Lecture Notes on Linear Algebra
A Second Course

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Fall 2012

Table of Contents

Computing the Inverse of an Invertible Matrix

If $A = (a_{ij})$ is an $n \times n$ invertible matrix, there is an $n \times n$ matrix B such that $AB = I$, in which case we write $A^{-1} = B$. To compute an unknown B, we must solve the matrix equation $AX = I$ for $X = (x_{i,j})$. Recall that the i^{th} entry in the j^{th} column of the product matrix AX is obtained by multiplying the i^{th} row of A and the j^{th} column of X in the following way:

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{bmatrix} = a_{i1}x_{1j} + a_{i2}x_{2j} + \cdots + a_{in}x_{nj}.$$

Let X_i denote the i^{th} column of X and write $X = [\mathbf{x}_1 \mid \mathbf{x}_2 \mid \cdots \mid \mathbf{x}_n]$. Then the j^{th} column of AX is

$$A\mathbf{x}_j = \begin{bmatrix} a_{11}x_{1j} + a_{12}x_{2j} + \cdots + a_{1n}x_{nj} \\ a_{21}x_{1j} + a_{22}x_{2j} + \cdots + a_{2n}x_{nj} \\ \vdots \\ a_{n1}x_{1j} + a_{n2}x_{2j} + \cdots + a_{nn}x_{nj} \end{bmatrix},$$

so that

$$AX = [A\mathbf{x}_1 \mid A\mathbf{x}_2 \mid \cdots \mid A\mathbf{x}_n].$$

Denote the j^{th} standard basis vector in $\mathbb{R}^n$ by

$$\mathbf{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow j^{th} \text{ position} \\ \\ \\ \end{matrix}$$

and express the identity matrix as

$$I = [\mathbf{e}_1 \mid \mathbf{e}_2 \mid \cdots \mid \mathbf{e}_n].$$

Then the matrix equation $AX = I$ can be rewritten in the form

$$[A\mathbf{x}_1 \mid A\mathbf{x}_2 \mid \cdots \mid A\mathbf{x}_n] = [\mathbf{e}_1 \mid \mathbf{e}_2 \mid \cdots \mid \mathbf{e}_n].$$

By equating columns we obtain n systems of linear equations

$$A\mathbf{x}_j = \mathbf{e}_j$$

whose solutions are the columns of the solution matrix $X = B$. Note that for each j, the row reduction is determined by A and is independent of j. So we may perform all n row reductions simultaneously by row-reducing the $n \times 2n$ matrix

$$[A \mid I],$$

by which we obtain

$$[I \mid A^{-1}].$$

Example 1 *Let* $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$. *The row reduction* $\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \end{array}\right]$ *implies* $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$.

8-30-2012

Simpson's Rule

Given n distinct data points $(a_1, b_1), \ldots, (a_n, b_n)$ no two of which lie on the same vertical line, there is a unique interpolating polynomial $p(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$, called the *interpolant,* whose graph passes through them. A proof of this fact in the case $n = 3$ appears in Example 3, p. 82 of Johnson, Riess, and Arnold. Simpson's Rule for numerically approximating the definite integral of a continuous function f on a closed interval $[a, b]$ is an important application.

For simplicity, let $h > 0$ and consider a continuous function f on $[0, 2h]$. Our goal is to find the quadratic interpolant $p(x) = a_0 + a_1 x + a_2 x^2$ of the data points $(0, f(0))$, $(h, f(h))$, and $(2h, f(2h))$. The coefficients a_0, a_1, and a_2 are the unique solution of the linear system

$$\begin{aligned} a_0 &= f(0) \\ a_0 + a_1 h + a_2 h^2 &= f(h) \\ a_0 + 2a_1 h + 4a_2 h^2 &= f(2h). \end{aligned}$$

Row-reducing the augmented coefficient matrix gives:

$$\begin{bmatrix} 1 & 0 & 0 & f(0) \\ 1 & h & h^2 & f(h) \\ 1 & 2h & 4h^2 & f(2h) \end{bmatrix} \overset{\text{row reduce}}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 & f(0) \\ 0 & 1 & 0 & \frac{1}{2h}[-3f(0) + 4f(h) - f(2h)] \\ 0 & 0 & 1 & \frac{1}{2h^2}[f(0) - 2f(h) + f(2h)] \end{bmatrix}.$$

