle (see Proposition 26, part 4). ■

The following remarkable characterization of congruent triangles is an immediate consequence of Theorems 108 and 109:

Corollary 110 *$\triangle PQR \cong \triangle ABC$ if and only if $\triangle ABC$ is the image of $\triangle PQR$ under a composition of three or fewer reflections.*

Corollary 111 *Two segments or two angles are congruent if and only if there exists an isometry mapping one onto the other.*

Proof. Two congruent segments or angles are contained in a pair of congruent triangles so such an isometry exists by Theorem 109. Since isometries preserve length and angle the converse also follows. ■

Now we can define a general notion of congruence for arbitrary plane figures.

Definition 112 *Two plane figures s_1 and s_2 are congruent if and only if there is an isometry α such that $s_2 = \alpha(s_1)$.*

Exercises

1. Let $P = \begin{bmatrix}0\\0\end{bmatrix}$; $Q = \begin{bmatrix}5\\0\end{bmatrix}$; $R = \begin{bmatrix}0\\10\end{bmatrix}$; $A = \begin{bmatrix}4\\2\end{bmatrix}$; $B = \begin{bmatrix}1\\-2\end{bmatrix}$; $C = \begin{bmatrix}12\\-4\end{bmatrix}$. Given that $\triangle PQR \cong \triangle ABC$, apply the algorithm in the proof of Theorem 109 to find three or fewer lines such that the image of $\triangle PQR$ under reflections in these lines is $\triangle ABC$.

2. Let $P = \begin{bmatrix}6\\7\end{bmatrix}$; $Q = \begin{bmatrix}3\\14\end{bmatrix}$; $R = \begin{bmatrix}8\\15\end{bmatrix}$. In each of the following, $\triangle PQR \cong \triangle ABC$. Apply the algorithm in the proof of Theorem 109 to find three or fewer lines such that the image of $\triangle PQR$ under reflections in these lines is $\triangle ABC$.

	A	B	C
a.	$\begin{bmatrix}12\\1\end{bmatrix}$	$\begin{bmatrix}9\\8\end{bmatrix}$	$\begin{bmatrix}14\\9\end{bmatrix}$
b.	$\begin{bmatrix}-1\\10\end{bmatrix}$	$\begin{bmatrix}-8\\7\end{bmatrix}$	$\begin{bmatrix}-9\\12\end{bmatrix}$
c.	$\begin{bmatrix}4\\-9\end{bmatrix}$	$\begin{bmatrix}7\\-16\end{bmatrix}$	$\begin{bmatrix}2\\-17\end{bmatrix}$
d.	$\begin{bmatrix}-4\\-15\end{bmatrix}$	$\begin{bmatrix}-11\\-12\end{bmatrix}$	$\begin{bmatrix}-12\\-17\end{bmatrix}$
e.	$\begin{bmatrix}-5\\-4\end{bmatrix}$	$\begin{bmatrix}-12\\-1\end{bmatrix}$	$\begin{bmatrix}-13\\-6\end{bmatrix}$

3.2 Classification of Isometries and Parity

In this section we prove our second major result–the classification of isometries. We also define the notion of "parity", which is a property of an isometry that helps us identify its type, and relate it to the notion of an "orientation" of the plane.

Theorem 113 ***(Classification of Isometries)*** *An isometry is exactly one of the following types: A reflection, a glide reflection, a rotation, or a non-identity translation.*

Proof. Every isometry is a composition of three or fewer reflections by Theorem 108. A composition of two reflections in distinct parallel lines is a translation by Theorem 80. A composition of two reflections in distinct intersecting lines is a rotation by Theorem 76. The identity is both a trivial translation (with vector $\mathbf{v} = \mathbf{0}$) and a trivial rotation (of angle $\Theta \in 0^\circ$). Non-identity translations are fixed point free, but fix every line in the direction of translation by Theorem 50. By definition, a non-identity rotation fixes exactly one point–its center–and a reflection fixes every point on its axis. Hence translations, non-identity rotations, and reflections form mutually exclusive familes of isometries. A composition of three reflections in concurrent or mutually parallel lines is a reflection by Corollary 93. A composition of three reflections in non-concurrent and non-mutually parallel lines is a glide reflection by Theorem 100. A glide reflection has no fixed points by Proposition 97, part b; consequently, a glide reflection is neither a rotation nor a reflection. A glide reflection fixes exactly one line–its axis–by Proposition 97, part d; consequently, a glide reflection is not a translation. ■

Figure 3.2 pictures the various configurations of three of fewer lines that may appear. Successively reflecting in the lines in these configurations gives various representations of the isometries as compositions of reflections. The first, second and third configurations represent reflections while the fourth and fifth configurations represent glide reflections. The sixth represents a non-identity rotation and the seventh represents a non-identity translation. Note that compositions of the form $\sigma_a \circ \sigma_b \circ \sigma_a$ also represent reflections since the axes of reflection are parallel when $a \| b$ and concurrent when $a \nparallel b$.

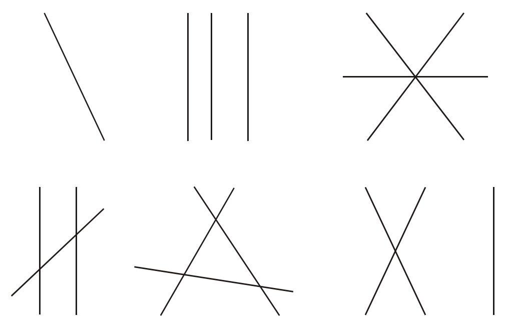


Figure 3.2. Configurations of lines representing reflections, translations, rotations, and glide reflections.

The Classification of Isometries Theorem is an example of mathematics par excellence–one of the crown jewels of this course. Indeed, the ultimate goal of any mathematical endeavor is to find and classify the objects studied. This is typically a profoundly difficult problem and rarely solved. Thus a complete solution of a classification problem calls for great celebration! And the Classification of Isometries Theorem is no exception.

"Parity" is a property of an isometry that helps us identify its type. Our next theorem will allow us to define the notion of parity precisely. But first we prove a lemma that will be useful in the proof of the theorem.

Lemma 114 *Let a and b be lines and let P be a point. There exist lines a' and b' with a' passing through P such that*

$$\sigma_{b'} \circ \sigma_{a'} = \sigma_b \circ \sigma_a.$$

Proof. If a and b intersect at P, there is nothing to prove. If a and b intersect at $Q \neq P$, let $a' = \overleftrightarrow{PQ}$. If $a \| b$, let a' be the line through P parallel to a. In either case, there is a unique line b' such that

$$\sigma_{b'} \circ \sigma_{a'} = \sigma_b \circ \sigma_a,$$

by Theorems 81 and 84. ■

Theorem 115 *A composition of four reflections reduces to a composition of two reflections, i.e., given lines p, q, r, and s, there exist lines ℓ and m such that*

$$\sigma_s \circ \sigma_r \circ \sigma_q \circ \sigma_p = \sigma_m \circ \sigma_\ell.$$

Proof. Choose a point P on line p and consider the composition $\sigma_r \circ \sigma_q$. By Lemma 114, there exist lines q' and r' with q' passing through P such that

$$\sigma_{r'} \circ \sigma_{q'} = \sigma_r \circ \sigma_q.$$

Next, consider the composition $\sigma_s \circ \sigma_{r'}$. By Lemma 114, there exist lines r'' and m' with r'' passing through P such that

$$\sigma_m \circ \sigma_{r''} = \sigma_s \circ \sigma_{r'}.$$

Now p, q', and r'' are concurrent at P. By Corollary 93, there is a unique line ℓ such that

$$\sigma_{r''} \circ \sigma_{q'} \circ \sigma_p = \sigma_\ell.$$

Therefore

$$\sigma_s \circ \sigma_r \circ \sigma_q \circ \sigma_p = \sigma_s \circ \sigma_{r'} \circ \sigma_{q'} \circ \sigma_p = \sigma_m \circ \sigma_{r''} \circ \sigma_{q'} \circ \sigma_p = \sigma_m \circ \sigma_\ell.$$

■

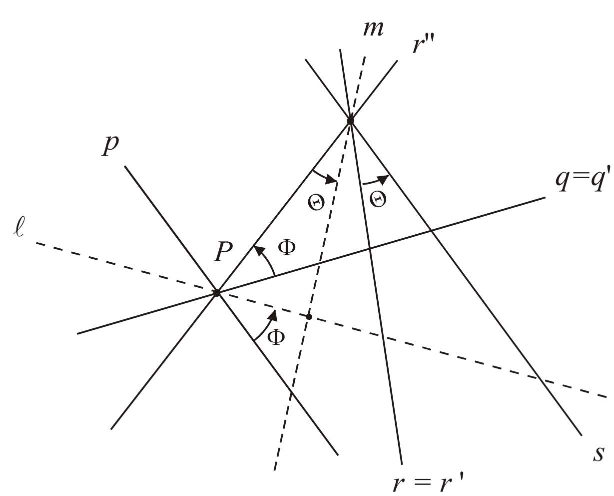


Figure 3.3. Here we chose $P = p \cap q$.

Repeated applications of Theorem 115 reduces a product of an odd number of reflections to three reflections or one, but never to a product of two. Likewise, a product of an even number of reflections reduces to two reflections or the identity, but never to a product of three or one. Since every isometry is a product of three or fewer reflections by the Fundamental Theorem, each isometry falls into one of two mutually exclusive families: (1) Those that can be represented as a product of an even number of reflections and (2) those that can be represented as a product of an odd number of reflections.

Definition 116 *An isometry is* <u>*even*</u> *if and only if it factors as a product of an even number of reflections; otherwise it is* <u>*odd*</u>*.*

Theorem 117 *An odd isometry is either a reflection or a glide reflection. An even isometry is either a translation or a rotation (the identity is a trivial rotation).*

Proof. Let α be an isometry. If α is odd and has a fixed point, it is a single reflection by Theorem 105. If α is odd and fixed point free, it is a glide reflection by Proposition 97 and Theorem 100. If α is even with exactly one fixed point, it is a rotation by Theorem 76. If α is even with more than one fixed point, it is the identity by Theorem 105. If α is even and fixed point free, it is a non-identity translation by Theorem 80. ■

Theorem 118 *Every even involutory isometry is a halfturn; every odd involutory isometry is a reflection.*

Proof. Reflections are involutory isometries, so consider an involutory isometry α that is not a reflection. We claim that α is a halfturn. Since $\alpha \neq \iota$, there exist distinct points P and P' such that $\alpha(P) = P'$. By applying α to both sides we obtain

$$\alpha^2(P) = \alpha(P').$$

Since α is an involution, $\alpha^2(P) = P$. Hence $\alpha(P') = P$ and α interchanges the points P and P'. Let M be the midpoint of P and P', then $P - M - P'$ and $PM = MP'$. Let $M' = \alpha(M)$; then $P - M' - P'$ and $PM' = M'P'$ since α is an isometry, a collineation, and preserves betweenness. It follows M is a fixed point; hence α is either a non-identity rotation about M or a reflection in some line containing M by Theorem 107. Since α is not a reflection, it is an involutory rotation about M, i.e., a halfturns by Corollary 67. Therefore $\alpha = \varphi_M$, and it follows that an involutory isometry is either a reflection or a halfturn. The conclusion follows from the fact that reflections are odd and halfturns are even. ■

The notion of an "orientation" of the plane is closely related to parity. As we shall see, even isometries preserve orientation and odd isometries reverse it. Recall that positive angle measure is defined counter clockwise. But equally well, we could have defined positive angle measure clockwise. Such a choice is called an "orientation" of the plane.

Recall that a basis for $\mathbb{R}^2$ is a pair of non-zero, non-parallel vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$. If $\mathbf{w}$ is an arbitrary vector, the system of linear equations $c\mathbf{v}_1 + d\mathbf{v}_2 = \mathbf{w}$ has a unique solution, and either

$$\det[\mathbf{v}_1 \mid \mathbf{v}_2] > 0 \text{ or } \det[\mathbf{v}_1 \mid \mathbf{v}_2] < 0.$$

Thus the sign of $\det[\mathbf{v}_1 \mid \mathbf{v}_2]$ places a given *ordered* basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ in one of two classes–those with positive determinant or those with negative determinant.

Definition 119 *An* *orientation* *of* $\mathbb{R}^2$ *is a choice of ordered basis* $\{\mathbf{v}_1, \mathbf{v}_2\}$. *An orientation* $\{\mathbf{v}_1, \mathbf{v}_2\}$ *is* *positive* *(respectively,* *negative*) *if and only if* $\det[\mathbf{v}_1 \mid \mathbf{v}_2] > 0$ *(respectively,* $\det[\mathbf{v}_1 \mid \mathbf{v}_2] < 0$*).*

When the orientation is positive, counter clockwise angle measure is positive; when the orientation is negative, clockwise angle measure is positive.

Example 120 *The standard ordered basis* $\{\mathbf{E}_1, \mathbf{E}_2\}$ *is a positive orientation of* $\mathbb{R}^2$, *and the (counter clockwise) angle measured from* $\mathbf{E}_1$ *to* $\mathbf{E}_2$ *is positive. However, the ordered basis* $\{\mathbf{E}_2, \mathbf{E}_1\}$ *is a negative orientation of* $\mathbb{R}^2$, *and the angle measured from* $\mathbf{E}_1$ *to* $\mathbf{E}_2$ *is negative.*

Theorem 121 *Given an orientation* $\{\mathbf{OA}, \mathbf{OB}\}$, *let* $\Theta = m\angle AOB$. *Then* $\{\mathbf{OA}, \mathbf{OB}\}$ *is positive if and only if some element of* Θ° *lies in* $(0, 180)$.

Proof. Let $E_1 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, let $\Theta_1 = m\angle E_1OA$, and let $\Theta_2 = m\angle E_1OB$. Then $\Theta = \Theta_2 - \Theta_1$, $\mathbf{v}_1 := \mathbf{OA} = \left[\begin{smallmatrix}\|\mathbf{v}_1\| \cos\Theta_1\\ \|\mathbf{v}_1\| \sin\Theta_1\end{smallmatrix}\right]$, and $\mathbf{v}_2 := \mathbf{OB} = \left[\begin{smallmatrix}\|\mathbf{v}_2\| \cos\Theta_2\\ \|\mathbf{v}_2\| \sin\Theta_2\end{smallmatrix}\right]$ so that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is positive if and only if

$$\begin{aligned}
0 &< \det[\mathbf{v}_1|\mathbf{v}_2] = (\|\mathbf{v}_1\| \cos\Theta_1)(\|\mathbf{v}_2\| \sin\Theta_2) - (\|\mathbf{v}_1\| \sin\Theta_1)(\|\mathbf{v}_2\| \cos\Theta_2) \\
&= \|\mathbf{v}_1\| \|\mathbf{v}_2\| (\cos\Theta_1 \sin\Theta_2 - \sin\Theta_1 \cos\Theta_2) \\
&= \|\mathbf{v}_1\| \|\mathbf{v}_2\| \sin(\Theta_2 - \Theta_1) = \|\mathbf{v}_1\| \|\mathbf{v}_2\| \sin\Theta
\end{aligned}$$

if and only if some element of Θ° lies in $(0, 180)$. ■

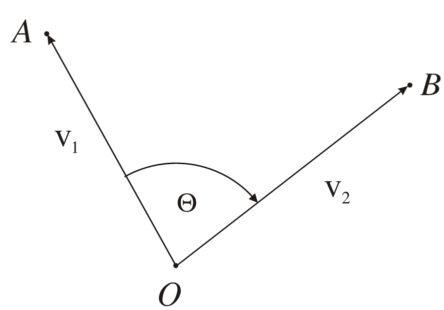


Figure 3.11. The orientation $\{\mathbf{v}_1, \mathbf{v}_2\}$ is negative.

To apply an isometry α to a vector $\mathbf{PQ}$, let $P' = \alpha(P)$ and $Q' = \alpha(Q)$ and define $\alpha(\mathbf{PQ}) = \mathbf{P'Q'}$.

Definition 122 *A transformation α is orientation-preserving if the orientations $\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\alpha(\mathbf{v}_1), \alpha(\mathbf{v}_2)\}$ are both positive or both negative; otherwise α is orientation-reversing. An orientation-preserving isometry is direct; an orientation-reversing isometry is indirect.*

Now given an orientation $\{\mathbf{v}_1, \mathbf{v}_2\}$, let Θ measure of the angle from $\mathbf{v}_1$ to $\mathbf{v}_2$ as defined above, and let Θ' measure the angle from $\mathbf{v}_1' = \alpha(\mathbf{v}_1)$ to $\mathbf{v}_2' = \alpha(\mathbf{v}_2)$. If α is orientation-preserving, $\Theta' = \Theta$; if α is orientation-reversing, $\Theta' = -\Theta$. Recall that the determinant of a matrix reverses sign when two of its columns (or rows) are interchanged. Thus if α is orientation-preserving, $\det[\mathbf{v}_1|\mathbf{v}_2] = \det[\mathbf{v}_1'|\mathbf{v}_2']$; otherwise $\det[\mathbf{v}_1|\mathbf{v}_2] = -\det[\mathbf{v}_1'|\mathbf{v}_2']$. In the later case, one can recover the original sign by interchanging the columns of $[\mathbf{v}_1'|\mathbf{v}_2']$, which is equivalent to replacing the original ordered basis with the ordered basis $\{\mathbf{v}_2', \mathbf{v}_1'\}$. In this way, an orientation-reversing isometry reverses the order of a given ordered basis.

The following fact is intuitively obvious, but the proof is somewhat tedious. We leave the proof to the reader as a series of exercises at the end of this section (see Exercises 11, 12, 13 and 14).

Proposition 123 *Direct isometries are even; indirect isometries are odd.*

Exercises

In problems 1-6, the respective equations of lines p, q, r and s are given. In each case, find lines ℓ and m such that $\sigma_s \circ \sigma_r \circ \sigma_q \circ \sigma_p = \sigma_m \circ \sigma_\ell$ and identify the isometry $\sigma_m \circ \sigma_\ell$ as a translation, a rotation or the identity.

1. $Y = X$, $Y = X + 1$, $Y = X + 2$ and $Y = X + 3$.

2. $Y = X$, $Y = X + 1$, $Y = X + 2$ and $Y = -X$.

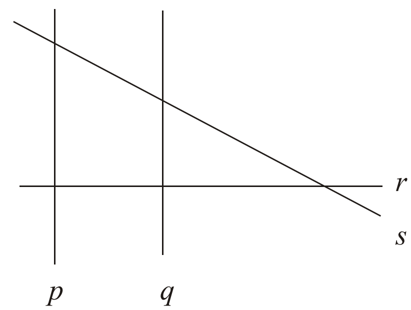
3. $X = 0$, $Y = 0$, $Y = X$ and $Y = -X$.

4. $X = 0$, $Y = 0$, $Y = 2$, and $X = 2$.

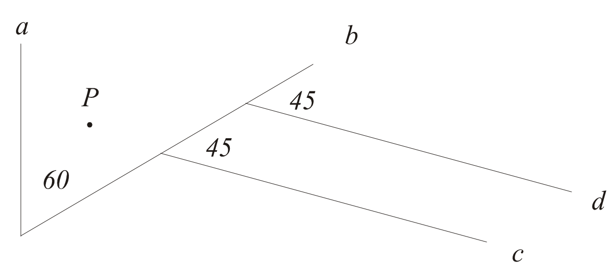
5. $X = 0$, $Y = 0$, $X = 1$ and $Y = X + 2$.

6. $X = 0$, $Y = 0$, $Y = X + 1$ and $Y = -X + 2$.

7. In the diagram below, use a MIRA to construct lines ℓ and m such that $\sigma_s \circ \sigma_r \circ \sigma_q \circ \sigma_p = \sigma_m \circ \sigma_\ell$.



8. Consider the following diagram:



   a. Use a MIRA to construct the point Q such that $\rho_{Q,\Theta} = \sigma_b \circ \varphi_P \circ \sigma_a$ and find the rotation angle Θ.

   b. Use a MIRA to construct the point R such that $\rho_{R,\Phi} = \sigma_d \circ \sigma_c \circ \sigma_b \circ \sigma_a$ and find the rotation angle Φ.

9. Prove that an even isometry with two distinct fixed points is the identity.

10. Identify the isometric dilatations and prove your answer.

11. Prove that translations are direct isometries.

12. Prove that the reflection in a line through the origin reverses orientation.

13. Use Exercises 11 and 12 together with Proposition 126 to prove that reflections are indirect isometries.

14. Use Exercise 13 to prove Proposition 123: Direct isometries are even; indirect isometries are odd.

3.3 The Geometry of Conjugation

In this section we give geometrical meaning to "algebraic conjugation", which plays an important role in our study of symmetry in the next chapter. You've seen conjugation before. For example, when rationalizing the denominator of $\frac{1}{3+\sqrt{2}}$ we *conjugate by* $3-\sqrt{2}$, i.e.,

$$\frac{1}{3+\sqrt{2}} = \left(\frac{1}{3+\sqrt{2}}\right)\left(\frac{3-\sqrt{2}}{3-\sqrt{2}}\right) = \frac{3-\sqrt{2}}{7}.$$

Since *multiplication of real numbers is commutative,* we can express this conjugation in the following seemingly awkward way:

$$\left(\frac{1}{3+\sqrt{2}}\right)\left(\frac{3-\sqrt{2}}{3-\sqrt{2}}\right) = \left(3-\sqrt{2}\right)\left(\frac{1}{3+\sqrt{2}}\right)\left(3-\sqrt{2}\right)^{-1} \tag{3.4}$$

When multiplication non-commutative, as it is when composing isometries, for example, conjugation always takes the form in (3.4) above.

Definition 124 *Let α and β be transformations. The* <u>conjugate of β by α</u> *is the composition*

$$\alpha \circ \beta \circ \alpha^{-1}.$$

Note that $\beta = \alpha \circ \beta \circ \alpha^{-1}$ if and only if $\alpha \circ \beta = \beta \circ \alpha$, i.e., α and β commute. Since translations commute we always have $\tau_{\mathbf{u}} = \tau_{\mathbf{v}} \circ \tau_{\mathbf{u}} \circ \tau_{\mathbf{v}}^{-1}$ for all vectors $\mathbf{u}$ and $\mathbf{v}$. We saw an important example of conjugation in the proof of Theorem 63 in which we derived the equations for a rotation about a general point C. Recall that $\rho_{C,\Theta}$ is equivalent to

1. *translating from C to O followed by*
2. *rotating about O through angle Θ followed by*
3. *translating from O to C.*

Thus

$$\rho_{C,\Theta} = \tau_{\mathbf{OC}} \circ \rho_{O,\Theta} \circ \tau_{\mathbf{OC}}^{-1}, \tag{3.5}$$

i.e., $\rho_{C,\Theta}$ *is the conjugate of* $\rho_{O,\Theta}$ *by* $\tau_{\mathbf{OC}}$. Indeed, since rotations and translation do not commute, the equations for rotations about C are quite different from those for rotations about O.

Here are some algebraic properties of conjugation.

Theorem 125

a. *The square of a conjugate is the conjugate of the square.*

b. *The conjugate of an involution is an involution.*

Proof. Let α and β be isometries.

(a) $\left(\alpha \circ \beta \circ \alpha^{-1}\right)^2 = \alpha \circ \beta \circ \alpha^{-1} \circ \alpha \circ \beta \circ \alpha^{-1} = \alpha \circ \beta^2 \circ \alpha^{-1}$.

(b) If β is an involution then $\alpha \circ \beta \circ \alpha^{-1} \neq \iota$ (otherwise $\alpha \circ \beta \circ \alpha^{-1} = \iota$ implies $\beta = \alpha^{-1} \circ \alpha = \iota$). By part (a), $\left(\alpha \circ \beta \circ \alpha^{-1}\right)^2 = \alpha \circ \beta^2 \circ \alpha^{-1} = \alpha \circ \iota \circ \alpha^{-1} = \alpha \circ \alpha^{-1} = \iota$ and $\alpha \circ \beta \circ \alpha^{-1}$ is an involution. ■

Here are some geometrical consequences.

