## Generalized Eigenvectors and Systems of Linear Differential Equations

Math 422

Consider a system of linear first order differential equations $\mathbf{x}'(t) = A\mathbf{x}(t)$ with initial value $\mathbf{x}(0) = \mathbf{x}_0$, and suppose that $A$ is similar to a Hessenberg matrix $H = S^{-1}AS$. Let

$$\mathbf{y}(t) = S^{-1}\mathbf{x}(t) \quad \text{and} \quad \mathbf{y}_0 = S^{-1}\mathbf{x}(0) = \mathbf{y}(0);$$

then

$$\mathbf{y}'(t) = S^{-1}\mathbf{x}'(t) = S^{-1}A\mathbf{x}(t) = \left(S^{-1}AS\right)S^{-1}\mathbf{x}(t) = H\mathbf{y}(t).$$

Thus the change of variables $\mathbf{y}(t) = S^{-1}\mathbf{x}(t)$ transforms the given initial value problem (IVP) into the equivalent one

$$\mathbf{y}'(t) = H\mathbf{y}(t) \quad \text{with} \quad \mathbf{y}(0) = \mathbf{y}_0. \tag{1}$$

For example, the matrix $A = \begin{bmatrix} -1 & -8 & 1 \\ -1 & -3 & 2 \\ -4 & -16 & 7 \end{bmatrix}$ is similar to

$$H = \begin{bmatrix} -1 & -4 & 1 \\ -1 & 5 & 2 \\ 0 & -8 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \begin{bmatrix} -1 & -8 & 1 \\ -1 & -3 & 2 \\ -4 & -16 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}.$$

Thus the change of variables above transforms the system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & -8 & 1 \\ -1 & -3 & 2 \\ -4 & -16 & 7 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix}$$

into

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & -4 & 1 \\ -1 & 5 & 2 \\ 0 & -8 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 9 \end{bmatrix}.$$

When $H$ is reduced, Jordan Canonical Form (the final topic in this course) is required to solve the IVP in (1). When $H$ is unreduced, however, we can solve the IVP in (1) by following the steps outlined below. But first we need some important theoretical results.

**Theorem 1** *Let $H = (h_{ij})$ be an unreduced $n \times n$ Hessenberg matrix. Then $\left\{\mathbf{e}_1, H\mathbf{e}_1, \ldots, H^{n-1}\mathbf{e}_1\right\}$ is linearly independent.*

**Proof.** First note that

$$H\mathbf{e}_1 = \begin{bmatrix} h_{11} \\ h_{21} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad H^2\mathbf{e}_1 = \begin{bmatrix} h_{11}^2 + h_{12}h_{21} \\ h_{21}h_{11} + h_{22}h_{21} \\ h_{32}h_{21} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 0 \\ \vdots \\ 0 \end{bmatrix};$$

since $H$ is unreduced, $h_{21} \neq 0$ and $b_3 = h_{32}h_{21} \neq 0$. Inductively, assume that $H^{i-1}\mathbf{e}_1 = [c_1 \cdots c_i \; 0 \cdots \; 0]^T$ with $c_i \neq 0$. Then

$$H^i\mathbf{e}_1 = H\left(H^{i-1}\mathbf{e}_1\right) = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1,i} & \cdots & * & * \\ h_{21} & h_{22} & \cdots & h_{2,i} & \cdots & * & * \\ 0 & h_{32} & \ddots & h_{3,i} & \cdots & \vdots & \vdots \\ 0 & 0 & \ddots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \ddots & h_{i+1,i} & \cdots & * & * \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & * & * \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} h_{11}c_1 + \cdots + h_{1i}c_i \\ h_{21}c_1 + \cdots + h_{2i}c_i \\ h_{32}c_2 + \cdots + h_{3i}c_i \\ \vdots \\ h_{i+1,i}c_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and $h_{i+1,i}c_i \neq 0$. Thus for $1 \leq i \leq n-1$ the $(i+1)^{st}$ component of $H^i\mathbf{e}_1$ is non-zero, and for $1 \leq i \leq n-2$, $i+2 \leq k \leq n$, the $k^{th}$ component of $H^i\mathbf{e}_1$ is zero. Hence the matrix $\left[\mathbf{e}_1 \mid H\mathbf{e}_1 \mid \cdots \mid H^{n-1}\mathbf{e}_1\right]$ is upper triangular with non-zero diagonal entries, and it follows that $\left\{\mathbf{e}_1, H\mathbf{e}_1, \ldots, H^{n-1}\mathbf{e}_1\right\}$ is linearly independent. ■