Thus

$$\int_0^{2h} f(x)\,dx \approx \int_0^{2h} p(x)\,dx = \int_0^{2h} a_0 + a_1 x + a_2 x^2 dx = (2h)\,a_0 + \left(2h^2\right) a_1 + \left(\frac{8h^3}{3}\right) a_2$$

$$= (2h)\,f(0) + h[-3f(0) + 4f(h) - f(2h)] + \frac{4h}{3}[f(0) - 2f(h) + f(2h)]$$

$$= \frac{h}{3}[f(0) + 4f(h) + f(2h)].$$

Now to obtain the familiar formula from calculus, let f be continuous on $[a, b]$ and let $\{a = x_0 < x_1 < \cdots < x_{2n} = b\}$ be the regular partition of $[a, b]$ with $2n$ subintervals. Set $h = x_i - x_{i-1} = (b - a)/n$ and apply the formula above on each of the subintervals $[x_{i-2}, x_i]$. Then

$$\int_{x_{i-2}}^{x_i} f(x)\,dx \approx \frac{b - a}{3n}[f(x_{i-2}) + 4f(x_{i-1}) + f(x_i)]$$

and we obtain

$$\int_a^b f(x)\,dx = \sum_{k=1}^{n} \int_{x_{2k-2}}^{x_{2k}} f(x)\,dx$$

$$\approx \frac{b - a}{3n}\{[f(x_0) + 4f(x_1) + f(x_2)] + [f(x_2) + 4f(x_3) + f(x_4)] + \cdots$$

$$+ [f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})]\}$$

$$= \frac{b - a}{3n}\{f(x_0) + f(x_n) + 2[f(x_2) + f(x_4) + \cdots + f(x_{2n-2})]$$

$$+4[f(x_1) + f(x_3) + \cdots + f(x_{2n-1})]\}.$$

10-30-2010

Vector Spaces and Linear Independence

Let F be a field. Roughly speaking, a vector space over F is an abelian group $(V,+)$ on which F acts by scalar multiplication, i.e., if $\mathbf{v} \in V$ and $a \in F$, then $a\mathbf{v} \in V$. But before we give the precise definition of a vector space, we need some preliminary definitions.

Definition 2 *A **group** is a non-empty set G equipped with a binary operation $*$ such that:*

1. *Closure: If $a, b \in G$, then $a * b \in G$.*
2. *Associativity: If $a, b, c \in G$, then $(a * b) * c = a * (b * c)$.*
3. *Identity element: There is an element $e \in G$ such that $a * e = a = e * a$ for all $a \in G$.*
4. *Inverses: If $a \in G$, there is an element $b \in G$ such that $a * b = e$. The element b is called the* inverse of *a and we write $b = a^{-1}$.*

Definition 3 *A group $(G, *)$ is **abelian** if $*$ is commutative: If $a, b \in G$, then $a * b = b * a$.*

Example 4 *Some familiar examples of abelian groups are the integers with its usual addition $(\mathbb{Z}, +)$ and the non-zero real numbers with its usual multiplication $(\mathbb{R} - \{0\}, \cdot)$.*

Definition 5 *A **field** is a set F containing the distinct elements 0 and 1, and equipped with an addition $+$ and a multiplication $\cdot$ such that:*

1. *$(F, +)$ is an abelian group.*
2. *$(F - \{0\}, \cdot)$ is an abelian group.*
3. *Multiplication distributes over addition: If $a, b, c \in F$, then $a(b + c) = ab + ac$.*

Example 6 *Some familiar examples of fields are the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, and the integers modulo a prime $\mathbb{Z}_p$. Note that $\mathbb{Z}_2 = \{0, 1\}$ is the smallest possible field.*

Definition 7 *Let F be a field. An abelian group $(V, +)$ is a **vector space over** F (with coefficients in F) if F acts on V in the following way:*

1. *If $a \in F$ and $\mathbf{v} \in V$, then $a\mathbf{v} = \mathbf{v}a \in V$.*
2. *If $\mathbf{v} \in V$, then $1\mathbf{v} = \mathbf{v}$.*
3. *If $a \in F$ and $\mathbf{v}, \mathbf{w} \in V$, then $a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}$.*
4. *If $a, b \in F$ and $\mathbf{v} \in V$, then $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.*
5. *If $a, b \in F$ and $\mathbf{v} \in V$, then $(ab)\mathbf{v} = a(b\mathbf{v})$.*

   *Elements of F are called **scalars**; elements of V are called **vectors**.*

Proposition 8 *Let V be a vector space over a field F. If $v \in V$, then $0\mathbf{v} = \mathbf{0}$.*

Proof. First, note that $0 = 0 + 0$ since 0 is the additive identity element in F. Then by Axiom 4 above we have

$$0\mathbf{v} = (0 + 0)\mathbf{v} = 0\mathbf{v} + 0\mathbf{v}. \tag{1}$$

Since $0\mathbf{v} \in \mathbf{V}$ by Axiom 1, we have $0\mathbf{v} - 0\mathbf{v} = \mathbf{0} \in V$. Thus subtracting $0\mathbf{v}$ from both sides of (1) gives $0\mathbf{v} = \mathbf{0}$.