Proposition 126 *Conjugation preserves parity, i.e., if α and β are isometries, then β and $\alpha \circ \beta \circ \alpha^{-1}$ have the same parity; both either preserve orientation or reverse orientation.*

Proof. By Theorem 108 we know that α factors as a product of reflections. Since the inverse of a product is the product of the inverses in reverse order, α and α^{-1} have the same parity and together contribute an even number of factors to every factorization of $\alpha \circ \beta \circ \alpha^{-1}$ as a product of reflections. Therefore the parity of an isometry β is the same as the parity of $\alpha \circ \beta \circ \alpha^{-1}$. The fact that β and $\alpha \circ \beta \circ \alpha^{-1}$ both either preserve orientation or reverse orientation follows immediately from Proposition 123. ■

Theorem 127 *Let α be an isometry.*

a. *The conjugate of a halfturn is a halfturn. Furthermore, if P is any point, then*
$$\alpha \circ \varphi_P \circ \alpha^{-1} = \varphi_{\alpha(P)}.$$

b. *The conjugate of a reflection is a reflection. Furthermore, if ℓ is any line, then*
$$\alpha \circ \sigma_\ell \circ \alpha^{-1} = \sigma_{\alpha(\ell)}.$$

Proof. (a) Since φ_P is even by Corollary 78, so is $\alpha \circ \varphi_P \circ \alpha^{-1}$ by Proposition 126. Furthermore, $\alpha \circ \varphi_P \circ \alpha^{-1}$ is an involution by Theorem 125. By Theorem 118, involutory isometries are either reflections (which are odd) or halfturns (which are even), so $\alpha \circ \varphi_P \circ \alpha^{-1}$ is a halfturn. To locate its center, observe that

$$\left(\alpha \circ \varphi_P \circ \alpha^{-1}\right)(\alpha(P)) = \left(\alpha \circ \varphi_P \circ \alpha^{-1} \circ \alpha\right)(P) = (\alpha \circ \varphi_P)(P) = \alpha(P).$$

Since $\alpha(P)$ is fixed by the halfturn $\alpha \circ \varphi_P \circ \alpha^{-1}$ we have

$$\alpha \circ \varphi_P \circ \alpha^{-1} = \varphi_{\alpha(P)}.$$

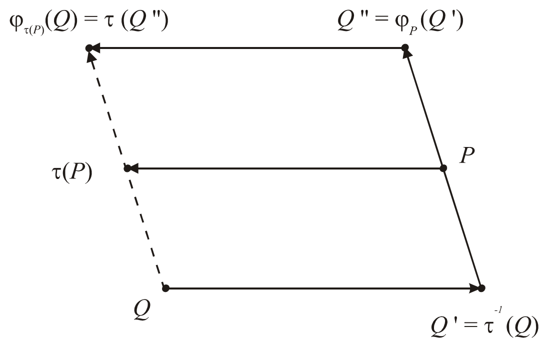


Figure 3.7: Conjugation of φ_P by τ.

(b) Since σ_ℓ is odd, so is $\alpha \circ \sigma_\ell \circ \alpha^{-1}$ by Proposition 126. Furthermore $\alpha \circ \sigma_\ell \circ \alpha^{-1}$

is an involution by Theorem 125. By Theorem 118 involutory isometries are either halfturns (which are even) or reflections (which are odd), so $\alpha \circ \sigma_\ell \circ \alpha^{-1}$ is a reflection. To determine its axis, observe that for every point P on ℓ,

$$\left(\alpha \circ \sigma_\ell \circ \alpha^{-1}\right)(\alpha(P)) = \left(\alpha \circ \sigma_\ell \circ \alpha^{-1} \circ \alpha\right)(P) = (\alpha \circ \sigma_\ell)(P) = \alpha(P).$$

Since every point $\alpha(P)$ on $\alpha(\ell)$ is fixed by the reflection $\alpha \circ \sigma_\ell \circ \alpha^{-1}$, its axis is $\alpha(\ell)$ and we have

$$\alpha \circ \sigma_\ell \circ \alpha^{-1} = \sigma_{\alpha(\ell)}.$$

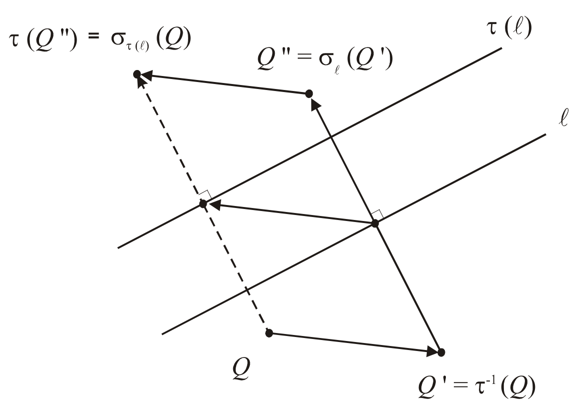


Figure 3.8: Conjugation of σ_ℓ by τ.

■

Theorem 128 *The conjugate of a translation is a translation. Furthermore, if α is an isometry, A and B are points, $A' = \alpha(A)$, and $B' = \alpha(B)$, then*

$$\alpha \circ \tau_{\mathbf{AB}} \circ \alpha^{-1} = \tau_{\mathbf{A'B'}}.$$

Proof. Let M be the midpoint of A and B; then

$$\tau_{\mathbf{AB}} = \varphi_M \circ \varphi_A$$

by Theorem 68. Since α is an isometry, $M' = \alpha(M)$ is the midpoint of $A' = \alpha(A)$ and $B' = \alpha(B)$. So again by Theorem 68,

$$\tau_{\mathbf{A'B'}} = \varphi_{M'} \circ \varphi_{A'}.$$

Therefore, by Theorem 127 (part a) we have

$$\begin{aligned} \alpha \circ \tau_{\mathbf{AB}} \circ \alpha^{-1} &= \alpha \circ (\varphi_M \circ \varphi_A) \circ \alpha^{-1} \\ &= (\alpha \circ \varphi_M \circ \alpha^{-1}) \circ (\alpha \circ \varphi_A \circ \alpha^{-1}) \\ &= \varphi_{M'} \circ \varphi_{A'} = \tau_{\mathbf{A'B'}}. \end{aligned}$$

■

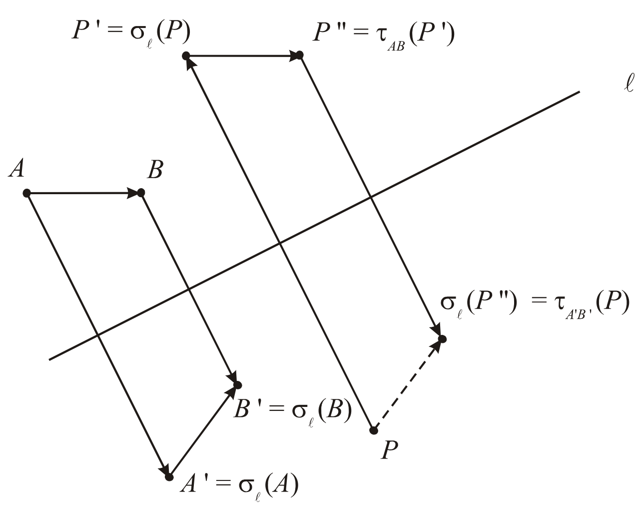


Figure 3.9. Conjugation of $\tau_{\mathbf{AB}}$ by σ_ℓ.

Our next theorem generalizes the remarks related to (3.5) above:

Theorem 129 *The conjugate of a rotation is a rotation. If α is an isometry, C is a point and $\Theta \in \mathbb{R}$, then*

$$\alpha \circ \rho_{C,\Theta} \circ \alpha^{-1} = \begin{cases} \rho_{\alpha(C),\Theta} & \text{if } \alpha \text{ is even} \\ \rho_{\alpha(C),-\Theta} & \text{if } \alpha \text{ is odd} \end{cases}.$$

Proof. Since α is the a product of three or fewer reflections, we consider each of the three possible factorizations of α separately.

Case 1: Let r be a line and let $\alpha = \sigma_r$. Then $\alpha = \alpha^{-1}$ is odd and we must show that $\sigma_r \circ \rho_{C,\Theta} \circ \sigma_r = \rho_{\sigma_r(C),-\Theta}$. If C is on r, there is a unique line s passing through C such that $\rho_{C,\Theta} = \sigma_s \circ \sigma_r$ by Corollary 82. Therefore

$$\sigma_r \circ \rho_{C,\Theta} \circ \sigma_r = \sigma_r \circ \sigma_s \circ \sigma_r \circ \sigma_r = \sigma_r \circ \sigma_s = \rho_{C,-\Theta} = \rho_{\sigma_r(C),-\Theta}.$$

If C is off r, let m be the line through C perpendicular to r. By Corollary 82, there is a (unique) line n passing through C such that

$$\rho_{C,\Theta} = \sigma_n \circ \sigma_m$$

(see Figure 3.10).

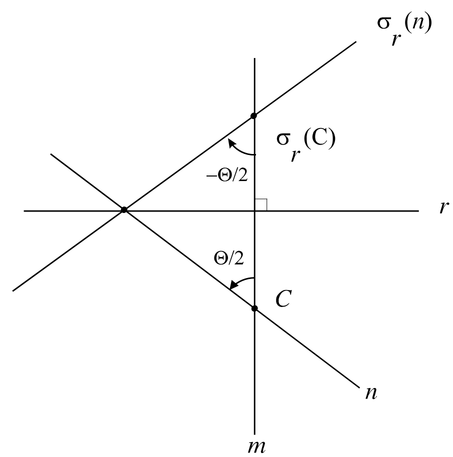


Figure 3.10.

Hence

$$\sigma_r \circ \rho_{C,\Theta} \circ \sigma_r = \sigma_r \circ \sigma_n \circ \sigma_m \circ \sigma_r = (\sigma_r \circ \sigma_n \circ \sigma_r) \circ (\sigma_r \circ \sigma_m \circ \sigma_r)$$

and Theorem 127 we have

$$(\sigma_r \circ \sigma_n \circ \sigma_r) \circ (\sigma_r \circ \sigma_m \circ \sigma_r) = \sigma_{\sigma_r(n)} \circ \sigma_{\sigma_r(m)}. \tag{3.6}$$

Since $m \perp r$ we know that

$$\sigma_r(m) = m.$$

Furthermore, $C = m \cap n$ so that

$$\sigma_r(C) = m \cap \sigma_r(n).$$

Since the measure of an angle from m to n is $\frac{1}{2}\Theta$, the measure of an angle from m to $\sigma_r(n)$ is $-\frac{1}{2}\Theta$. By Theorem 75, the right-hand side in (3.6) becomes

$$\sigma_{\sigma_r(n)} \circ \sigma_m = \rho_{\sigma_r(C),-\Theta}$$

and we conclude that

$$\sigma_r \circ \rho_{C,\Theta} \circ \sigma_r = \rho_{\sigma_r(C),-\Theta}. \tag{3.7}$$

Case 2: Let r and s be lines. Since $\alpha = \sigma_s \circ \sigma_r$ is even, we must show that $\alpha \circ \rho_{C,\Theta} \circ \alpha^{-1} = \rho_{\alpha(C),\Theta}$. But two successive applications of (3.7) give the desired result:

$$\begin{aligned} \alpha \circ \rho_{C,\Theta} \circ \alpha^{-1} &= (\sigma_s \circ \sigma_r) \circ \rho_{C,\Theta} \circ (\sigma_s \circ \sigma_r)^{-1} \\ &= \sigma_s \circ (\sigma_r \circ \rho_{C,\Theta} \circ \sigma_r) \circ \sigma_s \\ &= \sigma_s \circ \rho_{\sigma_r(C),-\Theta} \circ \sigma_s = \rho_{\alpha(C),\Theta}. \end{aligned}$$

Case 3: Let r, s, and t be lines. Since $\alpha = \sigma_t \circ \sigma_s \circ \sigma_r$ is odd, we must show that $\alpha \circ \rho_{C,\Theta} \circ \alpha^{-1} = \rho_{\alpha(C),-\Theta}$. This time, three successive applications of (3.7) in the manner of Case 2 give the result, as the reader can easily check. ■

Example 130 *Look again at the discussion on the equations for general rotations above. Equation (3.5) indicates that* $\tau_{\mathbf{OC}} \circ \rho_{O,\Theta} \circ \tau_{\mathbf{OC}}^{-1} = \rho_{C,\Theta}$. *Since* $\tau_{\mathbf{OC}}$ *is even and* $\tau_{\mathbf{OC}}(O) = C$ *we have*

$$\tau_{\mathbf{OC}} \circ \rho_{O,\Theta} \circ \tau_{\mathbf{OC}}^{-1} = \rho_{\tau_{\mathbf{OC}}(O),\Theta},$$

which confirms the conclusion of Theorem 129.

Theorem 131 *The conjugate of a glide reflection is a glide reflection. If* α *is an isometry and* γ *is a glide reflection with axis* c *and glide vector* $\mathbf{AB}$, *let* $A' = \alpha(A)$ *and* $B' = \alpha(B)$. *Then* $\alpha \circ \gamma \circ \alpha^{-1}$ *is a glide reflection with axis* $\alpha(c)$ *and glide vector* $\mathbf{A'B'}$.

Proof. Consider a glide reflection γ with axis c and glide vector $\mathbf{AB}$, and let α be an isometry. Since γ is odd, so is $\alpha \circ \gamma \circ \alpha^{-1}$, which is either a reflection or a glide reflection by Theorem 117. But $\gamma^2 = \tau_{2\mathbf{AB}}$ by Theorem 103 (part b) so that

$$\begin{aligned} \left(\alpha \circ \gamma \circ \alpha^{-1}\right)^2 &= \alpha \circ \gamma^2 \circ \alpha^{-1} = \alpha \circ \tau_{2\mathbf{AB}} \circ \alpha^{-1} = \alpha \circ \tau_{\mathbf{AB}}^2 \circ \alpha^{-1} \\ &= (\alpha \circ \tau_{\mathbf{AB}} \circ \alpha)^2 = \tau_{\mathbf{A'B'}}^2 = \tau_{2\mathbf{A'B'}} \neq \iota \end{aligned}$$

by Theorem 128. Since $\alpha \circ \gamma \circ \alpha^{-1}$ is not an involution, it is a glide reflection with glide vector $\mathbf{A'B'}$ by Exercise 8 at the end of this section. To determine the axis, note that $\alpha \circ \gamma \circ \alpha^{-1}$ fixes the line $\alpha(c)$:

$$\left(\alpha \circ \gamma \circ \alpha^{-1}\right)(\alpha(c)) = \left(\alpha \circ \gamma \circ \alpha^{-1} \circ \alpha\right)(c) = (\alpha \circ \gamma)(c) = \alpha(\gamma(c)) = \alpha(c).$$

Since the only line fixed by a glide reflection is its axis, hence $\alpha(c)$ is the axis. ■

Let's apply these techniques to draw some interesting geometrical conclusions.

Theorem 132 *Two non-identity rotations commute if and only if they have the same center of rotation.*

Proof. Let $\rho_{A,\Theta}$ and $\rho_{B,\Phi}$ be non-identity commuting rotations. Since $\rho_{A,\Theta}$ is even, apply Theorem 129 and obtain

$$\rho_{\rho_{A,\Theta}(B),\Phi} = \rho_{A,\Theta} \circ \rho_{B,\Phi} \circ \rho_{A,\Theta}^{-1} = \rho_{B,\Phi} \circ \rho_{A,\Theta} \circ \rho_{A,\Theta}^{-1} = \rho_{B,\Phi}$$

if and only if $\rho_{A,\Theta}(B) = B$ if and only if $A = B$. ■

Theorem 133 *Two reflections σ_m and σ_n commute if and only if $m = n$ or $m \perp n$.*

Proof. By Theorem 127 we have

$$\sigma_{\sigma_n(m)} = \sigma_n \circ \sigma_m \circ \sigma_n = \sigma_m \circ \sigma_n \circ \sigma_n = \sigma_m$$

if and only if $\sigma_n(m) = m$ if and only if $n = m$ or $n \perp m$. ■

Exercises

1. Given a line a and a point B off a, construct the line b such that $\varphi_B \circ \sigma_a \circ \varphi_B = \sigma_b$.

2. Given a line b and a point A off b, construct the point B such that $\sigma_b \circ \varphi_A \circ \sigma_b = \varphi_B$.

3. Given a line b and a point A such that $\sigma_b \circ \varphi_A \circ \sigma_b = \varphi_A$, prove that A lies on b.

4. Let A and B be distinct points and let c be a line. Prove that $\tau_{A,B} \circ \sigma_c = \sigma_c \circ \tau_{A,B}$ if and only if $\tau_{A,B}(c) = c$.

5. Let A and B be distinct points. Prove that if $\Theta + \Phi \notin 0^\circ$, $\rho_{B,\Phi} \circ \rho_{A,\Theta} = \rho_{C,\Theta+\Phi}$ and $\rho_{A,\Theta} \circ \rho_{B,\Phi} = \rho_{D,\Theta+\Phi}$, then $D = \sigma_{\overleftrightarrow{AB}}(C)$.

6. Let a be a line and let B be a point. Prove that $\varphi_B \circ \sigma_a \circ \varphi_B \circ \sigma_a \circ \varphi_B \circ \sigma_a \circ \varphi_B$ is a reflection in some line parallel to a.

7. Let A be a point and let τ be a non-identity translation. Prove that $\varphi_A \circ \tau \neq \tau \circ \varphi_A$.

8. Let γ be a glide reflection with axis c and glide vector $\mathbf{v}$. If $\gamma^2 = 2\mathbf{w}$, show that $\mathbf{v} = \mathbf{w}$.

9. Prove that $\gamma_c \circ \varphi_A \neq \varphi_A \circ \gamma_c$, for every glide reflection γ_c with axis c.

10. Let A and B be distinct points and let γ_c be a glide reflection with axis c. Prove that $\tau_{A,B} \circ \gamma_c = \gamma_c \circ \tau_{A,B}$ if and only if $\tau_{A,B}(c) = c$.

11. Complete the proof of Theorem 129 by proving Case 3.

Chapter 4

Symmetry

A "symmetry" of a plane figure F is an isometry that fixes F. If F is an equilateral triangle with centroid C, for example, there are six symmetries of F, one of which is the rotation $\rho_{C,120}$. In this chapter we observe that the set of symmetries of a given plane figure is a "group" under composition. The structure of these groups, called symmetry groups, encodes information pertaining to the "symmetry types" of plane figures.

The Classification Theorem of Plane Isometries (Theorem 113) assures us that symmetries are always reflections, translations, rotations or glide reflections. Consequently, we can systematically identify all symmetries of a given plane figure. Now if we restrict our attention to those plane figures with "finitely generated" symmetry groups, there are exactly five classes of symmetry types: (1) asymmetrical patterns, (2) patterns with only bilateral symmetry, (3) rosettes, (4) frieze patterns and (5) wallpaper patterns. Quite surprisingly, there are exactly seven symmetry types of frieze patterns and seventeen symmetry types of wallpaper patterns. Although there are infinitely many symmetry types of rosettes, their symmetry is simple and easy to understand. Furthermore, it is interesting to note that two rosettes with different symmetries have non-isomorphic symmetry groups. So for rosettes, the symmetry group is a perfect invariant. We begin our discussion with what little group theory we need.

4.1 Groups of Isometries

In this section we introduce the group of isometries $\mathcal{I}$ and some of its subgroups.

Definition 134 *A non-empty set G equipped with a binary operation $*$ is a group if and only if the following properties are satisfied:*

1. *Closure: If $a, b \in G$, then $a * b \in G$.*
2. *Associativity: If $a, b, c \in G$, then $a * (b * c) = (a * b) * c$.*

3. *Identity: For all* $a \in G$, *there exists an element* $e \in G$ *such that* $e * a = a * e = a$.

4. *Inverses: For each* $a \in G$, *there exists* $b \in G$ *such that* $a * b = b * a = e$.

A group G *is abelian (or commutative) if and only if for all* $a, b \in G$, $a * b = b * a$.

Theorem 135 *The set* $\mathcal{I}$ *of all isometries is a group under function composition.*

Proof. The work has already been done. Closure was proved in Exercise 1.1.3; the fact that composition of isometries is associative is a special case of Exercise 1.1.4; the fact that ι acts as an identity element in $\mathcal{I}$ was proved in Exercise 1.1.5; and the existence of inverses was proved in Exercise 1.1.7. ■

Since two halfturns with distinct centers of rotation do not commute and halfturns are elements of $\mathcal{I}$, the group $\mathcal{I}$ is *non-abelian.* On the other hand, some subsets of $\mathcal{I}$ (the translations for example) contain commuting elements. When such a subset is a group in its own right, it is abelian.

Definition 136 *Let* $(G, *)$ *be a group and let* H *be a non-empty subset of* G. *Then* H *is a* <u>*subgroup*</u> *of* G *if and only if* $(H, *)$ *is a group, i.e.,* H *is a group under the operation inherited from* G.

Given a non-empty subset H of a group G, is H itself a group under the operation in G? One could appeal to the definition and check all four properties, but it is sufficient to check just two.

Theorem 137 *Let* $(G, *)$ *be a group and let* H *be a non-empty subset of* G. *Then* H *is a subgroup of* G *if and only if the following two properties hold:*
a. Closure: If $a, b \in H$, *then* $a * b \in H$.
b. Inverses: For every $a \in H$, *there exists* $b \in H$ *such that* $a * b = b * a = e$.

Proof. If H is a subgroup of G, properties (a) and (b) hold by definition. Conversely, suppose that H is a non-empty subset of G in which properties (a) and (b) hold. *Associativity* is inherited from G, i.e., if $a, b, c \in H$, then as elements of G, $a * (b * c) = (a * b) * c$. *Identity:* Since $H \neq \varnothing$, choose an element $a \in H$. Then $a^{-1} \in H$ since H has inverses by property (b). Furthermore, operation $*$ is closed in H by property (a) so that $a * a^{-1} \in H$. But $a * a^{-1} = e$ since a and a^{-1} are elements of G, so $e \in H$ as required. Therefore H is a subgroup of G. ■

Proposition 138 *The set* $\mathcal{T}$ *of all translations is an abelian group.*

Proof. Closure and commutativity follow from Proposition **??**; the existence of inverses was proved in Exercise 7. Therefore $\mathcal{T}$ is an abelian subgroup of $\mathcal{I}$ by Theorem 137, and consequently $\mathcal{T}$ is an abelian group. ■

Proposition 139 *The set $\mathcal{R}_C$ of all rotations about a point C is an abelian group.*

Proof. The proof is left as an exercise for the reader. ■

Exercises

1. Prove that the set $\mathcal{R}_C$ of all rotations about a point C is an abelian group.

2. Prove that the set $\mathcal{E}$ of all even isometries is a non-abelian group.

3. Prove that the set $\mathcal{D}$ of all dilatations is a non-abelian group.

4.2 Groups of Symmetries

In this section we observe that the set of symmetries of a given plane figure F is a group, called the symmetry group of F. Consequently, symmetry groups are always subgroups of $\mathcal{I}$ (the group of all isometries).

Definition 140 *A plane figure is a non-empty subset of the plane.*

Definition 141 *Let F be a plane figure. An isometry α is a symmetry of F if and only if α fixes F.*

Theorem 142 *Let F be a plane figure. The set of all symmetries of F is a group, called the symmetry group of F.*

Proof. Let F be a plane figure and let $\mathcal{S} = \{\alpha : \alpha \text{ is a symmetry of } F\}$. Since the identity $\iota \in \mathcal{S}$, the set $\mathcal{S}$ is a non-empty subset of $\mathcal{I}$.

Closure: Let $\alpha, \beta \in \mathcal{S}$. By Exercise 1.1.3, the composition of isometries is an isometry. So it suffices to check that $\alpha \circ \beta$ fixes F. But since $\alpha, \beta \in \mathcal{S}$ we have $(\alpha \circ \beta)(F) = \alpha(\beta(F)) = \alpha(F) = F$.

Inverses: Let $\alpha \in \mathcal{S}$; we know that $\alpha^{-1} \in \mathcal{I}$ by Exercise 1.1.7; we must show that α^{-1} fixes F. But $\alpha^{-1}(F) = \alpha^{-1}(\alpha(F)) = \left(\alpha^{-1} \circ \alpha\right)(F) = F$ so that α^{-1} also fixes F. Thus $\alpha^{-1} \in \mathcal{S}$ whenever $\alpha \in \mathcal{S}$.

Therefore $\mathcal{S}$ is a group by Theorem 137. ■

Example 143 *(The Dihedral Group D_3) Let F denote an equilateral triangle positioned with its centroid at the origin and one vertex on the y-axis. There are exactly six symmetries of F, namely, the identity ι, two rotations ρ_{120} and ρ_{240} about the centroid, and three reflections σ_ℓ, σ_m and σ_n where ℓ, m and n have respective equations $\sqrt{3}x - 3y = 0$; $x = 0$; and $\sqrt{3}x + 3y = 0$ (see Figure 4.1).*

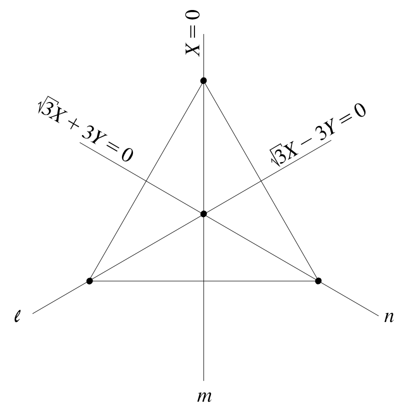


Figure 4.1: Lines of symmetry ℓ, m and n.