**Corollary 2** *Let $H$ be an unreduced $n \times n$ Hessenberg matrix. If $[a_0 \ \cdots \ a_{n-1}]^T \neq \mathbf{0}$, then*

$$a_0\mathbf{e}_1 + a_1 H\mathbf{e}_1 + \cdots + a_{n-1}H^{n-1}\mathbf{e}_1 \neq \mathbf{0}.$$

**Definition 3** *Let $A$ be an $n \times n$ matrix and let $\lambda$ be an eigenvalue of $A$. A vector $\mathbf{v} \in \mathbb{R}^n$ is a generalized eigenvector of order $r$ associated with $\lambda$ if $(A - \lambda I)^{r-1}\mathbf{v} \neq \mathbf{0}$ and $(A - \lambda I)^r \mathbf{v} = \mathbf{0}$.*

An usual eigenvector associated with $\lambda$ is a generalized eigenvector of order 1.

**Definition 4** *Let $A$ be an $n \times n$ matrix with characteristic polynomial $p(t) = (t - \lambda_1)^{r_1} \cdots (t - \lambda_k)^{r_k}$, where $r_1 + \cdots + r_k = n$ and $\lambda_i \in \mathbb{C}$. Then $r_i$ is the algebraic multiplicity of $\lambda_i$, and the dimension of the eigenspace associated with $\lambda_i$ is the geometric multiplicity of $\lambda_i$.*

The geometric multiplicity of an eigenvalue is at least 1 and at most its algebraic multiplicity.

**Definition 5** *An $n \times n$ matrix $A$ is defective if $A$ has an eigenvalue whose geometric multiplicity is less than its algebraic multiplicity.*

**Example 6** *The characteristic polynomial of the unreduced Hessenberg matrix*

$$H = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \quad \text{is} \quad p(t) = \det\begin{bmatrix} 1-t & -1 \\ 1 & 3-t \end{bmatrix} = (t-2)^2;$$

*the algebraic multiplicity of the eigenvalue $\lambda = 2$ is therefore 2. To find a basis for the corresponding eigenspace, solve the system*

$$(H - 2I)\mathbf{x} = 0$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \xrightarrow{row\ reduce} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix};$$

*then $\left\{\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ is a basis and the geometric multiplicity of the eigenvalue $\lambda = 2$ is 1. Consequently $H$ is defective and is not diagonalizable.*

**Exercise 7** *Show that the following matrices are defective:*

a. $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$ b. $\begin{bmatrix} -4 & 1 & 1 & 1 \\ -16 & 3 & 4 & 4 \\ -7 & 2 & 2 & 1 \\ -11 & 1 & 3 & 4 \end{bmatrix}$

**Theorem 8** *Let $\lambda$ be an eigenvalue of an unreduced $n \times n$ Hessenberg matrix $H$. If the algebraic multiplicity of $\lambda$ is $m$, then $H$ has a generalized eigenvector $\mathbf{v}_r$ of order $r$ associated with $\lambda$ for each $r = 1, 2, \ldots, m$.*

**Proof.** Let $p(t)$ be the characteristic polynomial of $H$. Repeatedly divide $p(t)$ by $t - \lambda$ and until

$$p(t) = (t - \lambda)^m q(t) \text{ with } q(\lambda) \neq 0.$$

Consider the polynomial

$$(t - \lambda)^{m-1} q(t) = a_0 + a_1 t + \cdots + a_{n-2} t^{n-2} + t^{n-1}$$

whose vector of coefficients $[a_0 \ \cdots \ a_{n-2} \ 1]^T \neq \mathbf{0}$. Thus

$$(H - \lambda I)^{m-1} q(H) \mathbf{e}_1 = a_0 \mathbf{e}_1 + a_1 H \mathbf{e}_1 + \cdots + a_{n-2} H^{n-2} \mathbf{e}_1 + H^{n-1} \mathbf{e}_1 \neq \mathbf{0}$$

by Corollary 2. Set $\mathbf{v}_m = q(H) \mathbf{e}_1$, then

$$(H - \lambda I)^{m-1} \mathbf{v}_m \neq \mathbf{0}.$$

On the other hand, $p(H) = (H - \lambda I)^m q(H)$ implies

$$(H - \lambda I)^m \mathbf{v}_m = (H - \lambda I)^m q(H) \mathbf{e}_1 = p(H) \mathbf{e}_1.$$