■

Example 9 *Some familiar examples of vector spaces from Linear Algebra I are $\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}\}$ and $\mathcal{P}_n = \{p(x) = a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{R} \text{ and } \deg p(x) \leq n\}$.*

Linear independence is a fundamentally important idea in linear algebra. If V is a vector space over a field F, an element of a linearly independent set $S \subset V$ cannot be written as a linear combination of the other elements of S. Note that for any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$, the equation $\mathbf{v}_1 x_1 + \cdots + \mathbf{v}_n x_n = \mathbf{0}$ always has the trivial solution $x_1 = \cdots = x_n = 0$.

Definition 10 *Let V be a vector space over a field F. A set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ is **linearly independent** if the equation $\mathbf{v}_1 x_1 + \cdots + \mathbf{v}_n x_n = \mathbf{0}$ has only the trivial solution. Otherwise S is **linearly dependent**.*

Linear independence in $\mathbb{R}^n$ is especially important. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \subset \mathbb{R}^n$ and express each $\mathbf{v}_i \in S$ as a column matrix. Form the $n \times m$ matrix $A = [\mathbf{v}_1 \mid \cdots \mid \mathbf{v}_m]$ and express the equation $\mathbf{v}_1 x_1 + \cdots + \mathbf{v}_m x_m = \mathbf{0}$ in matrix form as $A\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = [x_1 \ \cdots \ x_m]^T$. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is linearly independent if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if A row-reduces to $\begin{bmatrix} I^{m\times m} \\ \mathbf{0}^{(n-m)\times m} \end{bmatrix}$. Note that whenever A row-reduces to $\begin{bmatrix} I^{m\times m} \\ \mathbf{0}^{(n-m)\times m} \end{bmatrix}$, we automatically have $m \leq n$. It follows that *every* set containing more than n vectors in $\mathbb{R}^n$ is *linearly dependent.*

Example 11 *Since $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}$ row-reduces to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, the set $\left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$ is linearly independent in $\mathbb{R}^3$. On the other hand, since $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$ row-reduces to $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, the set $\left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is linearly dependent. The vector form of the solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$; by setting $t = 1$ we obtain the dependence relation $\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$ so that $\mathbf{v}_3 = \mathbf{v}_2 - \mathbf{v}_1$, which is clearly evident by inspection.*

9-6-2012

Matrix Representation of Linear Maps

Let V and W be vector spaces over a field F.

Definition 12 *A map $T : V \to W$ is* ***linear*** *if and only if*

$$T(a\mathbf{x} + b\mathbf{y}) = a\,T(\mathbf{x}) + b\,T(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in V$ and all $a, b \in F$.

Definition 13 *A subset $S \subset V$ is a* ***spanning set for*** *V if every element of V can be written as a linear combination of elements of S. A* ***basis for*** *V is a linearly independent spanning set. Since every basis for V has the same cardinality, we define the* ***dimension of*** *V to be the number of elements in any basis.*

When V and W are finite dimensional, we can evaluate a linear map $T : V \to W$ via matrix multiplication in the following way. Choose *ordered* bases $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for V and $\mathcal{C} = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ for W. Then each vector $\mathbf{x} \in V$ can be expressed as a unique linear combination of elements in $\mathcal{B}$, i.e.,

$$\mathbf{x} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n.$$

The (unique) coefficients x_i are called the *$\mathcal{B}$-coordinates of* $\mathbf{x}$ and the column matrix

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is called the *$\mathcal{B}$-coordinate matrix for* $\mathbf{x}$. Evaluate T at each basis vector $\mathbf{v}_i$, write $T(\mathbf{v}_i)$ in the basis $\mathcal{C}$ as

$$T(\mathbf{v}_i) = a_{1i}\mathbf{w}_1 + \cdots + a_{mi}\mathbf{w}_m,$$

and obtain the $\mathcal{C}$-coordinate matrix for $T(\mathbf{v}_i)$:

$$[T(\mathbf{v}_i)]_{\mathcal{C}} = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{bmatrix}.$$

Then

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n) \\ &= x_1\,T(\mathbf{v}_1) + \cdots + x_n\,T(\mathbf{v}_n) \\ &= x_1(a_{11}\mathbf{w}_1 + \cdots + a_{m1}\mathbf{w}_m) + \cdots + x_n(a_{1n}\mathbf{w}_1 + \cdots + a_{mn}\mathbf{w}_m) \\ &= (x_1a_{11} + \cdots + x_na_{1n})\,\mathbf{w}_1 + \cdots + (x_1a_{m1} + \cdots + x_na_{mn})\,\mathbf{w}_m, \end{aligned}$$

and the $\mathcal{C}$-coordinate matrix for $T(\mathbf{x})$ is

$$\begin{aligned} [T(\mathbf{x})]_{\mathcal{C}} &= \begin{bmatrix} x_1a_{11} + \cdots + x_na_{1n} \\ \\ x_1a_{m1} + \cdots + x_na_{mn} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ &= [[T(\mathbf{v}_1)]_{\mathcal{C}} \mid \cdots \mid [T(\mathbf{v}_n)]_{\mathcal{C}}]\,[\mathbf{x}]_{\mathcal{B}}. \end{aligned}$$