The multiplication table for composing these various symmetries appears in Table 4.1 below. Closure holds by inspection. Furthermore, since each row and column contains the identity element ι in exactly one position, each element has a unique inverse. By Theorem 137 these six symmetries form a group D_3 called the *Dihedral Group of order 6.*

$\circ$	ι	ρ_{120}	ρ_{240}	σ_ℓ	σ_m	σ_n
ι	ι	ρ_{120}	ρ_{240}	σ_ℓ	σ_m	σ_n
ρ_{120}	ρ_{120}	ρ_{240}	ι	σ_m	σ_n	σ_ℓ
ρ_{240}	ρ_{240}	ι	ρ_{120}	σ_n	σ_ℓ	σ_m
σ_ℓ	σ_ℓ	σ_n	σ_m	ι	ρ_{240}	ρ_{120}
σ_m	σ_m	σ_ℓ	σ_n	ρ_{120}	ι	ρ_{240}
σ_n	σ_n	σ_m	σ_ℓ	ρ_{240}	ρ_{120}	ι

Table 4.1: The Dihedral Group of Order 6

Look carefully at the upper left 4×4 block in Table 4.1 above. This is the multiplication table for the rotations $\{\iota, \rho_{120}, \rho_{240}\} \subset D_3$ (the identity is a rotation through angle 0); we shall denote this set by C_3. Once again we see that composition is closed in C_3 and the inverse of each element in C_3 is also in C_3. Therefore C_3 is a group; the symbol "C_3" stands for "*cyclic group of order 3*".

Definition 144 *A plane figure F has point symmetry if and only if some (non-identity) rotation is a symmetry of F. The center of a (non-identity) rotational symmetry of F is called a point of symmetry for F.*

Definition 145 *A plane figure F has line symmetry if and only if some reflection is a symmetry of F. The reflecting line of a reflection symmetry of F is called a line of symmetry for F.*

Definition 146 *A plane figure F has* <u>*bilateral symmetry*</u> *if and only if F has a unique line of symmetry.*

Corollary 147 *The two symmetries of a figure with bilateral symmetry form a group denoted by D_1. The four symmetries of a non-square rectangle form a group denoted by D_2. For $n \geq 3$, the $2n$ symmetries of a regular n-gon form a group denoted by D_n.*

Proof. A plane figure with bilateral symmetry has one line of symmetry and one rotational symmetry (the identity). A non-square rhombus has two lines of symmetry and two rotational symmetries about the centroid (including the identity). If $n \geq 3$, a regular n-gon has n lines of symmetry and n rotational symmetries about the centroid (including the identity). These sets of symmetries form groups by Theorem 142. ■

Note that the two lines of symmetry of a rectangle and n lines of symmetry of a regular n-gon are concurrent at the centroid.

Definition 148 *For $n \geq 1$, the group D_n, called the* <u>*dihedral group of order $2n$*</u>*, consists of n rotational symmetries with the same center C and n reflection symmetries whose axes are concurrent at C.*

Definition 149 *Let G be a group, let $a \in G$, and define $a^0 = \iota$. The group G is* <u>*cyclic*</u> *if and only if for all $a \in G$, there is an element $g \in G$ (called a generator) such that $a = g^k$ for some $k \in \mathbb{Z}$. A cyclic group with $n < \infty$ elements is said to be cyclic of order n. A cyclic group G with infinitely many elements is said to be infinite cyclic.*

Example 150 *Observe that the elements of $C_3 = \{\iota, \rho_{120}, \rho_{240}\}$ can be obtained as powers of either ρ_{120} or ρ_{240}. For example,*

$$\rho_{120} = \rho_{120}^1;\ \rho_{240} = \rho_{120}^2;\ \textit{and } \iota = \rho_{120} \circ \rho_{240} = \rho_{120}^3.$$

We say that ρ_{120} and ρ_{240} "generate" C_3. Also observe that this 3×3 block is symmetric with respect to the upper-left-to-lower-right diagonal. This indicates that C_3 is an abelian group. More generally, let C be a point, let n be a positive integer and let $\theta = \frac{360}{n}$. Then for each integer k, $\rho_{C,\theta}^k = \rho_{C,k\theta}$ and $\rho_{C,\theta}^n = \rho_{C,360} = \iota$. Therefore the group of rotations generated by $\rho_{C,\theta}$ is cyclic with n elements and is denoted by C_n.

If C is the centroid of a regular n-gon with $n \geq 3$, the finite cyclic group of rotations C_n introduced in Example 150 is the abelian subgroup of rotations in D_n. On the other hand, C_n can be realized as the symmetry group of a $3n$-gon constructed as follows: For $n = 4$, cut a square out of paper and draw its diagonals, thereby subdividing the square into four congruent isosceles right triangles with common vertex at the centroid of the square. From each of the four vertices, cut along the diagonals stopping midway between the vertices and the centroid. With the square positioned so that its edges are vertical or horizontal, fold the triangle at the top so that its right-hand vertex aligns with the centroid of the square. Rotate the paper 90° and fold the triangle now at the top in a similar way. Continue rotating and folding until you have what looks like a flattened pinwheel with four paddles (see Figure 4.2). The outline of this flattened pinwheel is a dodecagon (12-gon) whose symmetry group is C_4 generated by either ρ_{90} or ρ_{270}. For a general n, one can construct a $3n$-gon whose symmetry group is cyclic of order n by cutting and folding a regular n-gon in a similar way to obtain a pinwheel with n paddles.

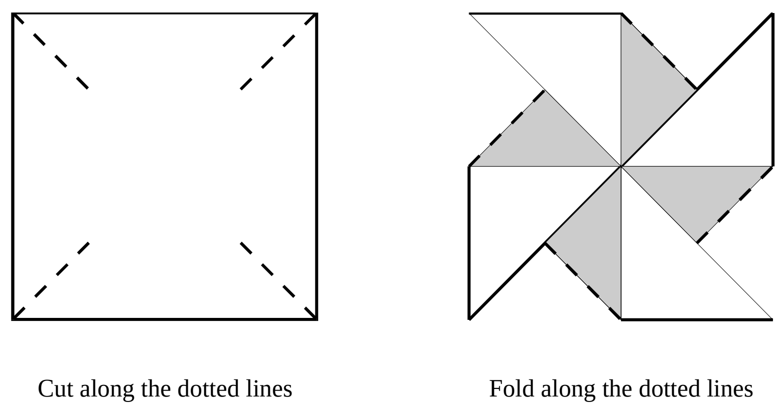


Figure 4.2: A polygon whose symmetry group is cyclic of order 4.

Example 151 *Let τ be a non-identity translation; let P be any point and let $Q = \tau(P)$. Then $\tau = \tau_{\mathbf{PQ}}$ and $\tau^2 = \tau_{\mathbf{PQ}} \circ \tau_{\mathbf{PQ}} = \tau_{2\mathbf{PQ}}$. Inductively, $\tau^n = \tau^{n-1} \circ \tau = \tau_{(n-1)\mathbf{PQ}} \circ \tau_{\mathbf{PQ}} = \tau_{n\mathbf{PQ}}$, for each $n \in \mathbb{N}$. Furthermore, $(\tau^n)^{-1} = (\tau_{n\mathbf{PQ}})^{-1} = \tau_{-n\mathbf{PQ}} = \tau^{-n}$, so distinct integer powers of τ are distinct translations. It follows that the set $G = \{\tau^n : n \in \mathbb{Z}\}$ is infinite. Note that G is a group: inverses were discussed above and closure follows from the fact that $\tau^n \circ \tau^m = \tau^{n+m}$. Since every element of G is an integer power of τ (or τ^{-1}), G is the infinite cyclic group generated by τ (or τ^{-1}).*

Let G be a group and let K be a non-empty subset of G. The symbol $\langle K \rangle$ denotes the set of all (finite) products of powers of elements of K and their inverses. If $K = \{k_1, k_2, \ldots\}$, we abbreviate and write $\langle k_1, k_2, \ldots \rangle$ instead of $\langle \{k_1, k_2, \ldots\} \rangle$. Thus $\langle K \rangle$ is automatically a subgroup of G since it is non-empty, the group operation is closed and contains the inverse of each element in $\langle K \rangle$.

Definition 152 *Let G be a group and let K be a non-empty subset of G. The subgroup $\langle K \rangle$ is referred to as the <u>subgroup of G generated by K</u>. A subset $K \subseteq G$ is said to be a <u>generating set</u> for G if and only if $G = \langle K \rangle$. A group G is <u>finitely generated</u> if and only if there exists a finite set K such that $G = \langle K \rangle$.*

Example 153 *A cyclic group G with generator $g \in G$ has the property that $G = \langle g \rangle$. So $\{g\}$ is a generating set for G.*

Example 154 *Let τ be a non-identity translation. Then $\langle \tau \rangle$ is infinite cyclic since $\tau^n \neq \iota$ for all $n \neq 0$ (see Example 151).*

Example 155 *Let K be the set of all reflections. Since every reflection is its own inverse, $\langle K \rangle$ consists of all (finite) products of reflections. By The Fundamental Theorem of Transformational Plane Geometry every isometry of the plane is a product of reflections. Therefore $\langle K \rangle = \mathcal{I}$, i.e., the group of all isometries, is infinitely generated by the set K of all reflections.*

Example 156 *Let H denote the set of all halfturns. Since the composition of two halfturns is a translation, the composition of two translations is a translation, and every translation can be written as a composition of two halfturns, $\mathcal{H} = \langle H \rangle$ is infinitely generated and is exactly the set of all translations and halfturns.*

Exercises

1. Recall that the six symmetries of an equilateral triangle form the dihedral group D_3 (see Example 143). Show that the set $K = \{\rho_{120},\ \sigma_\ell\}$ is a generating set for D_3 by writing each of the other four elements in D_3 as a product of powers of elements of K and their inverses. Compute all powers of each element in D_3 and show that no single element alone generates D_3. Thus D_3 is not cyclic.

2. The dihedral group D_4 consists of the eight symmetries of a square. When the square is positioned with its centroid at the origin and its vertices on the axes, the origin is a point of symmetry and the lines $a : Y = 0$, $b : Y = X$, $c : X = 0$ and $d : Y = -X$ are lines of symmetry. Construct a multiplication table for $D_4 = \{\iota, \rho_{90}, \rho_{180}, \rho_{270}, \sigma_a, \sigma_b, \sigma_c, \sigma_d\}$.

3. Find the symmetry group of

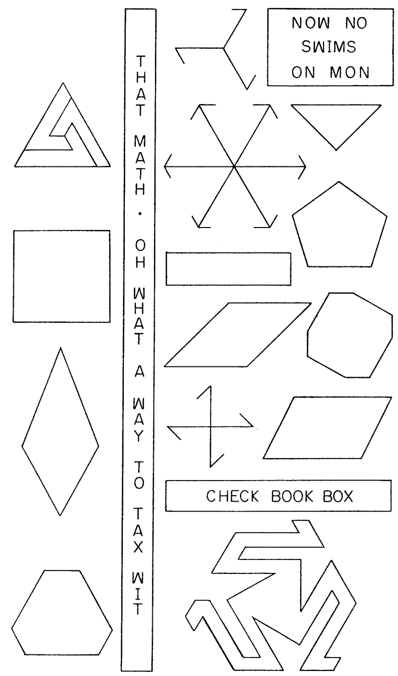
   a. A parallelogram that is neither a rectangle nor a rhombus.

   b. A rectangle that is not a square.

   c. A kite that is not a rhombus.

4. Find the symmetry group of each capital letter of the alphabet written in most symmetric form (write the letter "O" as an non-circular oval).

5. Determine the symmetry group of each figure below:



6. The discussion following Example 150 describes how to construct a $3n$-gon whose symmetry group is C_n, where $n \geq 3$. Alter this construction to obtain a $2n$-gon whose symmetry group is C_n.

7. Prove that a plane figure with bilateral symmetry has no points of symmetry.

8. Let C be a point. For which rotation angles Θ is $\langle \rho_{C,\Theta} \rangle$ an infinite group?

9. Consider the dihedral group D_n.

   (a) Prove that D_n contains a halfturn if and only if n is even.

   (b) Prove that if D_n contains a halfturn φ, then $\varphi \circ \alpha = \alpha \circ \varphi$ for all $\alpha \in D_n$.

   (c) Let $n \geq 3$ and let $\beta \in D_n$ such that $\beta \neq \iota$. Prove that if $\beta \circ \alpha = \alpha \circ \beta$ for all $\alpha \in D_n$, then β is a halfturn. (The subgroup of elements that commute with every element of a group G is called the *center of* G. Thus, if D_n contains a halfturn φ, the center of D_n is the subgroup $\{\iota, \varphi\}$.)

4.3 The Rosette Groups

Definition 157 *A rotational symmetry $\rho_{C,\Theta}$ of a plane figure R is* minimal *if $0^\circ < \Theta^\circ \leq \Phi^\circ$ for all non-identity rotational symmetries $\rho_{C,\Phi}$ of R.*

Definition 158 *A* rosette *is a plane figure R with the following properties:*

1. *There is a minimal rotational symmetry of R.*
2. *All non-identity rotational symmetries of R have the same center.*

The symmetry group of a rosette is called a rosette group.

Typically one thinks of a rosette as a pin-wheel (see Figures 4.2) or a flower with n-petals (See Figure 4.3). However, a regular polygon, a non-square rhombus, a yin-yang symbol and a pair of perpendicular lines are rosettes as well.

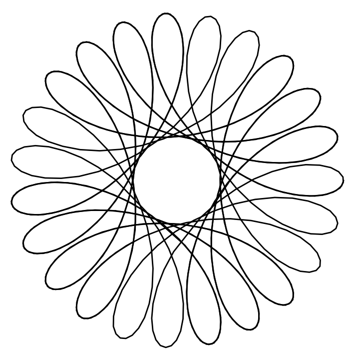


Figure 4.3: A typical rosette.

In the early sixteenth century, Leonardo da Vinci determined all possible finite groups of isometries; all but two of which are rosette groups. The two exceptions are C_1, which contains only the identity, and D_1, which contains the identity and one reflection. Note that D_1 is isomorphic to the rosette group C_2, which contains the identity and one halfturn.

Theorem 159 ***(Leonardo's Theorem):*** *Every finite group of isometries is either C_n or D_n for some $n \geq 1$.*

Proof. Let G be a finite group of isometries. Then G contains only rotations and reflections since non-identity translations and glide reflections would generate infinite subgroups. If $G = \{\iota\}$, then $G = C_1$. If σ is a reflection and $G = \{\iota, \sigma\}$, then $G = D_1$. Suppose G is neither C_1 nor D_1, and let E be the subset of rotations in G.

Claim 1: The non-identity rotations in E have the same center. On the contrary, suppose E contains non-identity rotations $\rho_{C,\Theta}$ and $\rho_{B,\Phi}$ with $B \neq C$.

Let $C' = \rho_{B,\Phi}(C)$; then $C' \neq C$ since C is not a fixed point. Conjugating $\rho_{C,\Theta}$ by $\rho_{B,\Phi}$ gives $\rho_{B,\Phi} \circ \rho_{C,\Theta} \circ \rho_{B,\Phi}^{-1} = \rho_{C',\Theta}$, which is an element of G by closure. Furthermore, $\rho_{C',\Theta} \circ \rho_{C,\Theta}^{-1} = \rho_{C',\Theta} \circ \rho_{C,-\Theta} \in G$, again by closure. But $\Theta + (-\Theta) \in 0^\circ$ so $\rho_{C',\Theta} \circ \rho_{C,-\Theta}$ is a non-identity translation by the Angle Addition Theorem (92), contradicting the fact that G contains no translations. Therefore $B = C$.

Claim 2: E is a cyclic subgroup of G. Recall that every congruence class of angles Φ° has a unique class representative in the range $0 \leq \Phi < 360$. Write each element in E uniquely in the form $\rho_{C,\Phi}$ with $0 \leq \Phi < 360$. Since E is finite, there is a rotation $\rho_{C,\Phi} \in E$ with the smallest positive rotation angle Φ. Thus if $\rho_{C,\Psi}$ is not the identity element of E, then $\Phi \leq \Psi < 360$ by the minimality of Φ, and there is a positive integer k such that $k\Phi \leq \Psi \leq (k+1)\,\Phi$. Thus $0 \leq \Psi - k\Phi \leq \Phi$. Now if both of these inequalities were strict, $\Psi - k\Phi$ would be a positive rotation angle strictly less than Φ, which violates the minimality of Φ. Therefore $\Psi = k\Phi$ or $\Psi = (k+1)\,\Phi$, i.e., $\Psi = m\Phi$ for some integer m. Consequently, $\rho_{C,\Psi} = \rho_{C,\Phi}^m$ and E is cyclic, i.e., for some $n \in \mathbb{N}$, there is a rotation $\rho \in E$ such that $E = \left\{\rho, \rho^2, \ldots, \rho^n = \iota\right\} = C_n$.

Now if $G = E$, we're done; otherwise, G contains reflections. Let $F = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$, $m \geq 1$, be the subset of reflections in G.

Claim 3: G has the same number of reflections as rotations (including the identity). Choose a reflection $\sigma \in F$ and note that the set $\sigma E = \left\{\sigma \circ \rho, \sigma \circ \rho^2, \ldots, \sigma \circ \rho^n\right\}$ contains n distinct odd isometries, which are reflections since G has no glide reflections. Therefore $\sigma E \subseteq F$ and $n \leq m$. On the other hand, the set $\sigma F = \{\sigma \circ \sigma_1, \sigma \circ \sigma_2, \ldots, \sigma \circ \sigma_m\}$ contains m distinct even isometries, which are rotations since G has no translations. Therefore $\sigma F \subseteq E$ and $m \leq n$. Thus $m = n$ as claimed.

Claim 4: G is the dihedral group D_n. First, $G = E \cup F = \left\{\rho, \rho^2, \ldots, \rho^n\right\} \cup \left\{\sigma \circ \rho, \sigma \circ \rho^2, \ldots, \sigma \circ \rho^n\right\} = \langle \sigma, \rho \rangle$. Second, given a non-identity element $\sigma \circ \sigma_i \in E = \{\sigma \circ \sigma_1, \sigma \circ \sigma_2, \ldots, \sigma \circ \sigma_n\}$, there is a positive integer $k \leq n$ such that $\rho_{C,\theta}^k = \sigma \circ \sigma_i$. Hence the axis of σ and the axis of σ_i intersect at C, whenever $\sigma \circ \sigma_i \neq \iota$. Thus all axes of reflection are concurrent at C so that $G = D_n$.

■

An immediate consequence of Leonardo's Theorem is the following:

Corollary 160 *The rosette groups are either dihedral D_n or finite cyclic C_n with $n \geq 2$.*

The symmetry group of a plane figure encodes some, but not all, of the geometrical information it its group structure. For example, a butterfly has line symmetry but no point symmetry whereas a yin-yang symbol has point symmetry but no line symmetry. Yet their symmetry groups, which contain very different symmetries, are isomorphic since both groups are cyclic of order two. Thus knowing that the symmetry group of some plane figure is cyclic, may not be enough information to determine the precise symmetries its elements represent. Nevertheless, we can be sure that two plane figures with non-isomorphic symmetry groups have different "symmetry types." Equivalently, two plane figures with the same symmetry type have isomorphic symmetry groups.

While the notion of "symmetry type" for general plane figures is quite subtle, we can make the idea precise for rosettes. Let R_1 and R_2 be rosettes with the same minimal positive rotation angle Θ and respective centers A and B. Let $\tau = \tau_{\mathbf{AB}}$; then $\tau \circ \rho_{A,\Theta} \circ \tau^{-1} = \rho_{\tau(A),\Theta} = \rho_{B,\Theta}$ and there is an isomorphism of cyclic groups $f : \langle \rho_{A,\Theta} \rangle \to \langle \rho_{B,\Theta} \rangle$ given by $f(\alpha) = \tau \circ \alpha \circ \tau^{-1}$. If R_1 and R_2 have no lines symmetry, then f is an isomorphism of symmetry groups. On the other hand, if the respective symmetry groups G_1 and G_2 have reflections $\sigma_\ell \in G_1$ and $\sigma_m \in G_2$, the lines ℓ and m are either intersecting or parallel. If parallel, $m = \tau(\ell)$ and $\tau \circ \sigma_\ell \circ \tau^{-1} = \sigma_{\tau(\ell)} = \sigma_m$, in which case $f(\alpha) = \tau \circ \alpha \circ \tau^{-1}$ is an isomorphism of symmetry groups. If ℓ and m intersect and the angle measure from ℓ to m is Φ°, then $(\tau \circ \rho_{A,\Phi}) \circ \sigma_\ell \circ (\tau \circ \rho_{A,\Phi})^{-1} = \sigma_{(\tau \circ \rho_{A,\Phi})(\ell)} = \sigma_m$ and $f(\alpha) = (\tau \circ \rho_{A,\Phi}) \circ \alpha \circ (\tau \circ \rho_{A,\Phi})^{-1}$ is an isomorphism of symmetry groups.

Now if G is any group and $g \in G$, the function $h : G \to G$ defined by $h(x) = gxg^{-1}$ is an isomorphism, as the reader can easily check. In particular, the map f defined above is the restriction to G_1 of an isomorphism $f : \mathcal{I} \to \mathcal{I}$, where $\mathcal{I}$ denotes the group of all plane isometries. We summarize this discussion in the definitions that follows:

Definition 161 *Let G be a group and let $g \in G$. The isomorphism $h : G \to G$ defined by $h(x) = gxg^{-1}$ is called an inner automorphism of G.*

Definition 162 *Let R_1 and R_2 be rosettes with respective symmetry groups G_1 and G_2. Rosettes R_1 and R_2 have the same symmetry type if and only if there is an inner automorphism of $\mathcal{I}$ that restricts to an isomorphism $f : G_1 \to G_2$.*

Corollary 163 *Two rosettes have the same symmetry type if and only if their respective symmetry groups are isomorphic.*

Exercises

1. Refer to Exercise 4 in Section 4.2 above. Which capital letters of the alphabet written in most symmetry form are rosettes?

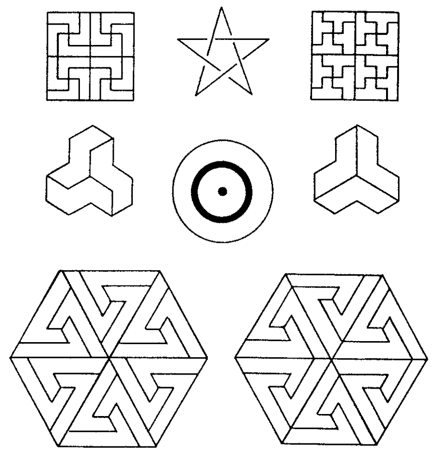
2. For $n \geq 2$, the graph of the equation $r = \cos n\theta$ in polar coordinates is a rosette.

   a. Find the rosette group of the graph for each $n \geq 2$.

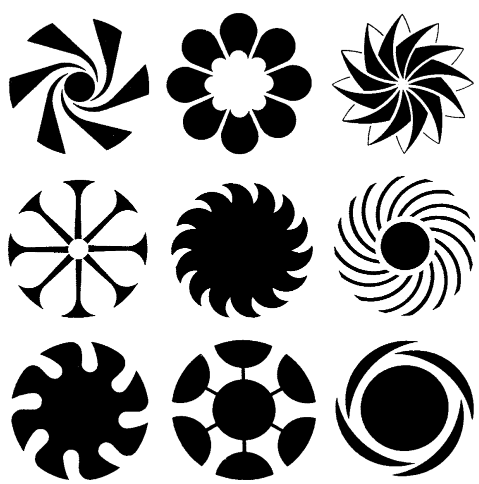
   b. Explain why the graph of the equation $r = \cos\theta$ in polar coordinates is *not* a rosette.

3. Find at least two rosettes in your campus architecture and determine their rosette groups.

4. Identify the rosette groups of the figures in the following that are rosettes:



5. Identify the rosette groups of the following rosettes:



4.4 The Frieze Groups

Frieze patterns are typically the familiar decorative borders often seen on walls or facades extended infinitely far in either direction (See Figure 4.4).

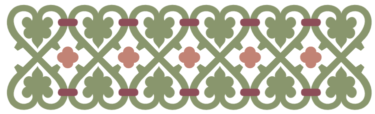


Figure 4.4: A typical frieze pattern.

In this section we identify all possible symmetries of frieze patterns and reach the startling conclusion that every frieze pattern is one of seven distinctive symmetry types.

Definition 164 *A basic translation of a plane figure F is a non-identity translational symmetry with the following property: If $\tau_{\mathbf{w}}$ is a non-identity translational symmetry of F such that $\mathbf{w} = k\mathbf{v}$, then $\|\mathbf{v}\| \leq \|\mathbf{w}\|$.*

Definition 165 *A frieze pattern is a plane figure F with the following properties:*

1. *There is a basic translation of F.*
2. *All non-identity translational symmetries of F fix the same lines.*

The symmetry group of a frieze pattern is called a frieze group.

Consider an row of equally spaced letter R's extending infinitely far in either direction (see Figure 4.5)

R R R R R R R R

Figure 4.5: Frieze pattern F_1.

This frieze pattern, denoted by F_1, has only translational symmetry. There are two basic translations of F_1—one shifting left; the other shifting right. Let τ be a basic translation; then $\tau^n \neq \iota$ for all $n \neq 0$ and the frieze group of F_1 is the infinite cyclic group $\mathcal{F}_1 = \langle \tau \rangle = \{\tau^n : n \in \mathbb{Z}\}$.