But $p(H) = \mathbf{0}$ by the Cayley-Hamilton Theorem, and we conclude that

$$(H - \lambda I)^m \mathbf{v}_m = \mathbf{0}.$$

Therefore $\mathbf{v}_m$ is a generalized eigenvector of order $m$ associated with $\lambda$. Finally, since $(H - \lambda I)^{m-1} q(H) \mathbf{e}_1 \neq \mathbf{0}$ and $(H - \lambda I)^m q(H) \mathbf{e}_1 = \mathbf{0}$, let

$$\mathbf{v}_r = (H - \lambda I)^{m-r} q(H) \mathbf{e}_1;$$

then $(H - \lambda I)^{r-1} \mathbf{v}_r \neq \mathbf{0}$ and $(H - \lambda I)^r \mathbf{v}_r = \mathbf{0}$. Thus $H$ has a generalized eigenvector of order $r$ *associated with* $\lambda$ for each $r = 1, 2, \ldots, m$. ■

**Theorem 9** *An unreduced $n \times n$ Hessenberg matrix has $n$ linearly independent generalized eigenvectors.*

**Proof.** Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the eigenvalues of $H$ and let $m_i$ be the algebraic multiplicity of $\lambda_i$. Then $p(t) = (t - \lambda_1)^{m_1} (t - \lambda_2)^{m_2} \cdots (t - \lambda_k)^{m_k}$ is the characteristic polynomial of $H$. By Theorem 8, each $\lambda_i$ has an associated generalized eigenvector $\mathbf{v}_i^r$ of order $r$ for each $r = 1, 2, \ldots, m_i$. Denote these by

$$\{\underbrace{\mathbf{v}_1^1, \ldots, \mathbf{v}_1^{m_1}}_{\lambda_1}, \underbrace{\mathbf{v}_2^1, \ldots, \mathbf{v}_2^{m_2}}_{\lambda_2}, \ldots, \underbrace{\mathbf{v}_k^1, \ldots, \mathbf{v}_k^{m_k}}_{\lambda_k}\}$$

and suppose that

$$a_1^1 \mathbf{v}_1^1 + \cdots + a_1^{m_1} \mathbf{v}_1^{m_1} + a_2^1 \mathbf{v}_2^1 + \cdots + a_2^{m_2} \mathbf{v}_2^{m_2} + \cdots + a_k^1 \mathbf{v}_k^1 + \cdots + a_k^{m_k} \mathbf{v}_k^{m_k} = \mathbf{0}. \tag{2}$$

Let $q(t) = (t - \lambda_2)^{m_2} \cdots (t - \lambda_k)^{m_k}$; since the factors of $q(t)$ commute, $q(H) \mathbf{v}_i^j = \mathbf{0}$ for all $i$ and all $j \geq 2$. Now multiply both sides of (2) by $q(H)$; then

$$q(H) \left(a_1^1 \mathbf{v}_1^1 + \cdots + a_1^{m_1} \mathbf{v}_1^{m_1}\right) + \underbrace{q(H) \left(a_2^1 \mathbf{v}_2^1 + \cdots + a_2^{m_2} \mathbf{v}_2^{m_2} + \cdots + a_k^1 \mathbf{v}_k^1 + \cdots + a_k^{m_k} \mathbf{v}_k^{m_k}\right)}_{\mathbf{0}} = \mathbf{0}$$

reduces to

$$a_1^1 q(H) \mathbf{v}_1^1 + \cdots + a_1^{m_1} q(H) \mathbf{v}_1^{m_1} = \mathbf{0}. \tag{3}$$

Keeping in mind that $(t - \lambda_1)^p q(t) = q(t) (t - \lambda_1)^p$, multiply both sides of (3) by $(H - \lambda_1 I)^{m_1 - 1}$; then

$$a_1^{m_1} \underbrace{(H - \lambda_1 I)^{m_1 - 1} q(H) \mathbf{v}_1^{m_1}}_{\neq \mathbf{0}} = \mathbf{0} \tag{4}$$

implies $a_1^{m_1} = 0$. Now multiply both sides of (4) by $(H - \lambda_1 I)^{m_1 - 2}$; then

$$a^1_{m_1-1} \underbrace{(H - \lambda_1 I)^{m_1-2} \, q(H) \, \mathbf{v}_1^{m_1-1}}_{\neq \mathbf{0}} = \mathbf{0}$$

implies $a^1_{m_1-1} = 0$, and so on inductively. Do this for each $\lambda_i$ and conclude that all coefficients vanish. ■