The second frieze pattern F_2 has a glide reflection symmetry (see Figure 4.6). Let γ be a glide reflection such that γ^2 is a basic translation. Then $\gamma^n \neq \iota$ for all $n \neq 0$ and γ^2 generates the translation subgroup. The frieze group of F_2 is the infinite cyclic group $\mathcal{F}_2 = \langle \gamma \rangle = \{\gamma^n : n \in \mathbb{Z}\}$. Note that while the elements of $\mathcal{F}_1$ and $\mathcal{F}_2$ are very different, the two groups are isomorphic.

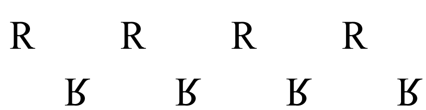


Figure 4.6: Frieze pattern F_2.

The third frieze pattern F_3 has vertical line symmetry (see Figure 4.7). Let ℓ be a line of symmetry. Choose a line m such that $\tau = \sigma_m \circ \sigma_\ell$ is a basic translation. Then m is also a line of symmetry since $\sigma_m = \tau \circ \sigma_\ell$ and the composition of symmetries is a symmetry (Theorem 142). In general, the reflection $\tau^n \circ \sigma_\ell$ is a symmetry for each $n \in \mathbb{Z}$; these reflections determine all lines of symmetry. The frieze group of F_3 is $\mathcal{F}_3 = \langle \tau, \sigma_\ell \rangle = \{\tau^n \circ \sigma_\ell^m : n \in \mathbb{Z};\ m = 0, 1\}$.

Figure 4.7: Frieze pattern F_3.

Frieze pattern F_4 has halfturn symmetry (see Figure 4.8). Let P be a point of symmetry. Choose a point Q such that $\tau = \varphi_Q \circ \varphi_P$ is a basic translation. Then Q is also a point of symmetry since $\varphi_Q = \tau \circ \varphi_P$. In general, the halfturn $\tau^n \circ \varphi_P$ is a symmetry for each $n \in \mathbb{Z}$; these halfturns determine all points of symmetry. The frieze group of F_4 is $\mathcal{F}_4 = \langle \tau, \varphi_P \rangle = \{\tau^n \circ \varphi_P^m : n \in \mathbb{Z};\ m = 0, 1\}$.

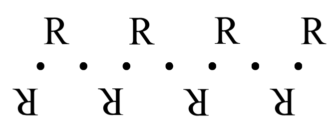


Figure 4.8: Frieze pattern F_4.

The fifth frieze pattern F_5 can be identified by its halfturn symmetry and glide reflection symmetry (see Figure 4.9). In addition, F_5 has vertical line symmetry, but as we shall see, these symmetries can be obtained by composing a glide reflection with a halfturn. Let P be a point of symmetry and let γ be a glide reflection such that γ^2 is a basic translation. Choose a point Q such that $\gamma^2 = \varphi_Q \circ \varphi_P$. Then Q is also a point of symmetry since $\varphi_Q = \gamma^2 \circ \varphi_P$. In general, the halfturn $\gamma^{2n} \circ \varphi_P$ is a symmetry for each $n \in \mathbb{Z}$; these halfturns determine all points of symmetry. Now the line symmetries can be obtained from γ and φ_P as follows: Let c be the horizontal axis of γ, let a be the vertical line through P, and let ℓ be the vertical line such that $\gamma = \sigma_\ell \circ \sigma_a \circ \sigma_c$. Then $\varphi_P = \sigma_c \circ \sigma_a$ so that $\gamma \circ \varphi_P = \sigma_\ell$ and the line symmetries are the reflections $\gamma^{2n} \circ \sigma_\ell$ with $n \in \mathbb{Z}$. The frieze group of F_5 is $\mathcal{F}_5 = \langle \gamma, \varphi_P \rangle = \{\gamma^n \circ \varphi_P^m : n \in \mathbb{Z};\ m = 0, 1\}$. Note that $\mathcal{F}_3$, $\mathcal{F}_4$, and $\mathcal{F}_5$ are isomorphic groups.

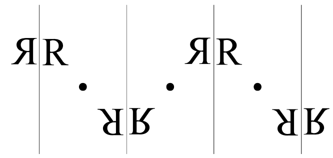


Figure 4.9: Frieze pattern F_5.

In Figure 4.10 we picture the frieze pattern F_6, which has a unique horizontal line of symmetry c. Thus the frieze group of F_6 is $\mathcal{F}_6 = \langle \tau, \sigma_c \rangle = \{\tau^n \circ \sigma_c^m : n \in \mathbb{Z};\, m = 0, 1\}$. The reader should check that $\mathcal{F}_6$ is abelian (see Exercise 5). Consequently $\mathcal{F}_6$ is not isomorphic to groups $\mathcal{F}_3$, $\mathcal{F}_4$ and $\mathcal{F}_5$.

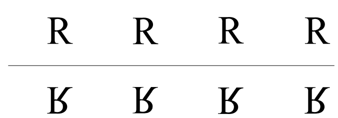


Figure 4.10: Frieze pattern F_6.

The final frieze group F_7 has vertical line symmetry and a unique horizontal line of symmetry c (see Figure 4.11). Let ℓ be a vertical line of symmetry and let τ be a basic translation. Then the vertical line symmetries are the reflections $\tau^n \circ \sigma_\ell$ with $n \in \mathbb{Z}$ and the point $P = c \cap \ell$ is a point of symmetry since $\varphi_P = \sigma_c \circ \sigma_\ell$. Thus the halfturn symmetries are the halfturns $\tau^n \circ \varphi_P$ with $n \in \mathbb{Z}$. The frieze group of F_7 is $\mathcal{F}_7 = \langle \tau, \sigma_c, \sigma_\ell \rangle = \left\{\tau^n \circ \sigma_c^m \circ \sigma_\ell^k : n \in \mathbb{Z};\, m, n = 0, 1\right\}$.

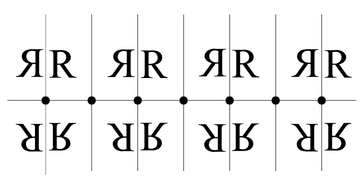


Figure 4.11: Frieze pattern F_7.

We collect the observations above as a theorem, however the proof that this list exhausts all possibilities is omitted:

Theorem 166 *Every frieze group is one of the following:*

$$\begin{array}{lll} \mathcal{F}_1 = \langle \tau \rangle & \mathcal{F}_2 = \langle \gamma \rangle & \\ \mathcal{F}_3 = \langle \tau, \sigma_\ell \rangle & \mathcal{F}_4 = \langle \tau, \varphi_P \rangle & \mathcal{F}_5 = \langle \gamma, \varphi_P \rangle \\ \mathcal{F}_6 = \langle \tau, \sigma_c \rangle & & \\ \mathcal{F}_7 = \langle \tau, \sigma_c, \sigma_\ell \rangle & & \end{array}$$

where τ is a basic translation, γ is a glide reflection such that $\gamma^2 = \tau$, ℓ is a vertical line of symmetry, P is a point of symmetry and c is the unique horizontal line of symmetry.

The following flowchart can be used to identify the frieze group associated with a particular frieze pattern:

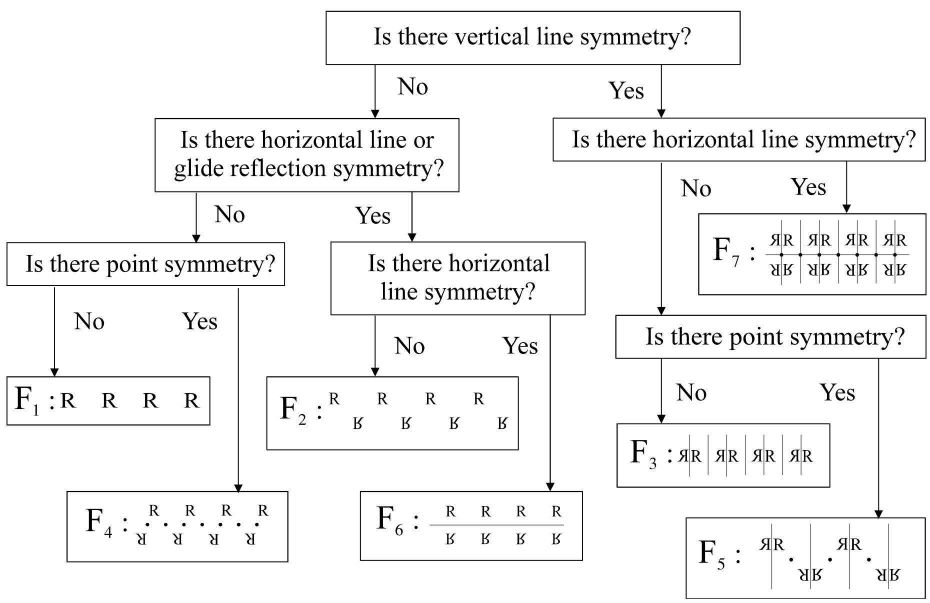


Figure 4.12. Recognition flowchart for frieze patterns

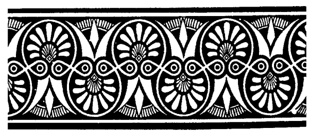
Exercises

1. Find at least two friezes in your campus architecture and identify their frieze groups.

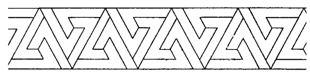
2. Find the frieze group for the pattern in Figure 4.4.

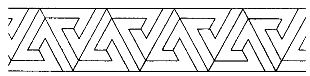
3. Prove that frieze group $\mathcal{F}_6$ is abelian.

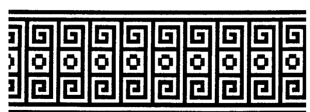
4. Identify the frieze groups for the following:

(a) 

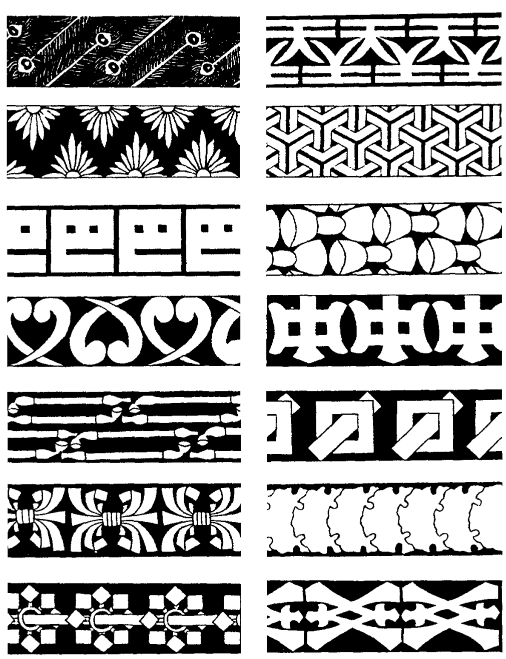
(b) 

(c) 

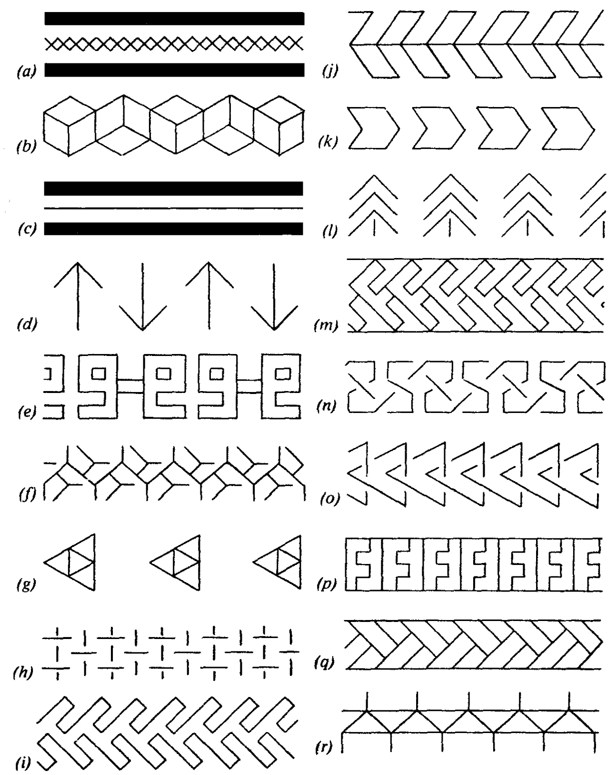
(d) 

(e) 

5. Identify the frieze groups of the following friezes taken from Theodore Menten's *Japanese Border Designs* in the Dover Pictorial Archive Series:



6. Identify the frieze groups for the following figures that are friezes:



4.5 The Wallpaper Groups

A *tessellation* (or *tiling*) of the plane is a collection of plane figures that fills the plane with no overlaps and no gaps. A "wallpaper pattern" is a special kind of tessellation, which we now define.

Definition 167 *Translations $\tau_{\mathbf{v}}$ and $\tau_{\mathbf{w}}$ are independent if and only if $\mathbf{v}$ and $\mathbf{w}$ are linearly independent.*

Definition 168 *A wallpaper pattern is a plane figure W with independent basic translations in two directions. A wallpaper group is the symmetry group of a wallpaper pattern.*

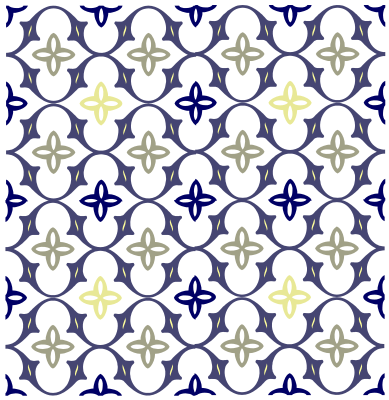


Figure 4.19: A typical wallpaper pattern.

Definition 169 *Let W be a wallpaper pattern with independent basic translations τ_1 and τ_2. Given any point A, let $B = \tau_1(A)$, $C = \tau_2(B)$, and $D = \tau_2(A)$. The unit cell with respect to A, τ_1, and τ_2 is the plane region bounded by parallelogram $\square ABCD$. The translation lattice determined by A, τ_1, and τ_2 is the set of points $L_{\tau_1,\tau_2}(A) = \{(\tau_2^n \circ \tau_1^m)(A) \mid m, n \in \mathbb{Z}\}$; this lattice is square, rectangular, or rhombic if and only if the unit cell with respect to A, τ_1 and τ_2 is square, rectangular or rhombic.*

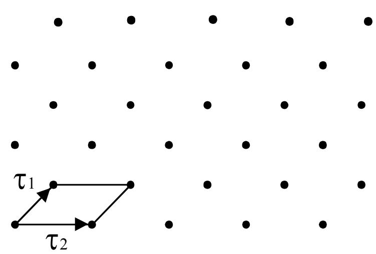


Figure 4.20: A typical translation lattice and unit cell.

Proposition 170 *Let W be a wallpaper pattern with independent basic translations τ_1 and τ_2. If τ is any translational symmetry of W, there exist integers m and n such that $\tau = \tau_2^n \circ \tau_1^m$.*

Proof. Let A be any point in a wallpaper pattern W, and let τ be a translational symmetry of W. Then τ fixes the translation lattice $L_{\tau_1,\tau_2}(A)$, and there exist integers m and n such that $\tau(A) = (\tau_2^n \circ \tau_1^m)(A)$. ■

Definition 171 *Let $n > 1$. A point P is an n-center of a wallpaper pattern if and only if the subgroup of rotational symmetries centered at P is C_n.*

For example, each vertex of the hexagonal tessellation pictured in Figure 4.14 is a 3-center.

Whereas our goal is to emphasize visual aspects and recognition techniques, we present a recognition algorithm but omit the proof of the fact that every wallpaper pattern is one of the seventeen specified types.

Theorem 172 *The symmetries of a wallpaper pattern fix the set of n-centers, i.e., if P is an n-center of W and α is a symmetry of W, then $\alpha(P)$ is an n-center of W.*

Proof. Let W be a wallpaper pattern with symmetry group $\mathcal{W}$ and let be P an n-center of W. Since C_n is the subgroup of rotational symmetries with center P, there is a smallest positive real number Θ such that $\rho^n_{P,\Theta} = \iota$. Now if $\alpha \in \mathcal{W}$ and $Q = \alpha(P)$, then $\alpha \circ \rho_{P,\Theta} \circ \alpha^{-1} = \rho_{Q,\pm\Theta} \in \mathcal{W}$ by closure and $\rho^n_{Q,\pm\Theta} = \left(\alpha \circ \rho_{P,\Theta} \circ \alpha^{-1}\right)^n = \alpha \circ \rho^n_{P,\Theta} \circ \alpha^{-1} = \iota$. But $\rho_{Q,\Theta} \in \mathcal{W}$ if and only if $\rho_{Q,-\Theta} \in \mathcal{W}$ so $\rho^n_{Q,\Theta} = \iota$. Thus Q is an m-center for some $m \leq n$. By the same reasoning, $\left(\alpha^{-1}\right) \circ \rho_{Q,\Theta} \circ \left(\alpha^{-1}\right)^{-1} = \rho_{P,\pm\Theta} \in \mathcal{W}$ implies that $\rho^m_{P,\Theta} = \iota$, in which case P is an n-center with $n \leq m$. Therefore $m = n$ and Q is an n-center as claimed. ■

Two n-centers in a wallpaper patterns cannot be arbitrarily close to one another.

Definition 173 *Let τ be a translation with glide vector $\mathbf{v}$. The length of τ, denoted by $\|\tau\|$, is the norm $\|\mathbf{v}\|$.*

Theorem 174 *Let W be a wallpaper pattern and let τ be a translational symmetry of shortest length. If A and B are distinct n-centers of W, then $AB \geq \frac{1}{2}\|\tau\|$.*

Proof. Let $n > 1$ and consider distinct n-centers A and B. Then $\rho_{A,360/n}$ and $\rho_{B,360/n}$ are elements of the wallpaper group $\mathcal{W}$. By closure and the Angle Addition Theorem, $\rho_{B,360/n} \circ \rho_{A,-360/n}$ is a non-identity translation in $\mathcal{W}$. Since every translation in $\mathcal{W}$ is generated by two basic translations τ_1 and τ_2, there exist integers i and j, not both zero, such that $\rho_{B,360/n} \circ \rho_{A,-360/n} = \tau_2^j \circ \tau_1^i$, or equivalently,

$$\rho_{B,360/n} = \tau_2^j \circ \tau_1^i \circ \rho_{A,360/n}.$$

Consider the point A_{ij} in the translation lattice determined by A given by

$$A_{ij} = \left(\tau_2^j \circ \tau_1^i\right)(A) = \left(\tau_2^j \circ \tau_1^i\right)\left(\rho_{A,360/n}(A)\right) = \rho_{B,360/n}(A).$$

Note that $A_{ij} \neq A$ since i and j are not both zero. Thus $AA_{ij} \geq \|\tau\|$. Now if $n = 2$, $A_{ij} = \varphi_B(A)$; and if $n > 2$, $\triangle ABA_{ij}$ is isosceles. In either case, $AB = BA_{ij}$. But $AB + BA_{ij} \geq AA_{ij}$ by the triangle inequality so it follows that $2AB \geq \|\tau\|$. ■

The next theorem, which was first proved by the Englishman W. Barlow in the late 1800's, is quite surprising. It tells us that wallpaper patterns cannot have 5-centers; consequently, crystalline structures cannot have pentagonal symmetry.

Theorem 175 *(The Crystallographic Restriction) If P is an n-center of a wallpaper pattern W, then $n \in \{2,3,4,6\}$.*

Proof. Let P be an n-center of W and let τ be a translation of shortest length. We first show that W has an n-center $Q \neq P$ whose distance from P is a minimum. Suppose no such Q exists and consider any n-center $Q_1 \neq P$. Since PQ_1 is not a minimum, there is an n-center $Q_2 \neq P$ such that $PQ_1 > PQ_2$. Continuing in this manner, then there is an infinite sequence of n-centers $\{Q_k \neq P\}$ such that $PQ_1 > PQ_2 > \cdots$. But $PQ_k \geq \frac{1}{2}\|\tau\|$ for all k by Theorem 174. Hence $\{PQ_k\}$ is a strictly decreasing sequence of positive real numbers converging to $M \geq \frac{1}{2}\|\tau\|$, i.e., given $\epsilon > 0$, there is a positive integer N such that if $k > N$ then $M < PQ_k < M + \epsilon$. Consequently, infinitely many n-centers Q_k lie within ϵ of the circle centered at P of radius M, which is impossible since $Q_iQ_j \geq \frac{1}{2}\|\tau\|$ for all i, j. So choose an n-center $Q \neq P$ such that PQ is a minimum and let $R = \rho_{Q,360/n}(P)$ and let $S = \rho_{R,360/n}(Q)$. Then R and S are n-centers by Theorem 172, and $PQ = QR = RS$. If $S = P$, then $\triangle PQR$ is equilateral in which case the rotation angle is $60°$ and $n = 6$. If $S \neq P$, then by the choice of Q, $SP \geq PQ = QR = RS$ in which case the rotation angle is at least $90°$ and $n \leq 4$. Therefore n is either $2, 3, 4$, or 6. ■

Corollary 176 *A wallpaper pattern with a 4-center has no 3 or 6-centers.*

Proof. If P is a 3-center and Q is a 4-center of a wallpaper pattern W, the corresponding wallpaper group $\mathcal{W}$ contains the rotations $\rho_{P,120}$ and $\rho_{Q,-90}$. By closure, $\mathcal{W}$ also contains the $30°$ rotation $\rho_{P,120} \circ \rho_{Q,-90}$, which generates C_{12}. Therefore there is an n-center of W with $n \geq 12$. But this contradicts Theorem 175. Similarly, if Q is a 4-center and R is a 6-center of W, there is also an n-center of W with $n \geq 12$ since $\rho_{R,-60} \circ \rho_{Q,90}$ is a $30°$ rotation. ■

In addition to translational symmetry, wallpaper patterns can have line symmetry, glide reflection symmetry, and $180°$, $120°$, $90°$ or $60°$ rotational symmetry. Since the only rotational symmetries in a frieze group are halfturns, it is not surprising to find more wallpaper groups than frieze groups. In fact, there are seventeen!

Throughout this discussion, W denotes a wallpaper pattern. We use the international standard notation to denote the various wallpaper groups. Each symbol is a string of letters and integers selected from p, c, m, g and $1, 2, 3, 4, 6$. The letter p stands for *primitive translation lattice.* The points in a primitive translation lattice are the vertices of parallelograms with no interior points of symmetry. When a point of symmetry lies at the center of some unit cell, we use the letter c. The letter m stands for *mirror* and indicates lines of symmetry;

the letter g indicates glide reflection symmetry. Integers indicate the maximum order of the rotational symmetries of W.

There are four symmetry types of wallpaper patterns with no n-centers. These are analyzed as follows: If W has no line symmetry or glide reflection symmetry, the corresponding wallpaper group consists only of translations and is denoted by $p1$. If W has glide reflection symmetry but no lines of symmetry, the corresponding wallpaper group is denoted by pg. There are two ways that both line symmetry and glide reflection symmetry can appear in W: (1) the axis of some glide reflection symmetry is not a line of symmetry and (2) the axis of every glide reflection symmetry is a line of symmetry. The corresponding wallpaper groups are denoted by cm and pm, respectively.

There are five symmetry types whose n-centers are all 2-centers. If W has neither lines of symmetry nor glide reflection symmetries, the corresponding wallpaper group is denoted by $p2$. If W has no line symmetry but has glide reflection symmetry, the corresponding group is denoted by pgg. If W has parallel lines of symmetry, the corresponding group is denoted by pmg. If W has lines of symmetry in two directions, there are two ways to configure them relative to the 2-centers in W: (1) all 2-centers lie on a line of symmetry and (2) not all 2-centers lie on a line of symmetry. The corresponding wallpaper groups are denoted by pmm and cmm, respectively.

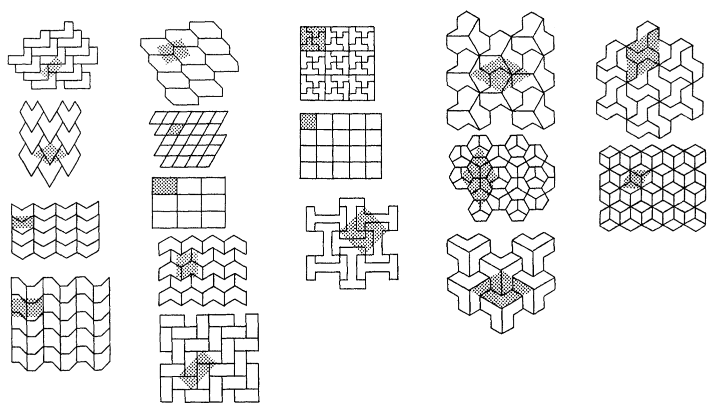
Three wallpaper patterns have n-centers whose smallest rotation angle is $90°$. Those with no lines of symmetry have wallpaper group $p4$. Those with lines of symmetry in four directions have wallpaper group $p4m$; other patterns with lines of symmetry have wallpaper group $p4g$.