**Example 10** *Continuing Example 6, Let's find a generalized eigenvector* $\mathbf{v}_2$ *associated with* $\lambda = 2$ *such that* $\{\mathbf{v}_1, \mathbf{v}_2\}$ *is linearly independent. Solve the system*

$$(H - 2I)\,\mathbf{x} = \mathbf{v}_1$$

$$\left[\begin{array}{cc|c} -1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right] \overset{row\ reduce}{\longrightarrow} \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right];$$

*the general solution is*

$$t\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

*Set* $t = 0$ *and obtain the particular solution* $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. *Note that*

$$(H - 2I)^2\,\mathbf{v}_2 = (H - 2I)\,\mathbf{v}_1 = 0;$$

*then* $\mathbf{v}_2$ *is a generalized eigenvector of order* 2 *associated with* $\lambda = 2$. *Thus we obtain two linearly independent generalized eigenvectors associated with* $\lambda = 2$ :

$$\left\{\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right\}.$$

**Problem:** Let $H$ be a complex $n \times n$ unreduced Hessenberg matrix. Solve the IVP $y' = Hy$ with $y(0) = y_0$.

**Steps in the solution:**

1. Using Krylov's method, compute the characteristic polynomial of $H$. Then

   $$p(t) = (t - \lambda_1)^{\alpha_1}(t - \lambda_2)^{\alpha_2} \cdots (t - \lambda_k)^{\alpha_k};$$

   the eigenvalue $\lambda_i$ has algebraic multiplicity $\alpha_i$.

2. Find a maximal linearly independent set of eigenvectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$. Then each $\mathbf{u}_j$ is associated with some $\lambda_i$. Set $\mathbf{y} = e^{\lambda_i t}\mathbf{u}_j$; then

   $$\mathbf{y}' = e^{\lambda_i t}\lambda_i \mathbf{u}_j = e^{\lambda_i t} H \mathbf{u}_j = H\left(e^{\lambda_i t}\mathbf{u}_j\right) = H\mathbf{y}$$

   so that $\mathbf{y}$ is a particular solution.

3. Let $m_i$ be the geometric multiplicity of $\lambda_i$. If $m_i < \alpha_i$, find $q_i = \alpha_i - m_i$ generalized eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_{q_i}\}$ of $H$ in the following way: Set $\mathbf{v}_1 = \mathbf{u}_1$ and solve the following successive sequence of linear equations in which $\mathbf{v}_r$ is known and $\mathbf{v}_{r+1}$ is unknown:

   $$\begin{aligned} (H - \lambda_i I)\,\mathbf{v}_2 &= \mathbf{v}_1 \\ (H - \lambda_i I)\,\mathbf{v}_3 &= \mathbf{v}_2 \\ (H - \lambda_i I)\,\mathbf{v}_4 &= \mathbf{v}_3 \\ &\vdots \\ (H - \lambda_i I)\,\mathbf{v}_{q_i} &= \mathbf{v}_{q_i - 1}. \end{aligned}$$

Then

$$(H-\lambda_i I)^{q_i-1}\mathbf{v}_{q_i} = (H-\lambda_i I)^{q_i-2}\mathbf{v}_{q_{i-1}} = \cdots = (H-\lambda_i I)\mathbf{v}_2 = \mathbf{v}_1. \tag{5}$$

Since $(H-\lambda_i I)\mathbf{v}_1 = \mathbf{0}$, multiplying each expression in (5) by $H-\lambda_i I$ gives

$$(H-\lambda_i I)^{q_i}\mathbf{v}_{q_i} = (H-\lambda_i I)^{q_i-1}\mathbf{v}_{q_{i-1}} = \cdots = (H-\lambda_i I)^2\mathbf{v}_2 = (H-\lambda_i I)\mathbf{v}_1 = 0.$$

Then $\{\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_{q_i}\}$ is linearly independent by Theorem 9 and $\mathbf{v}_r$ is a generalized eigenvector of order $r$ associated with $\lambda_i$ for each $r = 1, 2, \ldots, q_i$. Let

$$\mathbf{y}_r = e^{\lambda_i t}\left(\mathbf{v}_r + t\mathbf{v}_{r-1} + \frac{t^2}{2!}\mathbf{v}_{r-2} + \cdots + \frac{t^{r-1}}{(r-1)!}\mathbf{v}_1\right);$$

then

$$\begin{aligned}
\mathbf{y}_r' &= \lambda_i e^{\lambda_i t}\left(\mathbf{v}_r + t\mathbf{v}_{r-1} + \frac{t^2}{2!}\mathbf{v}_{r-2} + \cdots + \frac{t^{r-1}}{(r-1)!}\mathbf{v}_1\right) + e^{\lambda_i t}\left(\mathbf{v}_{r-1} + t\mathbf{v}_{r-2} + \cdots + \frac{t^{r-2}}{(r-2)!}\mathbf{v}_1\right) \\
&= e^{\lambda_i t}\Bigg((\mathbf{v}_{r-1} + \lambda_i\mathbf{v}_r) + t(\mathbf{v}_{r-2} + \lambda_i\mathbf{v}_{r-1}) + \frac{t^2}{2!}(\mathbf{v}_{r-3} + \lambda_i\mathbf{v}_{r-2}) + \cdots + \frac{t^{r-2}}{(r-2)!}(\mathbf{v}_1 + \lambda_i\mathbf{v}_2) \\
&\qquad + \frac{t^{r-1}}{(r-1)!}\lambda_i\mathbf{v}_1\Bigg) \\
&= e^{\lambda_i t}\left(H\mathbf{v}_r + tH\mathbf{v}_{r-1} + \frac{t^2}{2!}H\mathbf{v}_{r-2} + \cdots + \frac{t^{r-1}}{(r-1)!}H\mathbf{v}_1\right) = H\mathbf{y}_r
\end{aligned}$$

so that $\mathbf{y}_r$ is a particular solution.

4. Finally, the general solution of the given system of linear differential equations consists of all linear combinations of the solutions found in steps 2 and 3. Given this, one can solve a system of linear equations and find the particular solution that solves the IVP.

**Example 11** *Solve the system*

$$\begin{aligned} y_1' &= y_1 - y_2 \\ y_2' &= y_1 + 3y_2 \end{aligned}$$

*subject to the initial conditions* $y_1(0) = 5$, $y_2(0) = -7$. *Written in matrix form* $\mathbf{y}' = H\mathbf{y}$ *the IVP becomes*

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \text{ with } \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \end{bmatrix}. \tag{6}$$

*In Examples 6 and 10 we found the linearly independent generalized eigenvectors*

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

*associated with the eigenvalue* $\lambda = 2$ *of the unreduced Hessenberg coefficient matrix*

$$H = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}.$$

*Let*

$$\mathbf{y} = e^{2t}\mathbf{v}_1;$$

*differentiating we have*

$$\mathbf{y}' = e^{2t}(2\mathbf{v}_1) = e^{2t}H\mathbf{v}_1 = H\mathbf{y}$$

*so that* $\mathbf{y} = e^{2t}\mathbf{v}_1$ *is a particular solution. Now*

$$(H-2I)\mathbf{v}_2 = \mathbf{v}_1 \quad\Rightarrow\quad H\mathbf{v}_2 = \mathbf{v}_1 + 2\mathbf{v}_2.$$

*Let*

$$\mathbf{y} = e^{2t}\left(\mathbf{v}_2 + t\mathbf{v}_1\right);$$

*differentiating gives*

$$\begin{aligned}
\mathbf{y}' &= 2e^{2t}\left(\mathbf{v}_2 + t\mathbf{v}_1\right) + e^{2t}\mathbf{v}_1 = e^{2t}\left[\left(\mathbf{v}_1 + 2\mathbf{v}_2\right) + t\left(2\mathbf{v}_1\right)\right] \\
&= e^{2t}\left(H\mathbf{v}_2 + tH\mathbf{v}_1\right) = He^{2t}\left(\mathbf{v}_2 + t\mathbf{v}_1\right) = H\mathbf{y}
\end{aligned}$$

*so that* $\mathbf{y} = e^{2t}\left(\mathbf{v}_2 + t\mathbf{v}_1\right)$ *is a particular solution. Thus the general solution of system 6 is*

$$\mathbf{y} = ae^{2t}\mathbf{v}_1 + be^{2t}\left(\mathbf{v}_2 + t\mathbf{v}_1\right), \ a, b \in \mathbb{R}.$$

*To find the desired particular solution, solve* $a\mathbf{v}_1 + b\mathbf{v}_2 = \mathbf{y}(0):$

$$\begin{bmatrix} -1 & 1 & \vdots & 5 \\ 1 & 0 & \vdots & -7 \end{bmatrix} \overset{row\ reduce}{\longrightarrow} \begin{bmatrix} 1 & 0 & \vdots & -7 \\ 0 & 1 & \vdots & -2 \end{bmatrix}.$$

*The solution of the IVP is therefore*

$$\mathbf{y} = -7e^{2t}\mathbf{v}_1 - 2e^{2t}\left(\mathbf{v}_2 + t\mathbf{v}_1\right).$$

11-10-2010