Three symmetry types have n-centers whose smallest rotation angle is $120°$. Those with no lines of symmetry have wallpaper group $p3$. Those whose 3-centers lie on lines of symmetry have wallpaper group $p3m1$; those with some 3-centers off lines of symmetry have wallpaper group $p31m$.

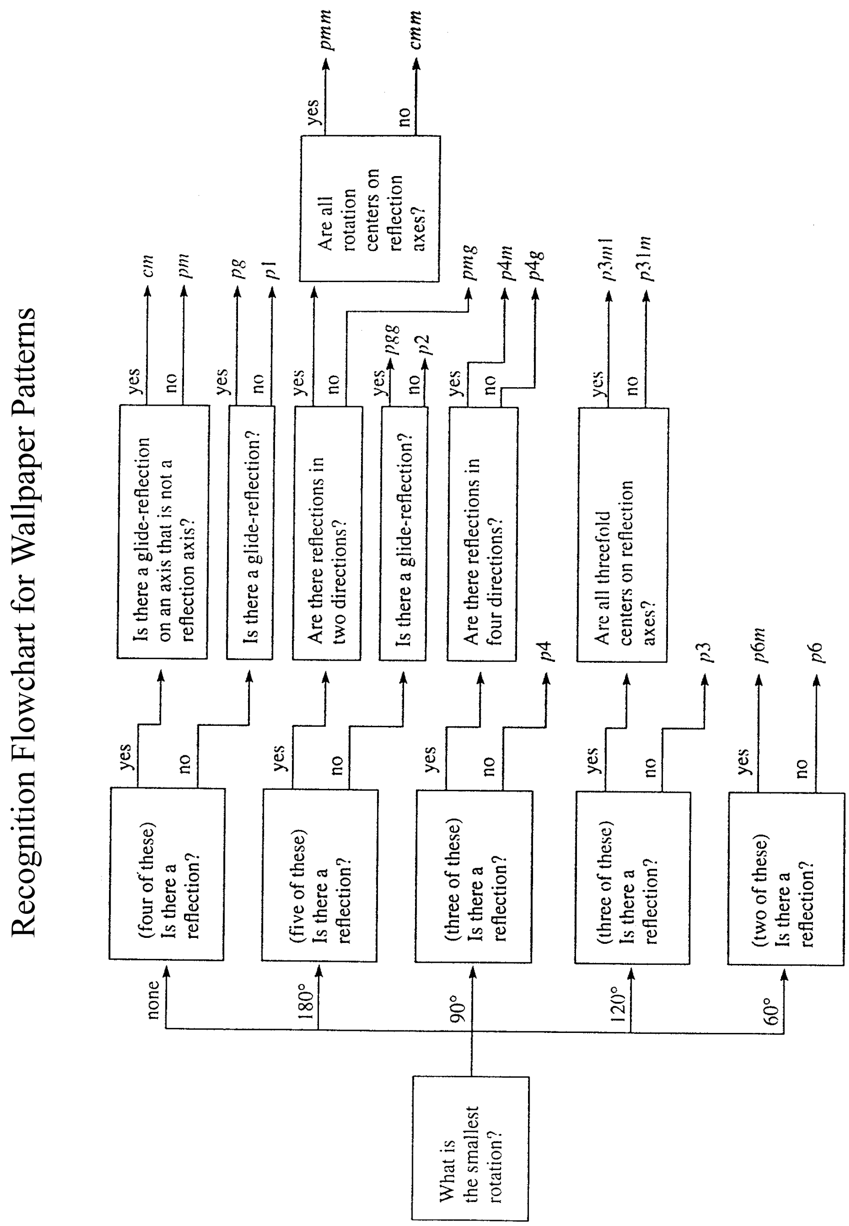
Finally, two symmetry types have n-centers whose smallest rotation angle is $60°$. Those with line symmetry have wallpaper group $p6m$; those with no line symmetry have wallpaper group $p6$.

Theorem 177 *Every wallpaper group is one of the following:*

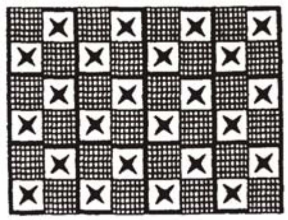
$p1$	$p2$	$p4$	$p3$	$p6$
cm	cmm	$p4m$	$p3m1$	$p6m$
pm	pmm	$p4g$	$p31m$	
pg	pmg			
	pgg			



The following flowchart can be used to identify the wallpaper group associated with a particular wallpaper pattern:



Example 178 *Here are some wallpaper patterns from around the world. Try your hand at identifying their respective wallpaper groups.*



FIJI p4m

SPAIN pmm

EGYPT pmg

JAPAN cm

GERMANY cmm

ITALY P4m

EGYPT P4g

CHINA p3m1

We conclude our discussion of wall paper patterns with a brief look at "edge tessellations", which have a simple and quite beautiful classification.

Definition 179 *An edge tessellation is tessellation of the plane is generated by reflecting a polygon in its edges.*

Obviously, regular hexigons, rectangles, and equilateral, 60-right and isosceles right triangles generate edge tessellations, but are there others? The complete answer was discovered by Millersville University students Andrew Hall, Joshua York, and Matthew Kirby in the spring of 2009, and we present it here as a theorem. The proof, which follows easily from the Crystallographic Restriction, is left to the reader (see Exercise 8 at the end of this section).

Theorem 180 *A polygon generating a edge tessellation is one of the following eight types: a rectangle; an equilateral,* 60*-right, isosceles right, or* 120*-isosceles triangle; a* 120*-rhombus; a* 60-90-120 *kite; or a regular hexagon.*

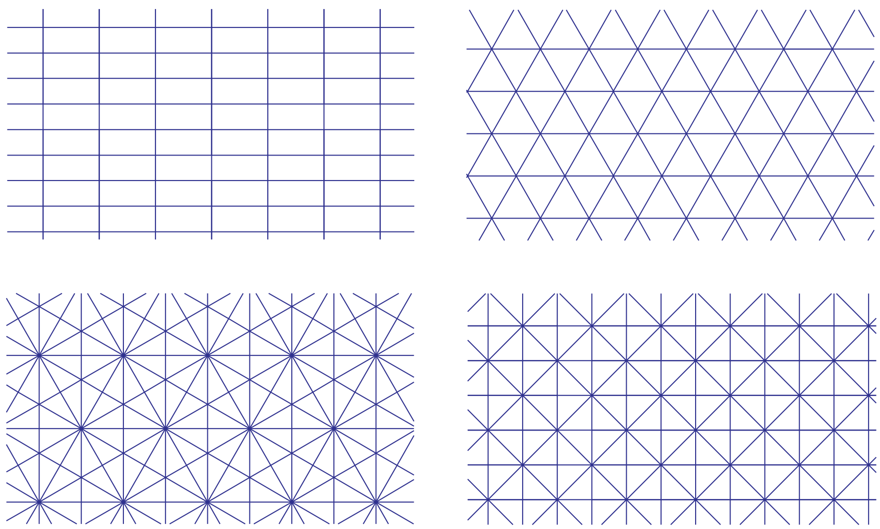


Figure 4.13. Edge tessellations generated by non-obtuse polygons.

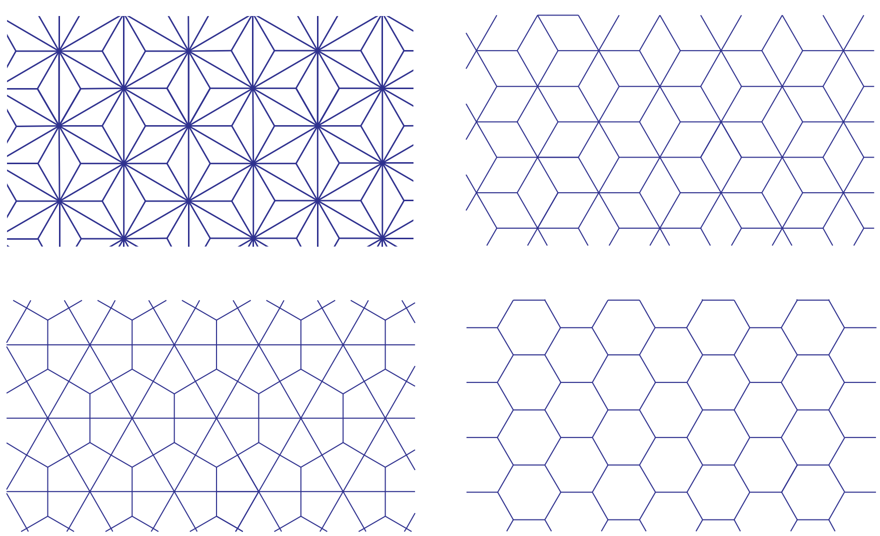


Figure 4.14. Edge tessellations generated by obtuse polygons.

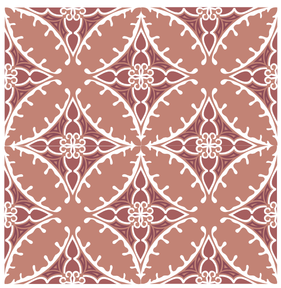
Edge tessellations represent 3 of the 17 symmetry types of wallpaper patterns. Non-square rectangles generate patterns of type pmm, isosceles right triangles and squares generate patterns of type $p4m$, and the other six edge tessellations have type $p6m$.

Exercises

1. Identify the wallpaper group for the pattern in Figure 4.12.

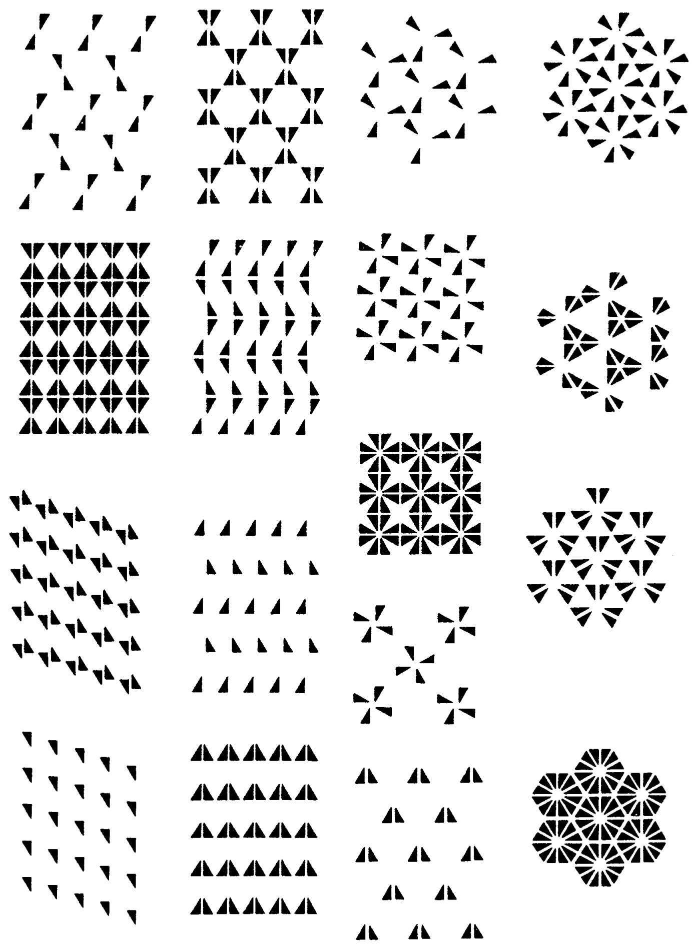
2. Find at least two different wallpaper patterns on your campus and identify their wallpaper groups.

3. Identify the wallpaper groups for the following patterns.

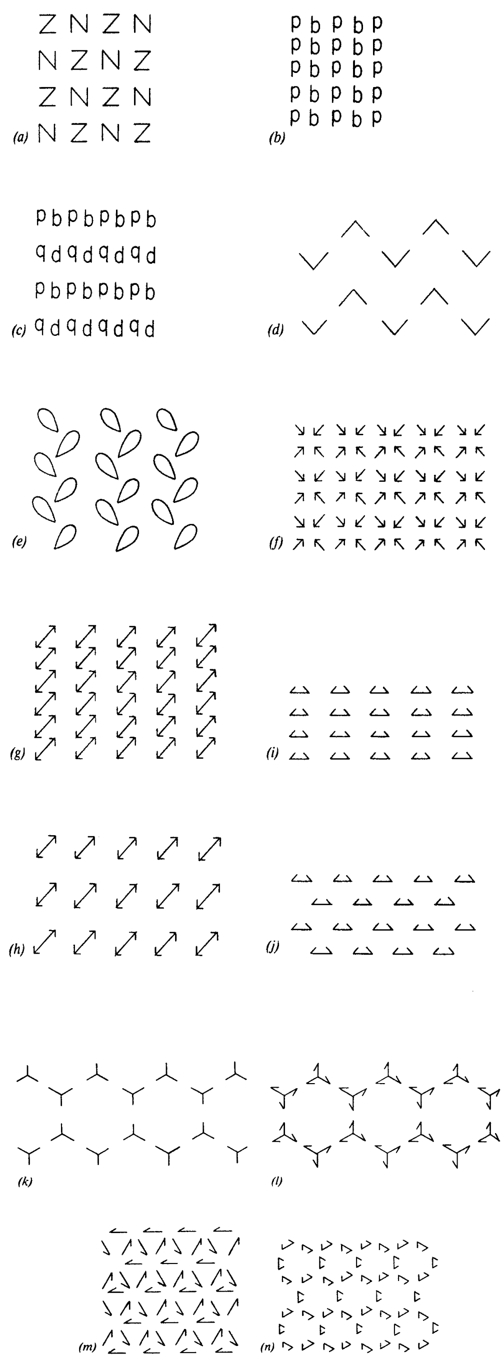



4. Prove that if A and B are distinct points of symmetry for a plane figure F, the symmetry group of F contains a non-identity translation, and consequently has infinite order. (Hint: Consider all possible combinations of n and m such that A is an n-center and B is an m-center.)

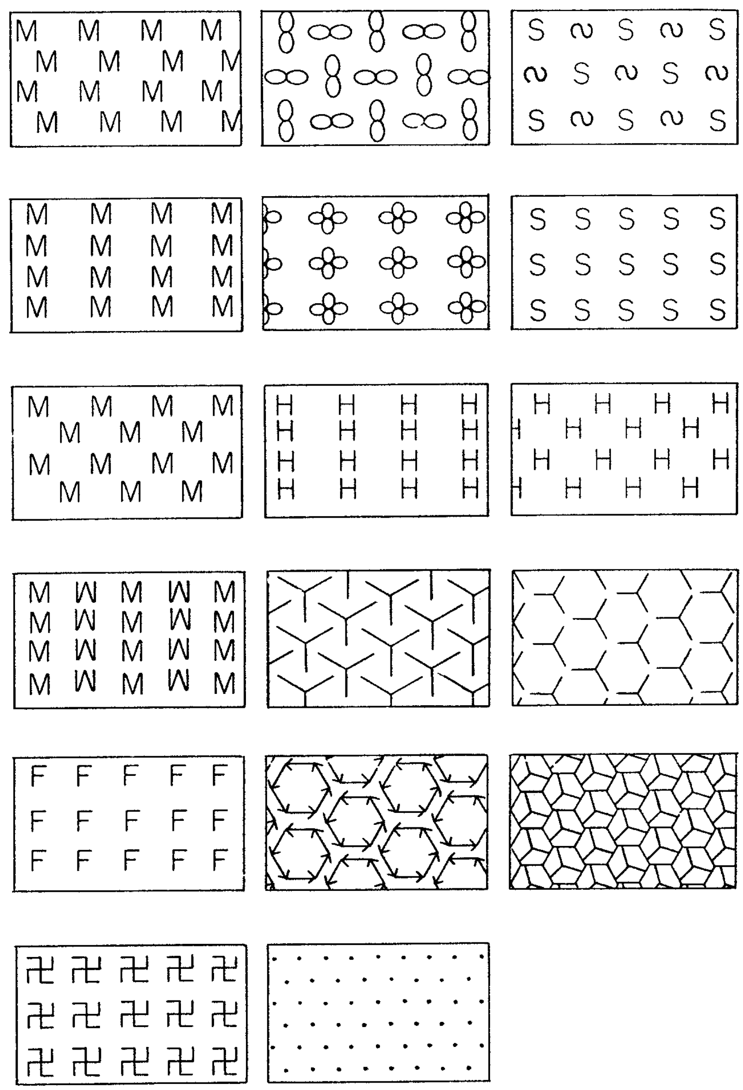
5. Identify the wallpaper groups for the following patterns:



6. Identify the wallpaper groups for the following patterns:



7. Identify the wallpaper groups for the following patterns:



8. Prove Theorem 5.2: A polygon generating a edge tessellation is one of the following eight types: a rectangle; an equilateral, 60-right, isosceles right, or 120-isosceles triangle; a 120-rhombus; a 60-90-120 kite; or a regular hexagon. (Hint: Using Crystallographic Restriction and the interior angle sum of a n-gon, set up and solve a system of two linear equations.whose

unknowns are the number of 30°, 45°, 60°, 90°, and 120° interior angles in the n-gon.)

Chapter 5

Similarity

In this chapter we consider transformations that magnify or stretch the plane. Such transformations are called "similarities" or "size transformations." One uses similarities to relate two similar triangles in much the same way one uses isometries to relate two congruent triangles. Particularly important are the "stretch" transformations, which linearly expand the plane radially outward from some fixed point. Stretch transformations are an essential component of every non-isometric similarity. Indeed, we shall prove that every similarity is one of the following four distinct types: an isometry, a stretch, a stretch reflection or a stretch rotation. We begin with another look at the family of dilatations, which we introduced in Section 1.3.

5.1 Plane Similarities

Definition 181 *Let $r > 0$. A similarity of ratio r is a transformation $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ with the following property: Given points P and Q, their images $P' = \alpha(P)$ and $Q' = \alpha(Q)$ satisfy $P'Q' = rPQ$.*

Note that a similarity of ratio 1 is an isometry. In particular, if a similarity α has distinct fixed points P and Q, then $P' = P$ and $Q' = Q$ so that $P'Q' = PQ$, .the ratio of similarity $r = 1$, and α is the identity or a reflection.. Furthermore, if α has three non-collinear fixed points, then $\alpha = \iota$ by Theorem 72. This proves:

Proposition 182 *A similarity of ratio 1 is an isometry; a similarity with two or more distinct fixed points is a reflection or the identity; a similarity with three non-collinear fixed points is the identity.*

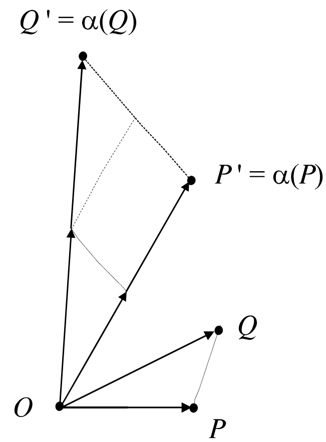


Figure 5.1: A similarity α of ratio 2.

Proposition 183 *The set of all similarities is a group under composition of functions.*

Proof. The proof is left to the reader. ■

Corollary 184 ***(Three Points Theorem for Similarities)*** *Two similarities that agree on three non-collinear points are equal.*

Proof. Let α and β be similarities, and let A, B, and C be points such that

$$\alpha(A) = \beta(A), \ \alpha(B) = \beta(B), \text{ and } \alpha(C) = \beta(C). \tag{5.1}$$

By applying β^{-1} to both sides of the equations in line (5.1) we obtain

$$\left(\beta^{-1} \circ \alpha\right)(A) = A, \ \left(\beta^{-1} \circ \alpha\right)(B) = B, \text{ and } \left(\beta^{-1} \circ \alpha\right)(C) = C.$$

But $\beta^{-1} \circ \alpha$ is a similarity by 183, hence $\beta^{-1} \circ \alpha = \iota$ by Proposition 182, and $\alpha = \beta$. ■

The proof of our next result is left as an exercise following Section 3.

Definition 185 *Let C be a point and let $r > 0$. A* <u>*stretch of ratio r about C*</u> *is the transformation $\xi_{C,r} : \mathbb{R}^2 \to \mathbb{R}^2$ with the following property: If P is a point, then $P' = \xi_{C,r}(P)$ is the unique point on $\overrightarrow{CP}$ such that $CP' = rCP$.*

Note that if $C' = \xi_{C,r}(C)$, then $CC' = rCC = 0$ and $C' = C$.

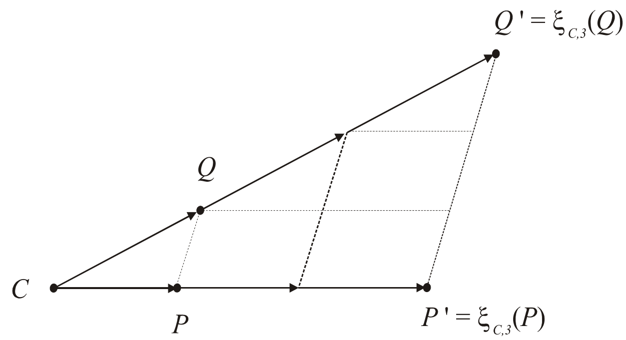


Figure 5.2: A stretch about C of ratio 3.

Of course, the identity is a stretch of ratio 1 about every point C. Furthermore, the equations of a stretch about the origin are

$$\xi_{O,r}:\begin{cases} x' = rx \\ y' = ry. \end{cases}$$

To obtain the equations of a stretch about $C = \begin{bmatrix} a \\ b \end{bmatrix}$, conjugate $\xi_{O,r}$ by the translation $\tau_{\mathbf{OC}}$, i.e.,

$$\xi_{C,r} = \tau_{\mathbf{OC}} \circ \xi_{O,r} \circ \tau_{\mathbf{OC}}^{-1}.$$

Composing equations gives

$$\xi_{C,r}:\begin{cases} x' = rx + (1-r)\,a \\ y' = ry + (1-r)\,b. \end{cases}$$

Proposition 186 *A stretch preserves orientation.*

Proof. First observe that a stretch about the origin preserves orientation. Choose an orientation $\left\{\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right\}$ of $\mathbb{R}^2$. Then $\mathbf{u}' = \xi_{O,r}\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} ru_1 \\ ru_2 \end{bmatrix}$ and $\mathbf{v}' = \xi_{O,r}\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} rv_1 \\ rv_2 \end{bmatrix}$. But $\det[\mathbf{u}'|\,\mathbf{v}'] = \det\begin{bmatrix} ru_1 & rv_1 \\ ru_2 & rv_2 \end{bmatrix} = r^2\det\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} = r^2\det[\mathbf{u}|\,\mathbf{v}]$. Since $\det[\mathbf{u}'|\,\mathbf{v}']$ and $\det[\mathbf{u}|\,\mathbf{v}]$ have the same sign, $\xi_{O,r}$ preserves orientation. A general stretch $\xi_{C,r} = \tau_{\mathbf{OP}} \circ \xi_{O,r} \circ \tau_{\mathbf{OP}}^{-1}$ preserves orientation since translations preserve oritention by Exercise 11 in Section 215. ■

Exercises

1. One can use the following procedure to determine the height of an object: Place a mirror flat on the ground and move back until you can see the top of the object in the mirror. Explain how this works.

2. Find the ratio of similarity r for a similarity α such that $\alpha\left(\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\right)=\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ and $\alpha\left(\left[\begin{smallmatrix}3\\4\end{smallmatrix}\right]\right)=\left[\begin{smallmatrix}3\\4\end{smallmatrix}\right]$.

3. Find the point P and ratio of similarity r such that $\xi_{P,r}\left(\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]\right)=\left[\begin{smallmatrix}3x+7\\3y-5\end{smallmatrix}\right]$.

4. Prove that a stretch is bijective.

5. If α is a similarity of ratio r and β is a similarity of ratio s, prove that $\alpha\circ\beta$ is a similarity of ratio rs.

6. If α is a similarity of ratio r, prove that α^{-1} is a similarity of ratio $\frac{1}{r}$.

7. Prove Proposition 183: The set of all similarities is a group under composition of functions.

8. If α is a similarity of ratio r and A, B, and C are distinct non-collinear points, let $A'=\alpha(A)$, $B'=\alpha(B)$, and $C'=\alpha(C)$. Prove that $\angle ABC\cong\angle A'B'C'$.

9. Prove that similarities preserve betweenness.

5.2 Classification of Dilatations

In this section we observe that every dilatation is either a translation, a stretch or a stretch followed by a halfturn. We begin with a review of some facts about similar triangles.

Definition 187 *Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar, written $\triangle ABC\sim\triangle A'B'C'$, if and only if all three pairs of corresponding angles are congruent.*

Our next two theorems state some important facts about similar triangles.

Theorem 188 (AA) *Two triangles are similar if and only if two pairs of corresponding angles are congruent.*

The proof of Theorem 188 follows immediately from the fact that the interior angle sum of a triangle is an element of 180°.

Theorem 189 (Similar Triangles) *Two triangles are similar if and only if the ratios of the lengths of their corresponding sides are equal, i.e.,*

$$\triangle ABC\sim\triangle A'B'C' \text{ if and only if } \frac{A'B'}{AB}=\frac{A'C'}{AC}=\frac{B'C'}{BC}.$$

Proof. Given $\triangle ABC$ and $\triangle A'B'C'$, let $r = A'B'/AB$ and consider the points $D = \xi_{A,r}(B)$ and $E = \xi_{A,r}(C)$. Since $\xi_{A,r}$ is a similarity of ratio r we have $AD = rAB = A'B'$, $AE = rAC$, and $DE = rBC$; since $\xi_{A,r}$ is a dilatation we have $\overleftrightarrow{DE} \| \overleftrightarrow{BC}$ and corresponding angles are congruent: $\angle D \cong \angle B$ and $\angle E \cong \angle C$.

If $\triangle ABC \sim \triangle A'B'C'$, then $\angle B \cong \angle B'$ and $\angle C \cong \angle C'$ by definition, so that $\angle D \cong \angle B'$ and $\angle E \cong \angle C'$. But $AD = A'B'$ implies $\triangle ADE \cong \triangle A'B'C'$ by SAA. Thus $A'C' = AE = rAC$ and $B'C' = DE = rBC$ by CPCTC so that $r = A'B'/AB = A'C'/AC = B'C'/BC$.

If $A'B'/AB = A'C'/AC = B'C'/BC$, then $AD = rAB = A'B'$, $AE = rAC = A'C'$, and $DE = rBC = B'C'$ so that $\triangle ADE \cong \triangle A'B'C'$ by SSS. Therefore $\angle A \cong \angle A'$, $\angle B \cong \angle D \cong \angle B'$ and $\angle C \cong \angle E \cong \angle C'$ by CPCTC, and $\triangle ABC \sim \triangle A'B'C'$ by definition. ■

Some authors use Theorem 189 as an alternative definition of similar triangles. Our next theorem, which is a standard result from Euclidean geometry, will be applied in the proof of Proposition 191:

Theorem 190 *Given $\triangle CP'Q'$, let P be a point on $\overline{CP'}$ and let Q be a point on $\overline{CQ'}$. Then $\overline{PQ}$ divides $\overline{CP'}$ and $\overline{CQ'}$ proportionally if and only if $\overleftrightarrow{PQ} \parallel \overleftrightarrow{P'Q'}$.*

Proof. If $\overleftrightarrow{PQ} \parallel \overleftrightarrow{P'Q'}$, then $\angle CPQ \cong \angle CP'Q'$ and $\angle CQP \cong \angle CQ'P'$ (corresponding angles). Thus $\triangle CP'Q' \sim \triangle CPQ$ by AA so that

$$= \frac{CQ'}{CQ}.$$

But $CP' = CP + PP'$ and $CQ' = CQ + QQ'$; therefore

$$\frac{PP'}{CP} = \frac{CP' - CP}{CP} = \frac{CP'}{CP} - 1 = \frac{CQ'}{CQ} - 1 = \frac{CQ' - CQ}{CQ} = \frac{QQ'}{CQ}$$

so that $\overline{PQ}$ divides sides $\overline{CP'}$ and $\overline{CQ'}$ of $\triangle CP'Q'$ proportionally. Conversely, assume that

$$\frac{PP'}{CP} = \frac{QQ'}{CQ} \tag{5.2}$$

and construct the line through Q' parallel to $\overleftrightarrow{PQ}$ intersecting $\overleftrightarrow{CP'}$ at point R (see Figure 5.3).

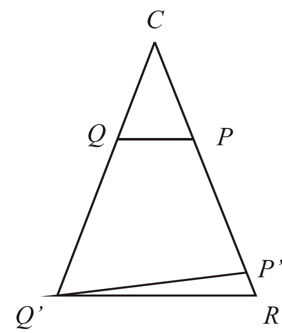


Figure 5.3.

Then $\angle CPQ \cong \angle CRQ'$ and $\angle CQP \cong \angle CQ'R$ (corresponding angles) so that $\triangle CRQ' \sim \triangle CPQ$ by AA and

$$\frac{CR}{CP} = \frac{CQ'}{CQ}.$$

But $CR = CP + PR$ and $CQ' = CQ + QQ'$; therefore

$$\frac{PR}{CP} = \frac{CR - CP}{CP} = \frac{CR}{CP} - 1 = \frac{CQ'}{CQ} - 1 = \frac{CQ' - CQ}{CQ} = \frac{QQ'}{CQ}. \tag{5.3}$$

Combining equations (5.2) and (5.3) we have

$$\frac{PP'}{CP} = \frac{QQ'}{CQ} = \frac{PR}{CP}$$

so that $PP' = PR = PP' + P'R$ and $P'R = 0$. Therefore $P' = R$ as desired. ■

Proposition 191 *Let C be a point and let $r > 0$. Then $\xi_{C,r}$ is both a dilatation and a similarity of ratio r.*

Proof. Let m be a line. If C is on m, then $\xi_{C,r}(m) = m$ by definition, and $\xi_{C,r}(\ell) \| \ell$. So assume C is off m, and choose distinct points P, Q, and R on m (see Figure 5.4).

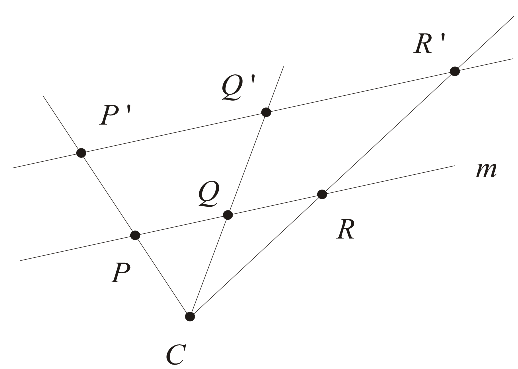


Figure 5.4.

By definition, $CP' = rCP$, $CQ' = rCQ$, and $CR' = rCR$. If $r < 1$, then $\overline{P'Q'}$ divides sides $\overline{CP}$ and $\overline{CQ}$ in $\triangle CPQ$ proportionally; on the other hand, if $r > 1$, then $\overline{PQ}$ divides sides $\overline{CP'}$ and $\overline{CQ'}$ in $\triangle CP'Q'$ proportionally. In either case, $\overleftrightarrow{PQ} \| \overleftrightarrow{P'Q'}$ by Theorem 190. Similarly, $\overleftrightarrow{PR} \| \overleftrightarrow{P'R'}$. But $\overleftrightarrow{P'Q'} \| \overleftrightarrow{P'R'}$ implies that P', Q' and R' are collinear. But $m = \overleftrightarrow{PR} = \overleftrightarrow{PQ}$ and $\xi_{C,r}(m) = \overleftrightarrow{P'R'} = \overleftrightarrow{P'Q'}$; therefore $\xi_{C,r}$ is a collineation. But $\xi_{C,r}$ is also a dilatation since $m \| \xi_{C,r}(m)$. Now consider parallel lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{P'Q'}$ cut by transversal $\overleftrightarrow{CP}$. Since corresponding angles are equal, $\triangle CPQ \sim \triangle CP'Q'$ by AA. But the lengths of corresponding sides in similar triangles are proportional, hence $P'Q'/PQ = CP'/CP = r$ and $P'Q' = rPQ$. Therefore $\xi_{C,r}$ is a similarity of ratio r as claimed. ■

Definition 192 *Let C be a point and let $r > 0$. A* <u>*dilation about C of ratio r*</u>*, denoted by $\delta_{C,r}$, is either a stretch about C of ratio r or a stretch about C of ratio r followed by a halfturn about C.*

Note that the identity and all halfturns are dilations of ratio 1. In general, if $P - C - Q$, there is a positive real number r such that $Q = \delta_{C,r}(P)$.

Theorem 193 *Let C be a point and let $r > 0$. Then $\delta_{C,r}$ is both a dilatation and a similarity of ratio r.*

Proof. If $\delta_{C,r} = \xi_{C,r}$, the conclusion is the statement in Proposition 191. So assume that $\delta_{C,r} = \varphi_C \circ \xi_{C,r}$. A halfturn is an isometry and hence a similarity; it is also a dilatation by Proposition 58, part 2. So on one hand, $\varphi_C \circ \xi_{C,r}$ is a composition of dilatations, which is a dilatation, and on the other hand $\varphi_C \circ \xi_{C,r}$ is a composition of similarities, which is a similarity by Proposition 183. Proof of the fact that $\varphi_C \circ \xi_{C,r}$ is a similarity of ratio r is left to the reader. ■

Theorem 194 *If $\overleftrightarrow{AB}$ and $\overleftrightarrow{DE}$ are distinct parallels, there is a unique dilatation α such that $D = \alpha(A)$ and $E = \alpha(B)$.*

Proof. First, we define a dilatation with the required property. Let $C = \tau_{\mathbf{AD}}(B)$, let $r = DE/DC$, and consider the dilation $\delta_{D,r}$ such that $E = \delta_{D,r}(C)$. Then

$$(\delta_{D,r} \circ \tau_{\mathbf{AD}})(A) = \delta_{D,r}(D) = D \text{ and } (\delta_{D,r} \circ \tau_{\mathbf{AD}})(B) = \delta_{D,r}(C) = E.$$

Since $\tau_{\mathbf{AD}}$ and $\delta_{D,r}$ are dilatations, $\delta_{D,r} \circ \tau_{\mathbf{AD}}$ is a dilatation with the required property. For uniqueness, let α be any dilatation such that $\alpha(A) = D$ and $\alpha(B) = E$. Let P be a point off $\overleftrightarrow{AB}$. To locate $P' = \alpha(P)$, note that P' is on the line ℓ through D parallel to $\overleftrightarrow{AP}$ and also on the line m through E parallel to $\overleftrightarrow{BP}$ (see Figure 5.5). Hence $P' = \ell \cap m$. Let Q be a point on $\overleftrightarrow{AB}$ distinct from A and let $Q' = \alpha(Q)$. Then Q' is on the line $\overleftrightarrow{DE}$ and also on the line n through P' parallel to $\overleftrightarrow{PQ}$. Hence $Q' = \overleftrightarrow{DE} \cap n$. In either case, P' and Q' are uniquely determined by the given points A, B, D, and E. Thus every similarity with the required property sends P to P' and Q to Q', and in particular,

$$(\delta_{D,r}\circ\tau_{\mathbf{AD}})(A) = \alpha(A), \quad (\delta_{D,r}\circ\tau_{\mathbf{AD}})(P) = \alpha(P), \text{ and } (\delta_{D,r}\circ\tau_{\mathbf{AD}})(Q) = \alpha(Q)$$

so that $\alpha = \delta_{D,r} \circ \tau_{\mathbf{AD}}$ by Corollary 184. ■

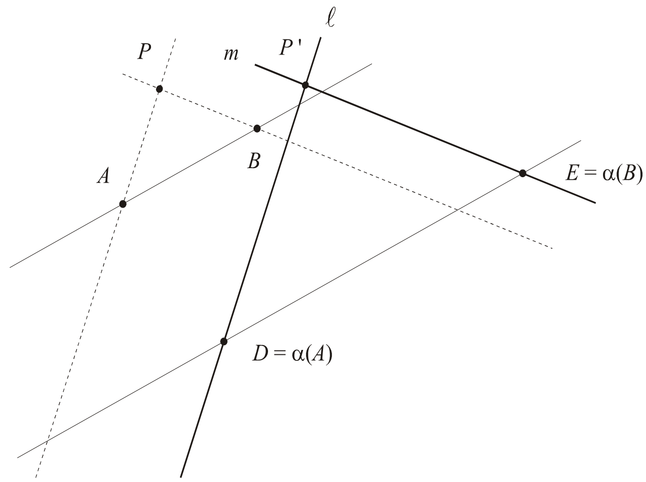


Figure 5.5.

Proposition 195 *If α is a dilatation A is a point distinct from $B = \alpha(A)$, then α fixes $\overleftrightarrow{AB}$.*

Proof. Let $C = \alpha(B)$ and note that $\overleftrightarrow{AB} \parallel \overleftrightarrow{BC}$ since α is a dilatation. Therefore $\overleftrightarrow{AB} = \overleftrightarrow{BC} = \alpha\left(\overleftrightarrow{AB}\right)$. ■

We are now able to determine all of the dilatations.

Theorem 196 *Every dilatation is a translation, halfturn or dilation.*

Proof. Let α be a dilatation. If α is an isometry, it either a translation or a halfturn by Theorem 47, Proposition 58, and Exercise 10 in Section 3.1. The identity, which is a trivial translation, is also a dilatation, so assume that $\alpha \neq \iota$. Choose a line ℓ distinct from $\ell' = \alpha(\ell)$, its parallel image, and choose distinct points A and B on ℓ. Since dilatations are collineations and collineations are bijective, $A' = \alpha(A)$ and $B' = \alpha(B)$ are distinct points on ℓ', and furthermore, α is the unique dilatation such that $A' = \alpha(A)$ and $B' = \alpha(B)$ by Theorem 194. We consider two cases:

<u>*Case 1:*</u> Assume $\overleftrightarrow{AA'} \parallel \overleftrightarrow{BB'}$. Then $\square AA'B'B$ is a parallelogram so that

$$\tau_{\mathbf{AA'}}(A) = A' \text{ and } \tau_{\mathbf{AA'}}(B) = B'.$$

Since translations are dilatations by Theorem 47, $\alpha = \tau_{\mathbf{AA'}}$ by uniqueness in Theorem 194.

<u>*Case 2:*</u> Assume $\overleftrightarrow{AA'} \cap \overleftrightarrow{BB'} = C$. Then $\alpha\left(\overleftrightarrow{AA'}\right) \cap \alpha\left(\overleftrightarrow{BB'}\right) = \overleftrightarrow{AA'} \cap \overleftrightarrow{BB'} = C$ by Proposition 195, and it follows that $\alpha(C) = C$. Since $\overleftrightarrow{AB} \neq \overleftrightarrow{A'B'}$ by assumption, C is off $\overleftrightarrow{AB}$; since A, A' and C are collinear C is also off $\overleftrightarrow{A'B'}$ (see Figures 6.5 and 6.6). Thus $\triangle ABC \sim \triangle A'B'C$ (by AA) with ratio of similarity $r = CA'/CA = CB'/CB$.

<u>*Subcase 2a:*</u> Assume $r = 1$. Then C is the midpoint of $\overline{AA'}$ and $\overline{BB'}$, in which case $\varphi_C(A) = A'$ and $\varphi_C(B) = B'$. Since halfturns are dilatations by Proposition 58, $\alpha = \varphi_C$ by uniqueness in Theorem 194 (see Figure 5.6).

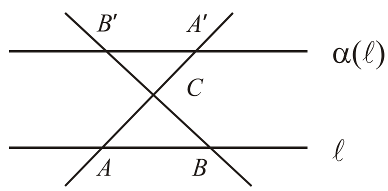


Figure 5.6.

<u>*Subcase 2b:*</u> Assume $r \neq 1$. Let $D = \xi_{C,r}(A)$ and $E = \xi_{C,r}(B)$; then D is the unique point on $\overrightarrow{CA}$ such that $CD = rCA$ and E is the unique point on $\overrightarrow{CB}$ such that $CE = rCB$. If $A - C - A'$, then C is the midpoint of $\overline{DA'}$ and $\overline{EB'}$, in which case $\left(\varphi_C \circ \xi_{C,r}\right)(A) = A'$ and $\left(\varphi_C \circ \xi_{C,r}\right)(B) = B'$. Otherwise, $\xi_{C,r}(A) = A'$ and $\xi_{C,r}(B) = B'$. Thus $\alpha = \delta_{C,r}$ by Theorem 193 and uniqueness in Theorem 194 (see Figure 5.7). ■

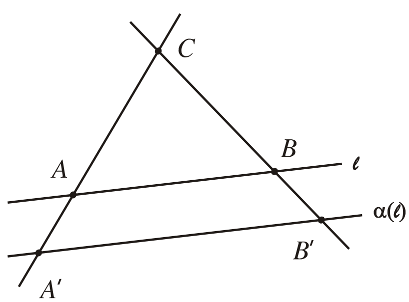


Figure 5.7.

Exercises

1. Let C be a point and let $r > 0$. Prove that the dilation $\varphi_C \circ \xi_{C,r}$ has ratio r.

2. Let $A = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $A' = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $B' = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Identify the (unique) dilatation α such that $\alpha(A) = A'$ and $\alpha(B) = B'$ as a translation, stretch or dilation. Determine the ratio of similarity and any fixed points.

3. Prove the AA Theorem for similarity: Two triangles are similar if and only if two pairs of corresponding angles are congruent.

5.3 Similarities as an Isometry and a Stretch

Given two congruent triangles, the Classification Theorem for Plane Isometries (Theorem 109) tells us there is exactly one isometry that maps one of two congruent triangles onto the other. A similar statement, which appears as Theorem 197 below, can be made for a pair of similar triangles, namely, there is exactly one similarity that maps one of two similar triangles onto the other. In this section we also observe that every similarity is a stretch followed by an isometry. This important fact will lead to the complete classification of all similarities in the next section.

Theorem 197 $\triangle ABC \sim \triangle A'B'C'$ *if and only if there is a unique similarity* α *such that* $A' = \alpha(A)$, $B' = \alpha(B)$, *and* $C' = \alpha(C)$.

Proof. Given similar triangles $\triangle ABC$ and $\triangle A'B'C'$, we first define a similarity that sends A to A', B to B' and C to C' then prove its uniqueness. The ratio of similarity $r = A'B'/AB$; let $D = \xi_{A,r}(B)$ and $E = \xi_{A,r}(C)$. Then $AD = rAB = A'B'$ and $AE = rAC = A'C'$. Furthermore, $\angle DAE = \angle BAC \cong \angle B'A'C'$ since corresponding angles are congruent and $\triangle ADE \cong \triangle A'B'C'$ by SAS. Let β be the isometry that maps $\triangle ADE$ congruently onto $\triangle A'B'C'$. Then $\beta \circ \xi_{A,r}$ is a similarity such that

$$\begin{aligned}
(\beta \circ \xi_{A,r})(A) &= \beta(A) = A', \\
(\beta \circ \xi_{A,r})(B) &= \beta(D) = B', \\
(\beta \circ \xi_{A,r})(C) &= \beta(E) = C'.
\end{aligned}$$

For uniqueness, let α be any similarity such that $A' = \alpha(A)$, $B' = \alpha(B)$, and $C' = \alpha(C)$. Then

$$(\beta \circ \xi_{A,r})(A) = \alpha(A), \quad (\beta \circ \xi_{A,r})(B) = \alpha(B), \text{ and } (\beta \circ \xi_{A,r})(C) = \alpha(C)$$

so that $\alpha = \beta \circ \xi_{A,r}$ by Corollary 184.

Conversely, suppose that α is a similarity of ratio r such that $A' = \alpha(A)$, $B' = \alpha(B)$, and $C' = \alpha(C)$. Then by definition, $A'B' = rAB$, $B'C' = rBC$, and $C'A' = rCA$ so that

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'},$$

and $\triangle ABC \sim \triangle A'B'C'$ by Theorem 189. ■

Definition 198 *Two plane figures s_1 and s_2 are <u>similar</u> if and only if there is a similarity α such that $\alpha(s_1) = s_2$.*

Note that if s_1 and s_2 in Definition 198 are lines, infinitely many similarities α satisfy $\alpha(s_1) = s_2$. Thus α is not necessarily unique. The proof of Theorem 197 seems to suggest that a similarity is a stretch about some point P followed by an isometry. In fact, this is true and very important.

Theorem 199 *If α is a similarity of ratio r and C is any point, there exists an isometry β such that*

$$\alpha = \beta \circ \xi_{C,r}.$$

Proof. Let α be a similarity of ratio r and let C be any arbitrarily chosen point. Then $\beta = \alpha \circ \xi_{C,r}^{-1}$ is an isometry since $\xi_{C,r}^{-1}$ has ratio $\frac{1}{r}$ and the composition $\alpha \circ \xi_{C,r}^{-1}$ has ratio $r \cdot \frac{1}{r} = 1$ by Exercise 5. Therefore $\alpha = \beta \circ \xi_{C,r}$. ■

Definition 200 *A <u>stretch rotation</u> is a non-identity stretch about some point C followed by a non-identity rotation about C.*

For example, a stretch rotation with rotation angle 180 is a particular kind of dilation.

Definition 201 *A <u>stretch reflection</u> is a non-identity stretch about some point C followed by a reflection in some line through C.*

Exercises

1. Which points and lines are fixed by a stretch rotation?

2. Which points and lines are fixed by a stretch reflection?

3. Prove that similarities are bijective. (HINT: Apply Exercise 4 in Section 5.1 and Theorem 199.)

4. Let C be a point, let $\Theta \in \mathcal{R}$, and let $r > 0$. Prove that $\rho_{C,\Theta} \circ \xi_{C,r} = \xi_{C,r} \circ \rho_{C,\Theta}$.

5. Let ℓ be a line, let C be a point on ℓ, and let $r > 0$. Prove that $\sigma_\ell \circ \xi_{C,r} = \xi_{C,r} \circ \sigma_\ell$.

5.4 Classification of Similarities

In this section we prove our premier result–the Classification Theorem for Similarities, which states that every similarity is either an isometry, a stretch, a stretch reflection, or a stretch rotation. The Classification Theorem is a consequence of our next theorem.

Theorem 202 *Every non-isometric similarity has a fixed point.*

Proof. Let α be a similarity. If α is a non-isometric dilatation, it is a dilation by Theorem 196. Since a dilation has a fixed point, it is sufficient to show that the statement holds when α is neither an isometry nor a dilatation. Choose a line ℓ that intersects its image $\ell' = \alpha(\ell)$ at the point $A = \ell \cap \ell'$ and let $A' = \alpha(A)$. If $A' = A$, then α has a fixed point as claimed. So assume that $A' \neq A$; then A' is on ℓ' and off ℓ. Let m be the line through A' parallel to ℓ; since A' is off ℓ, the lines ℓ and m are distinct parallels. Let $m' = \alpha(m)$. I claim that ℓ' and m' are distinct parallels. If not, either $m' \nparallel \ell'$ or $m' = \ell'$, but in either case there exist points $Q' \in m' \cap \ell'$, Q_1 on m, and Q_2 on ℓ such that $\alpha(Q_1) = \alpha(Q_2) = Q'$. But ℓ and m are distinct parallels and α is injective, so this is impossible. Let $B = m \cap m'$ and let $B' = \alpha(B)$. Then B' is on m' and $B' \neq A'$ since A' is on ℓ'. If $B' = B$, then α has a fixed point as claimed. So assume that $B' \neq B$ Then $\ell' = \overleftrightarrow{AA'}$, $m' = \overleftrightarrow{BB'}$, and $\overleftrightarrow{AA'} \| \overleftrightarrow{BB'}$. If $\overleftrightarrow{AB} \| \overleftrightarrow{A'B'}$ as in Figure 5.8, then $\square ABB'A'$ is a parallelogram and $AB = A'B'$, in which case α is an isometry, contrary to our hypothesis.

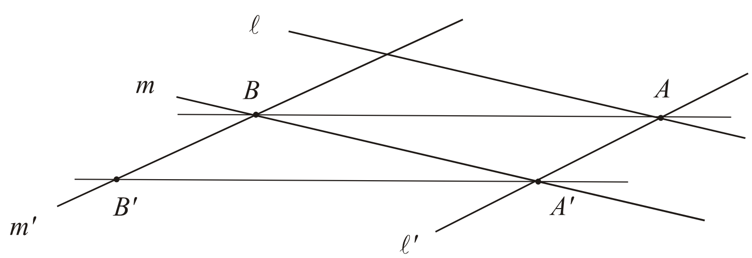


Figure 5.8.

Thus lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ must intersect at some point P off parallels $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$. Let $P' = \alpha(P)$; if $P' = P$, the proof is complete. Either $A - P - B$, $A - B - P$, or $P - A - B$. So consider three cases:

<u>*Case 1*</u>: Assume $A - P - B$ as in Figure 5.9.

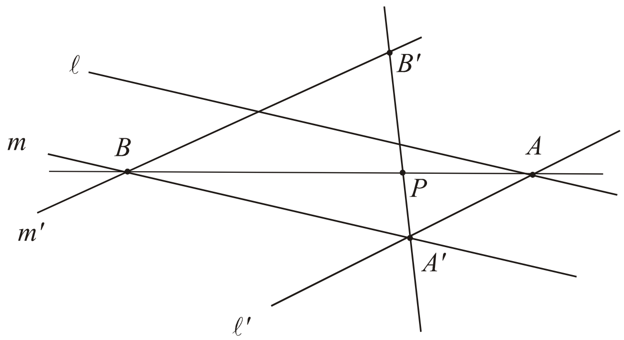


Figure 5.9.

First, $A' - P' - B'$ since α preserves betweenness (see Exercise 9 in Section 5.1). Second, $A' - P - B'$ since $\overleftrightarrow{AA'} \| \overleftrightarrow{BB'}$. But $\triangle APA' \sim \triangle BPB'$ by AA, so that

$$\frac{AP}{PB} = \frac{A'P}{PB'}$$

$(CPSTP)$. Let r be the ratio of α; then

$$\frac{AP}{PB} = \frac{rAP}{rPB} = \frac{A'P'}{P'B'}$$

so that

$$\frac{A'P}{PB'} = \frac{A'P'}{P'B'}. \tag{5.4}$$

I claim that $A'P = A'P'$, in which case $P = P'$. But if $A'P < A'P'$, then $\frac{A'P}{P'B'} < \frac{A'P'}{P'B'} = \frac{A'P}{PB'}$ implies $PB' < P'B'$ so that $A'B' = A'P + PB' < A'P' + P'B' = A'B'$, which is a contradiction, and similarly for $A'P > A'P'$.

The cases $A - B - P$ and $P - A - B$ are similar and left as exercises for the reader. ■

We can now prove our premier result:

Theorem 203 ***(Classification of Plane Similarities)*** *A similarity is exactly one of the following: an isometry, a stretch, a stretch rotation, or a stretch reflection.*

Proof. If α is a non-isometric similarity, then α has a fixed point C by Theorem 202. By Theorem 199, there is an isometry β and a stretch ξ about C such that $\alpha = \beta \circ \xi$, or equivalently, $\alpha \circ \xi^{-1} = \beta$. But ξ^{-1} is also a stretch about C so

$$\beta(C) = (\alpha \circ \xi^{-1})(C) = \alpha(C) = C.$$

Since the isometry β has fixed point C, β is one of the following: the identity, a rotation about C or a reflection in some line passing through C. Hence α is one of the following: a stretch, a stretch rotation, or a stretch reflection. Proof of the fact that α is exactly one of these four is left to the reader. ■

Definition 204 *A* <u>*direct*</u> *similarity preserves orientation; an* <u>*opposite*</u> *similarity reverses orientation.*

In light of Proposition 186 and Theorem 203, direct similarities are even isometries, stretches and stretch rotations and opposite similarities are odd isometries and stretch reflections. Thus the equations of a similarity are easy to obtain.

Theorem 205 *A direct similarity has equations of form*

$$\left\{\begin{array}{rl} x' = & ax - by + c \\ y' = & bx + ay + d \end{array}\right., \ a^2 + b^2 > 0;$$

an opposite similarity has equations of form

$$\left\{\begin{array}{rl} x' = & ax - by + c \\ y' = & -bx - ay + d \end{array}\right., \ a^2 + b^2 > 0.$$

Conversely, a transformation with equations of either form is a similarity.

Proof. One can easily check that an even isometry has equations

$$\left\{\begin{array}{rl} x' = & ax - by + c \\ y' = & bx + ay + d \end{array}\right., \ a^2 + b^2 = 1;$$

and an odd isometry has equations

$$\left\{\begin{array}{rl} x' = & ax - by + c \\ y' = & -bx - ay + d \end{array}\right., \ a^2 + b^2 = 1.$$

By Theorem 203, every non-isometric similarity is a stretch, a stretch reflection or a stretch rotation. The equations of a stretch of ratio r have the form

$$\left\{\begin{array}{rl} x' = & rx + c \\ y' = & ry + d \end{array}\right., \ r > 0.$$

Composing the equations of a reflection or a rotation with those of a stretch gives the result. The converse is left to the reader. ■

We conclude with some observations about conjugation by a stretch.

Theorem 206 *If C and P are arbitrary points, $r > 0$ and $\Theta \notin 0°$, then*

$$\xi_{P,r} \circ \rho_{C,\Theta} \circ \xi_{P,r}^{-1} = \rho_{\xi_{P,r}(C),\Theta}.$$

Proof. Note that $\beta = \xi_{P,r} \circ \rho_{C,\Theta} \circ \xi_{P,r}^{-1}$ is an isometry since it is a similarity of ratio 1. Let $C' = \xi_{P,r}(C)$. Then $\beta(C') = \left(\xi_{P,r} \circ \rho_{C,\Theta} \circ \xi_{P,r}^{-1}\right)(C') = \left(\xi_{P,r} \circ \rho_{C,\Theta}\right)(C) = \xi_{P,r}(C) = C'$, so β fixes C'. I claim C' is the unique fixed point of β. If Q is any fixed point of β, then $Q = \left(\xi_{P,r} \circ \rho_{C,\Theta} \circ \xi_{P,r}^{-1}\right)(Q)$ implies $\xi_{P,r}^{-1}(Q) = \rho_{C,\Theta}\left(\xi_{P,r}^{-1}(Q)\right)$. Hence $\xi_{P,r}^{-1}(Q) = C$ and $Q = \xi_{P,r}(C) = C'$, proving the claim. Therefore β is a rotation about C'. Let A' be a point distinct from C' and let $B' = \beta(A')$. Let $A = \xi_{P,r}^{-1}(A')$ and $B = \xi_{P,r}^{-1}(B')$. Then $B' = \left(\xi_{P,r} \circ \rho_{C,\Theta} \circ \xi_{P,r}^{-1}\right)(A')$ implies $\xi_{P,r}^{-1}(B') = \rho_{C,\Theta}\left(\xi_{P,r}^{-1}(A')\right)$ so that $B = \rho_{C,\Theta}(A)$. Since a stretch preserves orientation and $\xi_{P,r}(\triangle ACB) = \triangle A'C'B'$, we have $m\angle A'C'B' = m\angle ACB = \Theta$. Therefore $\beta = \rho_{C',\Theta}$. ■

Theorem 207 *If C is any point, ℓ is any line and $r > 0$, then*

$$\xi_{C,r} \circ \sigma_\ell \circ \xi_{C,r}^{-1} = \sigma_{\xi_{C,r}(\ell)}.$$

Proof. The proof is left to the reader. ■

In light of Theorems 206 and 207, it is immediately clear that a stretch about C commutes with rotations about C and reflections in lines through C (c.f. Exercises 4 and 5 in Section 5.3).

Exercises

1. Consider an equilateral triangle $\triangle ABC$ and the line $\ell = \overleftrightarrow{BC}$. Find all points and lines fixed by the similarity $\sigma_\ell \circ \xi_{A,2}$.

2. A dilation with center P and ratio r has equations $x' = -2x + 3$ and $y' = -2y - 4$. Find P and r.

3. Let α be a similarity such that $\alpha\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}$, $\alpha\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\2\end{bmatrix}$ and $\alpha\left(\begin{bmatrix}2\\2\end{bmatrix}\right) = \begin{bmatrix}-1\\6\end{bmatrix}$.

   a. Find the equations of α.

   b. Find $\alpha\left(\begin{bmatrix}-1\\6\end{bmatrix}\right)$.

4. Prove that if C is any point, ℓ is any line and $r > 0$, then $\xi_{C,r} \circ \sigma_\ell \circ \xi_{C,r}^{-1} = \sigma_{\xi_{C,r}(\ell)}$.

5. (*The fixed point of a stretch-reflection or stretch-rotation*) Let α be a stretch-reflection or a stretch-rotation. Choose distinct points A and B such that $m = \overleftrightarrow{AB}$ and $m' = \alpha(m)$ intersect at point P. Let $A' = \alpha(A)$, $B' = \alpha(B)$ and choose a point C off m. Let n be the line through C parallel to m; let n' be the line through $C' = \alpha(C)$ parallel to m'. Then lines n and n' intersect at point Q. Similarly, lines $\ell = \overleftrightarrow{AC}$ and $\ell' = \overleftrightarrow{A'C'}$ intersect at point R. Let k be the line through B parallel to ℓ; let k' be the line through B' parallel to ℓ'. Then lines k and k' intersect at point S. Finally, let $F = \overleftrightarrow{PQ} \cap \overleftrightarrow{RS}$ and prove that $\alpha(F) = F$.

6. Complete the proof of Theorem 203: Prove that the sets $\mathcal{I} = \{\text{isometries}\}$, $\mathcal{J} = \{\text{stretches}\}$, $\mathcal{K} = \{\text{stretch rotations}\}$,and $\mathcal{L} = \{\text{stretch reflections}\}$ are mutually disjoint.

7. Prove that the set of all *direct* similarities forms a group under composition of functions.

8. Which group properties fail for the set of opposite similarities?

9. Complete the proof of Theorem 202:

   (a) If $A - B - P$, prove that $P = P'$

   (b) If $P - A - B$, prove that $P = P'$.

10. Prove the converse of Theorem 205, i.e., a transformation with equations of either indicated form is a similarity.

Chapter 6

Billiards: An Application of Symmetry

The trajectory of a billiard ball in motion on a frictionless billiards table is completely determined by its initial position, direction, and speed. When the ball strikes a bumper, we assume that the angle of incidence equals the angle of reflection. Once released, the ball continues indefinitely along its trajectory with constant speed unless it strikes a vertex, at which point it stops. If the ball returns to its initial position with its initial velocity direction, it retraces its trajectory and continues to do so repeatedly; we call such trajectories *periodic.* Nonperiodic trajectories are either *infinite* or *singular*; in the later case the trajectory terminates at a vertex.

More precisely, think of a billiards table as a plane region R bounded by a polygon G. A *nonsingular trajectory on G* is a piecewise linear constant speed curve $\alpha : \mathbb{R} \to R$, where $\alpha(t)$ is the position of the ball at time t. An *orbit* is the restriction of some nonsingular trajectory to a closed interval; this is distinct from the notion of "orbit" in discrete dynamical systems.

A nonsingular trajectory α is *periodic* if $\alpha(a+t) = \alpha(b+t)$ for some $a < b$ and all $t \in \mathbb{R}$; its restriction to $[a, b]$ is a *periodic orbit.* A periodic orbit retraces the same path exactly $n \geq 1$ times. If $n = 1$, the orbit is *primitive*; otherwise it is an *n-fold iterate.* If α is primitive, α^n denotes its n-fold iterate. The *period* of a periodic orbit is the number of times the ball strikes a bumper as it travels along its trajectory. If α is primitive of period k, then α^n has period kn.

In this article we give a complete solution to the following billiards problem: *Find, classify, and count the classes of periodic orbits of a given period on an equilateral triangle.* While periodic orbits are known to exist on all nonobtuse and certain classes of obtuse triangles, existence in general remains a long-standing open problem. The first examples of periodic orbits were discovered by Fagnano in 1745. Interestingly, his orbit of period 3 on an acute triangle, known as the "Fagnano orbit," was not found as the solution of a billiards problem, but rather as the triangle of least perimeter inscribed in a given acute triangle.

This problem, known as "Fagnano's problem," is solved by the orthic triangle, whose vertices are the feet of the altitudes of the given triangle (see Figure 1). The orthic triangle is a periodic trajectory since its angles are bisected by the altitudes of the triangle in which it is inscribed; the proof given by Coxeter and Greitzer uses exactly the "unfolding" technique we apply below. Coxeter credits this technique to H. A. Schwarz and mentions that Frank and F. V. Morley extended Schwarz's treatment on triangles to odd-sided polygons. For a discussion of some interesting properties of the Fagnano orbit on any acute triangle.

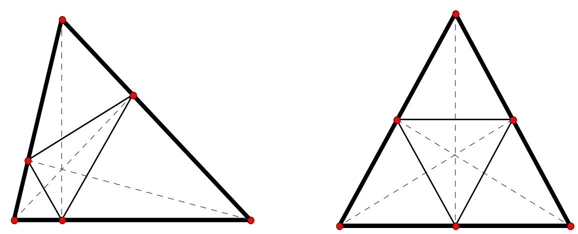


Figure 6.1: Fagnano's period 3 orbit.

Much later, in 1986, Masur proved that every *rational* polygon (one whose interior angles are rational multiples of π) admits infinitely many periodic orbits with distinct periods, but he neither constructed nor classified them. A year later Katok proved that the number of periodic orbits of a given period grows subexponentially. Existence results on various polygons were compiled by Tabachnikov in 1995.

This article is organized as follows: In Section 2 we introduce an equivalence relation on the set of all periodic orbits on an equilateral triangle and prove that every orbit with odd period is an odd iterate of Fagnano's orbit. In Section 3 we use techniques from analytic geometry to identify and classify all periodic orbits. The paper concludes with Section 4, in which we derive two counting formulas: First, we establish a bijection between classes of orbits with period $2n$ and partitions of n with 2 or 3 as parts and use it to show that there are $\mathcal{O}(n) = \lfloor \frac{n+2}{2} \rfloor - \lfloor \frac{n+2}{3} \rfloor$ classes of orbits with period $2n$ (counting iterates). Second, we show that there are $\mathcal{P}(n) = \sum_{d|n} \mu(d)\mathcal{O}(n/d)$ classes of primitive orbits with period $2n$, where μ denotes the Möbius function.

6.1 Orbits and Tessellations

Consider an equilateral triangle $\triangle ABC$. We begin with some key observations.

Proposition 208 *Every nonsingular trajectory strikes some side of* $\triangle ABC$ *with an angle of incidence in the range* $30° \leq \theta \leq 60°$.

Proof. Given a nonsingular trajectory α, choose a point P_1 at which α strikes $\triangle ABC$ with angle of incidence θ_1. If θ_1 lies in the desired range, set $\theta = \theta_1$. Otherwise, let α_1 be the segment of α that connects P_1 to the next strike point P_2 and label the vertices of $\triangle ABC$ so that P_1 is on side $\overline{AC}$ and P_2 is on side $\overline{BC}$ (see Figure 2). If $0° < \theta_1 < 30°$, then $\theta_2 = m\angle P_1P_2B = \theta_1 + 60°$ so that $60° < \theta_2 < 90°$. Let α_2 be the segment of α that connects P_2 to the next strike point P_3. Then the angle of incidence at P_3 satisfies $30° < \theta_3 < 60°$; set $\theta = \theta_3$. If $60° < \theta_1 \leq 90°$ and θ_1 is an interior angle of $\triangle P_1P_2C$, then the angle of incidence at P_2 is $\theta_2 = m\angle P_1P_2C = 120° - \theta_1$ and satisfies $30° \leq \theta_2 < 60°$; set $\theta = \theta_2$. But if $60° < \theta_1 \leq 90°$ and θ_1 is an exterior angle of $\triangle P_1P_2C$, then the angle of incidence at P_2 is $\theta_2 = m\angle P_1P_2C = \theta_1 - 60°$, in which case $0° < \theta_2 \leq 30°$. If $\theta_2 = 30°$ set $\theta = \theta_2$; otherwise continue as above until $30° < \theta_4 < 60°$ and set $\theta = \theta_4$. ■

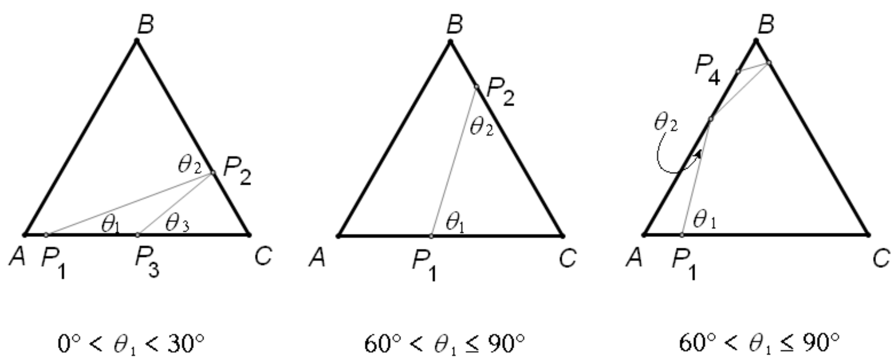


Figure 6.2: Incidence angles in the range $30° \leq \theta \leq 60°$.

Let α be an orbit of period n on $\triangle ABC$ oriented so that $\overline{BC}$ is horizontal. Since Proposition 208 applies equally well to periodic orbits, choose a point P at which α strikes $\triangle ABC$ with angle of incidence in the range $30° \leq \theta \leq 60°$. If necessary, relabel the vertices of $\triangle ABC$, change initial points, and reverse the parameter so that side $\overline{BC}$ contains P, α begins and ends at P, and the components of α' as the ball departs from P are positive. Let $\mathcal{T}$ be a regular tessellation of the plane by equilateral triangles, each congruent to $\triangle ABC$, and positioned so that one of its families of parallel edges is horizontal. Embed $\triangle ABC$ in $\mathcal{T}$ so that its base $\overline{BC}$ is collinear with a horizontal edge of $\mathcal{T}$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ denote the directed segments of α, labelled sequentially; then α_1 begins at P and terminates at P_1 on side s_1 of $\triangle ABC$ with angle of incidence θ_1. Let σ_1 be the reflection in the edge of $\mathcal{T}$ containing s_1. Then α_1 and $\sigma_1(\alpha_2)$ are collinear segments and $\sigma_1(\alpha)$ is a periodic orbit on $\sigma_1(\triangle ABC)$, which is the basic triangle of $\mathcal{T}$ sharing side s_1 with $\triangle ABC$. Follow $\sigma_1(\alpha_2)$ from P_1 until it strikes side s_2 of $\sigma_1(\triangle ABC)$ at P_2 with incidence angle θ_2. Let σ_2 be the reflection in the edge of $\mathcal{T}$ containing s_2; then α_1, $\sigma_1(\alpha_2)$ and $(\sigma_2\sigma_1)(\alpha_3)$ are collinear segments and $(\sigma_2\sigma_1)(\alpha)$ is a periodic orbit on $(\sigma_2\sigma_1)(\triangle ABC)$.

Continuing in this manner for $n-1$ steps, let θ_n be the angle of incidence at $Q = (\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)(P)$. Then $\theta_1, \theta_2, \ldots, \theta_n$ is a sequence of incidence angles with $30° \leq \theta_n \leq 60°$, and $\alpha_1, \sigma_1(\alpha_2), \ldots, (\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)(\alpha_n)$ is a sequence of collinear segments whose union is the directed segment from P to Q. Let $\underline{PQ}$ denote the directed segment from P to Q. Then $\underline{PQ}$ has the same length as α and enters and exits the triangle $(\sigma_i\cdots\sigma_1)(\triangle ABC)$ with angles of incidence θ_i and θ_{i+1}. We refer to $\underline{PQ}$ as an *unfolding* of α and to θ_n as its *representation angle.*

Proposition 209 *A periodic orbit strikes the sides of $\triangle ABC$ with at most three incidence angles, exactly one of which lies in the range $30° \leq \theta \leq 60°$. In fact, exactly one of the following holds:*

1. *All incidence angles measure $60°$.*
2. *There are exactly two distinct incidence angles measuring $30°$ and $90°$.*
3. *There are exactly three distinct incidence angles ϕ, θ, and ψ such that $0° < \phi < 30° < \theta < 60° < \psi < 90°$.*

Proof. Let α be a periodic orbit and let $\underline{PQ}$ be an unfolding. By construction, $\underline{PQ}$ cuts each horizontal edge of $\mathcal{T}$ with angle of incidence in the range $30° \leq \theta \leq 60°$. Consequently, $\underline{PQ}$ cuts a left-leaning edge of $\mathcal{T}$ with angle of incidence $\phi = 120° - \theta$ and cuts a right-leaning edge of $\mathcal{T}$ with angle of incidence $\psi = 60° - \theta$ (see Figure 3). In particular, if $\theta = 60°$, $\underline{PQ}$ cuts only left-leaning and horizontal edges, and all incidence angles are equal. In this case, α is either the Fagnano orbit, a primitive orbit of period 6, or some iterate of these. If $\theta = 30°$, then $\phi = 90°$ and $\psi = 30°$, and α is either primitive of period 4 or some iterate thereof (see Figure 4). When $30° < \theta < 60°$, clearly $0° < \phi < 30°$ and $60° < \psi < 90°$. ■

Corollary 210 *Any two unfoldings of a periodic orbit are parallel.*

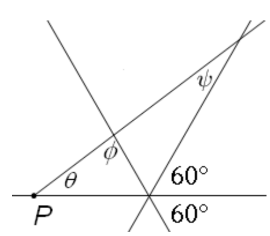


Figure 6.3: Incidence angles θ, ϕ, and ψ.

Our next result plays a pivotal role in the classification of orbits.

Theorem 211 *If an unfolding of a periodic orbit α terminates on a horizontal edge of $\mathcal{T}$, then α has even period.*

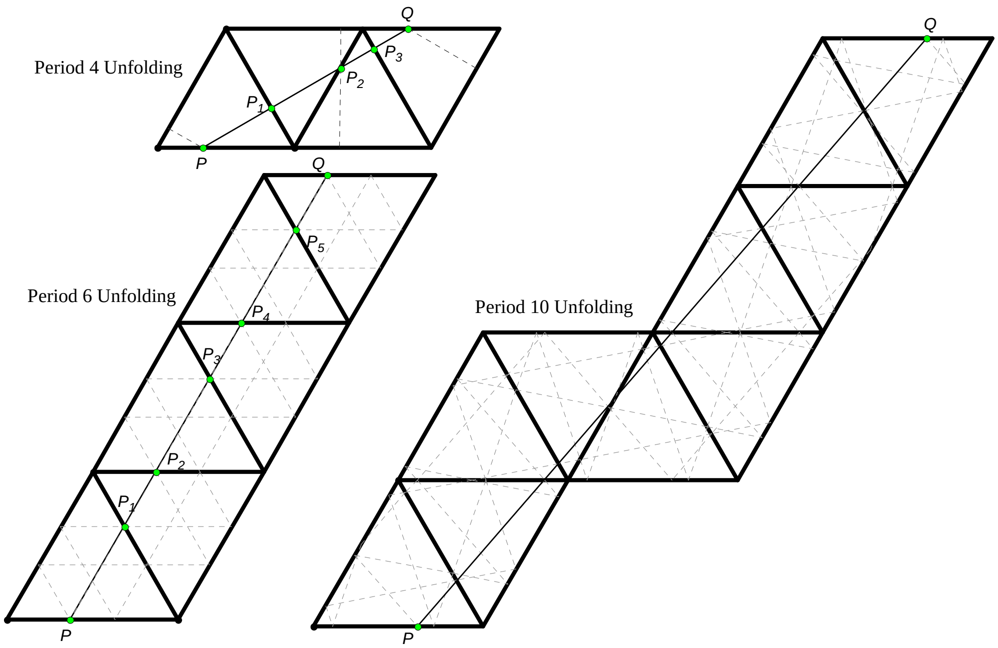


Figure 6.4: Unfolded orbits of period 4, 6, and 10.

Proof. Let $\underline{PQ}$ be an unfolding of α. Then both P and Q lie on horizontal edges of $\mathcal{T}$, and the basic triangles of $\mathcal{T}$ cut by $\underline{PQ}$ pair off and form a polygon of rhombic tiles containing $\underline{PQ}$ (see Figure 5). As the path $\underline{PQ}$ traverses this polygon, it enters each rhombic tile through an edge, cuts a diagonal of that tile (collinear with a left-leaning edge of $\mathcal{T}$), and exits through another edge. Since each exit edge of one tile is the entrance edge of the next and the edge containing P is identified with the edge containing Q, the number of distinct edges of $\mathcal{T}$ cut by $\underline{PQ}$ is twice the number of rhombic tiles. It follows that α has even period. ■

Let γ denote the Fagnano orbit.

Theorem 212 *If α is a periodic orbit and $\alpha \neq \gamma^{2k-1}$ for all $k \geq 1$, then every unfolding of α terminates on a horizontal edge of $\mathcal{T}$.*

Proof. We prove the contrapositive. Suppose there is an unfolding $\underline{PQ}$ of α that does not terminate on a horizontal edge of $\mathcal{T}$. Let θ be the angle of incidence at Q; then θ is also the angle of incidence at P and $\theta \in \{30°, 60°\}$ by the proof of Proposition 209. But if $\theta = 30°$, then α is some iterate of the period 4 orbit whose unfoldings terminate on a horizontal edge of $\mathcal{T}$ (see Figure 4). So $\theta = 60°$. But α is neither an iterate of a period 6 orbit nor an even iterate of γ since their unfoldings also terminate on a horizontal edge of $\mathcal{T}$ (see Figure 4). It follows that $\alpha = \gamma^{2k-1}$ for some $k \geq 1$. ■

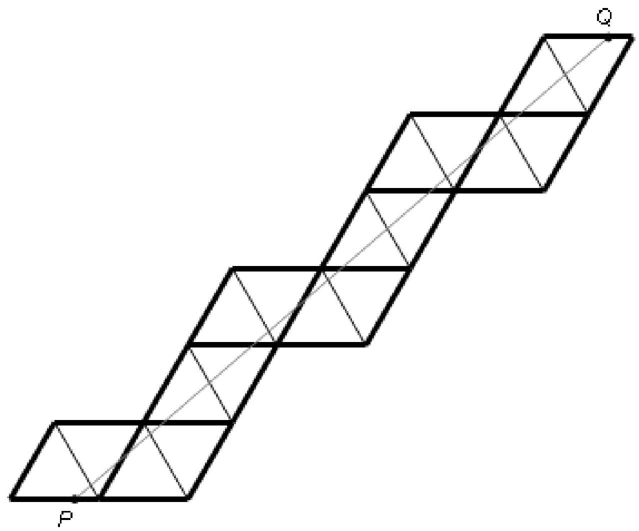


Figure 6.5: A typical rhombic tiling.

Combining the contrapositives of Theorems 211 and 212 we obtain the following characterization:

Corollary 213 *If α is an orbit with odd period, then $\alpha = \gamma^{2k-1}$ for some $k \geq 1$, in which case the period is $6k-3$.*

Let α be an orbit with even period and let $\underline{PQ}$ be an unfolding. Let G be the group generated by all reflections in the edges of $\mathcal{T}$. Since the action of G on $\overline{BC}$ generates a regular tessellation $\mathcal{H}$ of the plane by hexagons, α terminates on some horizontal edge of $\mathcal{H}$. As in the definition of an unfolding, let $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ be the reflections in the lines of $\mathcal{T}$ cut by $\underline{PQ}$ (in order) and σ_n be the reflection in the line of $\mathcal{T}$ containing Q. Then the composition $f = \sigma_n \sigma_{n-1} \cdots \sigma_1$ maps P to Q and maps the hexagon whose base $\overline{BC}$ contains P to the hexagon whose base $\overline{B'C'}$ contains Q. Then n (the period of α) is even and f is either a translation by vector $\overrightarrow{PQ}$ or a rotation of $120°$ or $240°$. But $\overline{BC} \| \overline{B'C'}$ so f is a translation and the position of Q on $\overline{B'C'}$ is exactly the same as the position of P on $\overline{BC}$.

Periodic orbits represented by horizontal translations of an unfolding $\underline{PQ}$ are generically distinct, but have the same length and incidence angles (up to permutation) as α. Hence it is natural to think of them as equivalent.

Definition 214 *Periodic orbits α and β are equivalent if there exist respective unfoldings $\underline{PQ}$ and $\underline{RS}$ and a horizontal translation τ such that $\underline{RS} = \tau\left(\underline{PQ}\right)$. The symbol $[\alpha]$ denotes the equivalence class of α. The period of a class $[\alpha]$ is the period of its elements; a class is even if and only if it has even period.*

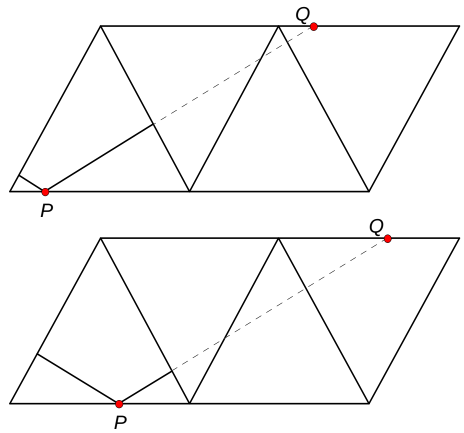


Figure 6.6: Unfoldings of equivalent period 4 orbits.

Consider an unfolding $\underline{PQ}$ of a periodic orbit α. If $[\alpha]$ is even, let R be a point on $\overline{BC}$ and let τ is the translation from P to R. We say that the point R is *singular for* $[\alpha]$ if $\tau(\underline{PQ})$ contains a vertex of $\mathcal{T}$; then $\tau(\underline{PQ})$ is an unfolding of a periodic orbit whenever R is non-singular for $[\alpha]$. Furthermore, α strikes $\underline{BC}$ at finitely many points and at most finitely many points on $\overline{BC}$ are singular for $[\alpha]$. Therefore $[\alpha]$ has cardinality $\mathfrak{c}$ (the cardinality of an interval). On the other hand, Corollary 213 tells us that an orbit of odd period is γ^{2k-1} for some $k \geq 1$. But if $k \neq \ell$, then γ^{2k-1} and $\gamma^{2\ell-1}$ have different periods and cannot be equivalent. Therefore $[\gamma^{2k-1}]$ is a singleton class for each k. We have proved:

Proposition 215 *The cardinality of a class is determined by its parity; in fact, α has odd period if and only if $[\alpha]$ is a singleton class.*

Proposition 215 and Corollary 213 completely classify orbits with odd period. The remainder of this article considers orbits with even period. Our strategy is to represent the classes of all such orbits as lattice points in some "fundamental region," which we now define. First note that any two unfoldings whose terminal points lie on the same horizontal edge of $\mathcal{H}$ are equivalent. Since $\mathcal{H}$ has countably many horizontal edges, there are countably many even classes of orbits. Furthermore, since at most finitely many points in $\overline{BC}$ are singular for each even class, there is a point O on $\overline{BC}$ other than the midpoint that is nonsingular for every class. Therefore, given an even class $[\alpha]$, there is a point S and an element $x \in [\alpha]$ such that $\underline{OS}$ is an unfolding of x. Note that if $\underline{PQ}$ is an unfolding of α, then $\underline{OS}$ is the horizontal translation of $\underline{PQ}$ by $\overrightarrow{PO}$. Therefore α uniquely determines the point S, denoted henceforth by S_α, and we refer to $\underline{OS_\alpha}$ as the *fundamental unfolding of* $[\alpha]$. The *fundamental region at* O, denoted by Γ_O, is the polar region $30° \leq \theta \leq 60°$ centered at O; the points S_α are called *lattice points* of Γ_O.

Since O is not the midpoint of $\overline{BC}$, odd iterates of Fagnano's orbit γ have no fundamental unfoldings. On the other hand, the fundamental unfolding of γ^{2n} represents the n-fold iterate of a primitive period 6 orbit. Nevertheless, with the notable exception of $[\gamma^2]$, "primitivity" is a property common to all orbits of the same class (see Figure 7). Indeed, the fundamental unfolding of $[\gamma^2]$ represents a primitive orbit. So we define a *primitive class* to be either $[\gamma^2]$ or a class of primitives.

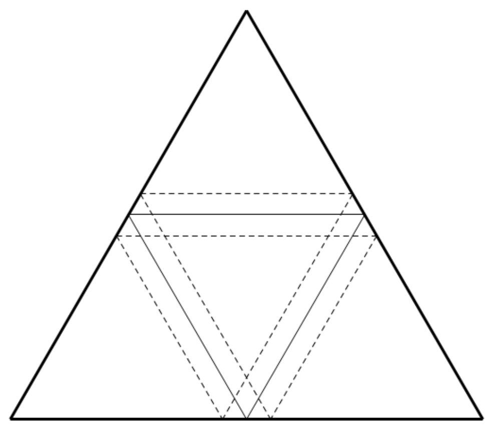


Figure 6.7: The Fagnano orbit and an equivalent period 6 orbit (dotted).

To complete the classification, we must determine exactly which directed segments in Γ_O with initial point O represent orbits with even period. We address this question in the next section.

6.2 Orbits and Rhombic Coordinates

In this section we introduce the analytical structure we need to complete the classification and to count the distinct classes of orbits of a given even period. Expressing a fundamental unfolding $\underline{OS}$ as a vector $\overrightarrow{OS}$ allows us to exploit the natural rhombic coordinate system given by $\mathcal{T}$. Let O be the origin and take the x-axis to be the horizontal line containing it. Take the y-axis to be the line through O with inclination $60°$ and let BC be the unit of length (see Figure 8). Then in rhombic coordinates

$$\Gamma_O = \{(x, y) \mid 0 \le x \le y\}.$$

Since the period of $[\alpha]$ is twice the number of rhombic tiles cut by $\underline{OS_\alpha}$, and the rhombic coordinates of S_α count these rhombic tiles, we can strengthen Theorem 211:

Corollary 216 *If $S_\alpha = (x, y)$, then α has period $2(x + y)$.*

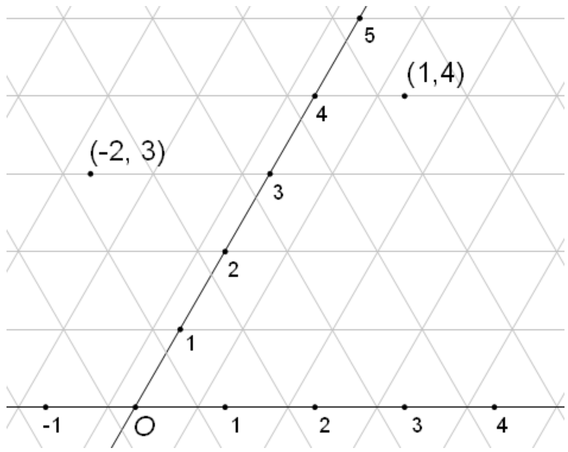


Figure 6.8: Rhombic coordinates.

Points in the integer sublattice $\mathcal{L}$ of points on the horizontals of $\mathcal{H}$ that are images of O under the action of G have the following simple characterization: Let H be the hexagon of $\mathcal{H}$ with base $\overline{BC}$, and let τ_1 and τ_2 denote the translations by the vectors $(1,1)$ and $(0,3)$, respectively. Then the six hexagons adjacent to H are its images $\tau_2^b\tau_1^a(H)$, $(a,b) \in \{\pm(1,0), \pm(1,-1), \pm(2,-1)\}$. Inductively, if H' is any hexagon of $\mathcal{H}$, then $H' = \tau_2^b\tau_1^a(H)$ for some $a, b \in \mathbb{Z}$. Note that $a(1,1)+b(0,3)$ defines the translation $\tau_2^b\tau_1^a$. Hence $\mathcal{L}$ is generated by the vectors $(1,1)$ and $(0,3)$ and it follows that $(x,y) \in \mathcal{L}$ if and only if $x \equiv y \pmod 3$.

Now recall that if $\underline{PQ}$ is an unfolding, then Q lies on a horizontal of $\mathcal{H}$. Hence $\underline{OS}$ is a fundamental unfolding if and only if $S \in \mathcal{L} \cap \Gamma_O - O$ if and only if $S \in \{(x,y) \in \mathbb{Z}^2 \cap \Gamma_O \mid x \equiv y (\text{mod } 3), x+y=n\}$. We have proved:

Theorem 217 *Given an even class $[\alpha]$, let $(x,y)_\alpha = S_\alpha$. There is a bijection*

$$\Phi : \{[\alpha] \mid [\alpha] \text{ has period } 2n\} \to \{(x,y) \in \mathbb{Z}^2 \cap \Gamma_O \mid x \equiv y (mod\ 3), x+y=n\}$$

given by $\Phi([\alpha]) = (x,y)_\alpha$.

Taken together, Proposition 215, Corollary 216 and Theorem 217 classify all periodic orbits on an equilateral triangle.

Theorem 218 *(Classification) Let α be a periodic orbit on an equilateral triangle.*

1. *If α has period $2n$, then $[\alpha]$ has cardinality $\mathfrak{c}$ and contains exactly one representative whose unfolding $\underline{OS}$ satisfies $S = (x,y)$, $0 \le x \le y$, $x \equiv y$ $(\text{mod } 3)$, and $x+y=n$.*
2. *Otherwise, $\alpha = \gamma^{2k-1}$ for some $k \ge 1$, in which case its period is $6k-3$.*

In view of Theorem 217, we may count classes of orbits of a given period $2n$ by counting integer pairs (x, y) such that $0 \leq x \leq y$, $x \equiv y \pmod 3$ and $x + y = n$. This is the objective of the next and concluding section.

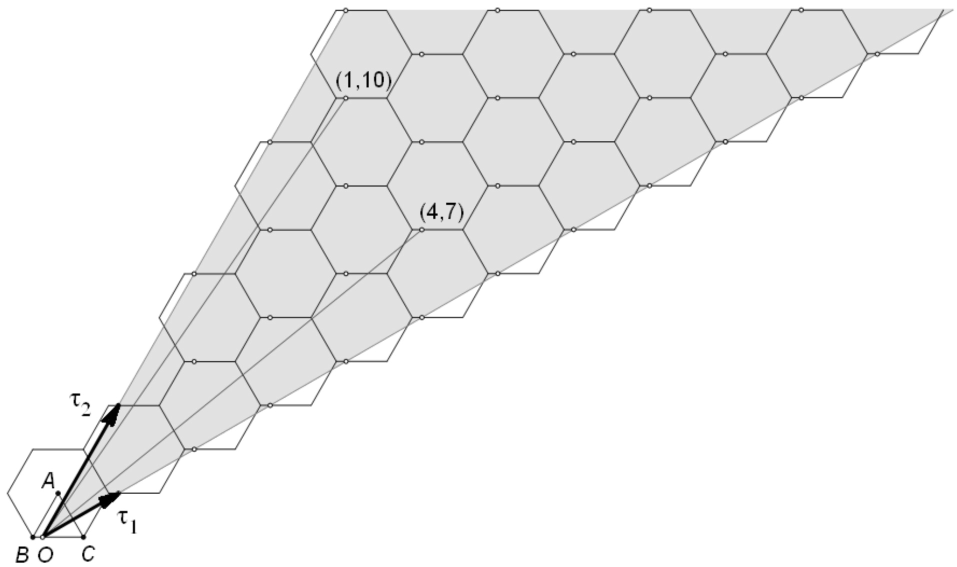


Figure 6.9: Translated images of O in Γ_O and unfoldings of period 22 orbits.

6.3 Orbits and Integer Partitions

We will often refer to an ordered pair (x, y) as an "orbit" when we mean the even class of orbits to which it corresponds. Two questions arise: (1) Is there an orbit with period $2n$ for each $n \in \mathbb{N}$? (2) If so, exactly how many distinct classes of orbits with period $2n$ are there?

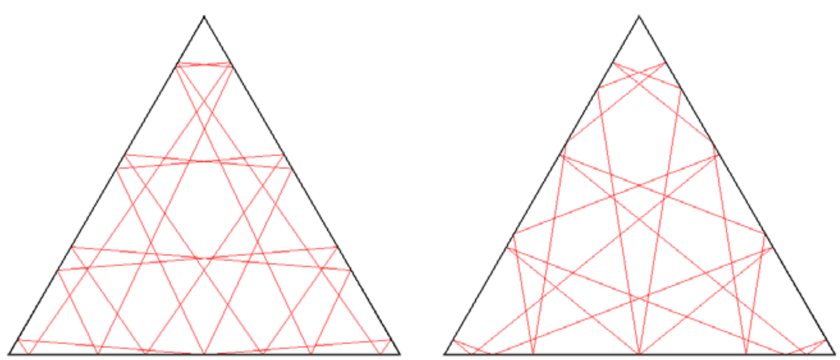


Figure 15: Period 22 orbits $(1, 10)$ and $(4, 7)$.

If we admit iterates, question (1) has an easy answer. Clearly there are no period 2 orbits since no two sides of $\triangle ABC$ are parallel — alternatively, if (a,b) is a solution of the system $x \equiv y \pmod 3$ and $x+y=1$, either a or b is negative. For each $n>1$, the orbit

$$\alpha = \begin{cases} \left(\frac{n}{2}, \frac{n}{2}\right), & n \text{ even} \\ \left(\frac{n-1}{2}-1, \frac{n-1}{2}+2\right), & n \text{ odd.} \end{cases}$$

has period $2n$. Note that the period 22 orbits $(1,10)$ and $(4,7)$ are not equivalent since they have different lengths and representation angles (see Figures 9 and 10).

To answer to question (2), we reduce the problem to counting partitions by constructing a bijection between classes of orbits with period $2n$ and partitions of n with 2 and 3 as parts. For a positive integer n, a *partition* of n is a nonincreasing sequence of nonnegative integers whose terms sum to n. Such a sequence has finitely many nonzero terms, called the *parts*, followed by infinitely many zeros. Thus, we seek pairs of nonnegative integers (a,b) such that $n = 2a+3b$. The reader can easily prove:

Lemma 219 *For each $n \in \mathbb{N}$, let*

$$X_n = \left\{(x,y) \in \mathbb{Z}^2 \mid 0 \le x \le y,\ x \equiv y \pmod 3),\ x+y=n\right\} \text{ and}$$

$$Y_n = \{(a,b) \in \mathbb{Z}^2 \mid a,b \ge 0 \text{ and } 2a+3b=n\}.$$

The function $\varphi : Y_n \to X_n$ given by $\varphi(a,b) = (a, a+3b)$ is a bijection.

Combining Theorem 218 and Lemma 219, we have:

Corollary 220 *For each $n \in \mathbb{N}$, there is a bijection between period $2n$ orbits and the partitions of n with 2 and 3 as parts.*

Counting partitions of n with specified parts is well understood (e.g., Sloane's A103221). The number of partitions of n with 2 and 3 as parts is the coefficient of x^n in the generating function

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \mathcal{O}(n) x^n \\ &= (1+x^2+x^4+x^6+\cdots)(1+x^3+x^6+x^9+\cdots) \\ &= \frac{1}{(1-x^2)(1-x^3)}. \end{aligned}$$

To compute this coefficient, let ω be a primitive cube root of unity and perform a partial fractions decomposition. Then

$$\begin{aligned} f(x) &= \frac{1}{4(1+x)} + \frac{1}{4(1-x)} + \frac{1}{6(1-x)^2} + \frac{1}{9}\left(\frac{1+2\omega}{\omega - x} + \frac{1+2\omega^2}{\omega^2 - x}\right) \\ &= \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n x^n + \frac{1}{4}\sum_{n=0}^{\infty} x^n + \frac{1}{6}\sum_{n=0}^{\infty}(n+1)x^n \\ &+ \frac{1}{9}\sum_{n=0}^{\infty}(\omega^{2n+2} + 2\omega^{2n} + \omega^{n+1} + 2\omega^n)x^n, \end{aligned}$$

and we have

$$\mathcal{O}(n) = \frac{(-1)^n}{4} + \frac{n}{6} + \frac{5}{12} + \frac{1}{9}\left(\omega^{2n+2} + 2\omega^{2n} + \omega^{n+1} + 2\omega^n\right).$$

By easy induction arguments, one can obtain the following simpler formulations:

Theorem 221 *The number of distinct classes of period* $2n$ *is exactly*

$$\mathcal{O}(n) = \begin{cases} \lfloor \frac{n}{6} \rfloor, & n \equiv 1 \pmod{3} \\ \lfloor \frac{n}{6} \rfloor + 1, & \textit{otherwise} \end{cases}$$

$$= \left\lfloor \frac{n+2}{2} \right\rfloor - \left\lfloor \frac{n+2}{3} \right\rfloor.$$

Let us refine this counting formula by counting only primitives. For every divisor d of n, the (n/d)-fold iterate of a primitive period $2d$ orbit has period $2n$. Hence, if $\mathcal{P}(n)$ denotes the number of primitive classes of period $2n$, then

$$\mathcal{O}(n) = \sum_{d|n} \mathcal{P}(d).$$

A formula for $\mathcal{P}(n)$ is a direct consequence of the Möbius inversion formula. The Möbius function $\mu : \mathbb{N} \to \{-1, 0, 1\}$ is defined by

$$\mu(d) = \begin{cases} 1, & d = 1 \\ (-1)^r, & d = p_1 p_2 \cdots p_r \text{ for distinct primes } p_i \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 222 *For each* $n \in \mathbb{N}$, *there are exactly*

$$\mathcal{P}(n) = \sum_{d|n} \mu(d)\mathcal{O}(n/d)$$

primitive classes of period $2n$.

Theorems 221 and 222, together with Example 227 below, imply:

Corollary 223 *$\mathcal{O}(n) = 0$ if and only if $n = 1$; $\mathcal{P}(n) = 0$ if and only if $n = 1, 4, 6, 10$.*

Corollary 224 *The following are equivalent:*

1. *The integer n is 1 or prime.*
2. *$\mathcal{P}(n) = \mathcal{O}(n)$.*
3. *All classes of period $2n$ are primitive.*

Table 1 displays some values of $\mathcal{O}$ and $\mathcal{P}$. The values $\mathcal{O}(4) = 1$, $\mathcal{P}(4) = 0$, and $\mathcal{P}(2) = 1$, for example, indicate that the single class of period 8 contains only 2-fold iterates of the primitive orbits in the single class of period 4.

We conclude with an example of a primitive class of period $2n$ for each $n \in \mathbb{N} - \{1, 4, 6, 10\}$. But first we need the following self-evident lemma:

Lemma 225 *Given an orbit $(x, y) \in \Gamma_O$, let $d \in \mathbb{N}$ be the largest value such that $x/d \equiv y/d \pmod 3$. Then (x, y) is primitive if and only if $d = 1$; otherwise (x, y) is a d-fold iterate of the primitive orbit $(x/d, y/d)$.*

Although d is difficult to compute, it is remarkably easy to check for primitivity.

Theorem 226 *An orbit $(x, y) \in \Gamma_O$ is primitive if and only if either*

1. $\gcd(x, y) = 1$ *or*
2. $(x, y) = (3a, 3b)$, $\gcd(a, b) = 1$, *and* $a \not\equiv b \pmod 3$ *for some* $a, b \in \mathbb{N} \cup \{0\}$.

Proof. If $\gcd(x, y) = 1$, the orbit (x, y) is primitive. On the other hand, if $(x, y) = (3a, 3b), a \not\equiv b \pmod 3$, and $\gcd(a, b) = 1$ for some a, b, let d be as in Lemma 225. Then $d \neq 3$ since $a \not\equiv b \pmod 3$. But $\gcd(a, b) = 1$ implies $d = 1$, so (x, y) is also primitive when (2) holds.

Conversely, given a primitive orbit (x, y), let $c = \gcd(x, y)$. Then $cm = x \leq y = cn$ for some $m, n \in \mathbb{N} \cup \{0\}$; thus $m \leq n$, $\gcd(m, n) = 1$ and $cm \equiv cn \pmod 3$. Suppose (2) fails. The reader can check that $3 \nmid c$, in which case $m \equiv n \pmod 3$. But $x/c \equiv y/c \pmod 3$ and the primitivity of (x, y) imply $c = 1$. ■

Example 227 *Using Theorem 226, the reader can check that the following orbits of period $2n$ are primitive:*

- $n = 2k + 1, k \geq 1 : (k - 1, k + 2)$
- $n = 2 : (1, 1)$
- $n = 4k + 4, k \geq 1 : (2k - 1, 2k + 5)$
- $n = 4k + 10, k \geq 1 : (2k - 1, 2k + 11)$.

Since $\mathcal{P}(n)$ tells us there are *no* primitive orbits of period 2, 8, 12 or 20, Example 227 exhibits a primitive orbit of every possible even period.

n	$2n$	$\mathcal{O}(n)$	$\mathcal{P}(n)$	n	$2n$	$\mathcal{O}(n)$	$\mathcal{P}(n)$
1	2	0	0	31	62	5	5
2	4	1	1	32	64	6	3
3	6	1	1	33	66	6	3
4	8	1	0	34	68	6	2
5	10	1	1	35	70	6	4
6	12	2	0	36	72	7	2
7	14	1	1	37	74	6	6
8	16	2	1	38	76	7	3
9	18	2	1	39	78	7	4
10	20	2	0	40	80	7	2
11	22	2	2	41	82	7	7
12	24	3	1	42	84	8	2
13	26	2	2	43	86	7	7
14	28	3	1	44	88	8	4
15	30	3	1	45	90	8	4
16	32	3	1	46	92	8	3
17	34	3	3	47	94	8	8
18	36	4	1	48	96	9	3
19	38	3	3	49	98	8	7
20	40	4	2	50	100	9	4
21	42	4	2	51	102	9	5
22	44	4	1	52	104	9	4
23	46	4	4	53	106	9	9
24	48	5	1	54	108	10	3
25	50	4	3	55	110	9	6
26	52	5	2	56	112	10	4
27	54	5	3	57	114	10	6
28	56	5	2	58	116	10	4
29	58	5	5	59	118	10	10
30	60	6	2	60	120	11	2

Table 1. Sample values of $\mathcal{O}$ and $\mathcal{P}$.

n	$2n$	$\mathcal{O}(n)$	$\mathcal{P}(n)$	n	$2n$	$\mathcal{O}(n)$	$\mathcal{P}(n)$
1	2	0	0	31	62	5	5
2	4	1	1	32	64	6	3
3	6	1	1	33	66	6	3
4	8	1	0	34	68	6	2
5	10	1	1	35	70	6	4
6	12	2	0	36	72	7	2
7	14	1	1	37	74	6	6
8	16	2	1	38	76	7	3
9	18	2	1	39	78	7	4
10	20	2	0	40	80	7	2
11	22	2	2	41	82	7	7
12	24	3	1	42	84	8	2
13	26	2	2	43	86	7	7
14	28	3	1	44	88	8	4
15	30	3	1	45	90	8	4
16	32	3	1	46	92	8	3
17	34	3	3	47	94	8	8
18	36	4	1	48	96	9	3
19	38	3	3	49	98	8	7
20	40	4	2	50	100	9	4
21	42	4	2	51	102	9	5
22	44	4	1	52	104	9	4
23	46	4	4	53	106	9	9
24	48	5	1	54	108	10	3
25	50	4	3	55	110	9	6
26	52	5	2	56	112	10	4
27	54	5	3	57	114	10	6
28	56	5	2	58	116	10	4
29	58	5	5	59	118	10	10
30	60	6	2	60	120	11	2

Table 6.1: Sample Values for $\mathcal{O}(n)$ and $\mathcal{P}(n)$.