The *matrix for T in the bases $\mathcal{B}$ and $\mathcal{C}$* is

$$[T]_{\mathcal{C},\mathcal{B}} = [[T(\mathbf{v}_1)]_{\mathcal{C}} \mid \cdots \mid [T(\mathbf{v}_n)]_{\mathcal{C}}] = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

In summary, we have

$$[T]_{\mathcal{C},\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}.$$

Example 14 *Let $V = \mathcal{P}_3$ with ordered basis $\mathcal{B}_3 = \{1, x, x^2, x^3\}$ and let $W = \mathcal{P}_2$ with ordered basis $\mathcal{B}_2 = \{1, x, x^2\}$. Then polynomial differentiation is a linear map $D : \mathcal{P}_3 \to \mathcal{P}_2$, which acts on basis elements as follows:*

$$D(1) = 0, \quad D(x) = 1, \quad D(x^2) = 2x, \quad D(x^3) = 3x^2.$$

Thus

$$[D(1)]_{\mathcal{B}_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad [D(x)]_{\mathcal{B}_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [D(x^2)]_{\mathcal{B}_2} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad [D(x^3)]_{\mathcal{B}_2} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix},$$

and the matrix for D in the bases $\mathcal{B}_3$ and $\mathcal{B}_2$ is

$$[D]_{\mathcal{B}_2,\mathcal{B}_3} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Let $p(x) = 6 + 5x + 4x^2 + 3x^3$; then

$$[p(x)]_{\mathcal{B}_3} = \begin{bmatrix} 6 \\ 5 \\ 4 \\ 3 \end{bmatrix}$$

and

$$[D(p(x))]_{\mathcal{B}_2} = [D]_{\mathcal{B}_2,\mathcal{B}_3}[p(x)]_{\mathcal{B}_3} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 9 \end{bmatrix}.$$

Indeed, by translating this column matrix into a polynomial in $\mathcal{P}_2$ we see that

$$D(6 + 5x + 4x^2 + 3x^3) = 5 + 8x + 9x^2.$$

9-11-2012

Geometry of Linear Operators on $\mathbb{R}^2$

Definition 15 *A **linear operator** on a vector space V is a linear map $T : V \to V$.*

Let T be a linear operator on $\mathbb{R}^2$. To represent T in the standard basis $\left\{\mathbf{e}_1 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \mathbf{e}_2 = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right\}$, we evaluate $T(\mathbf{e}_1) = \left[\begin{smallmatrix}a\\c\end{smallmatrix}\right]$ and $T(\mathbf{e}_2) = \left[\begin{smallmatrix}b\\d\end{smallmatrix}\right]$ and write

$$[T] = \left[\begin{array}{c|c} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{array}\right] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Note that $[T]$ has no subscript specifying the basis (the standard basis is the default). Then for $\mathbf{x} = \left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] \in \mathbb{R}^2$, we have

$$[T(\mathbf{x})] = [T][\mathbf{x}] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Proposition 16 *Let T be a linear operator on $\mathbb{R}^2$ and let l be a line. Then $T(l)$ is either a point or a line.*

Proof. The equation of the line l is $\alpha x + \beta y + \gamma = 0$, where α and β are not both zero. We set

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

and consider two cases.

<u>Case 1</u>. Assume $[T]$ is invertible. Then

$$\begin{bmatrix} x \\ y \end{bmatrix} = [T]^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} (dx' - by')/(ad - bc) \\ (-cx' + ay')/(ad - bc) \end{bmatrix}.$$

Substituting for x and y in the equation of l gives

$$\alpha \left(\tfrac{1}{ad-bc} (dx' - by') \right) + \beta \left(\tfrac{1}{ad-bc} (-cx' + ay') \right) + \gamma = 0$$

or equivalently,

$$(d\alpha - c\beta) x' + (-b\alpha + a\beta) y' + (ad - bc)\gamma = 0.$$

Thus $\left[\begin{smallmatrix}x'\\y'\end{smallmatrix}\right]$ satisfies the equation

$$(d\alpha - c\beta) x + (-b\alpha + a\beta) y + (ad - bc)\gamma = 0,$$

which is the equation of a line if $d\alpha - c\beta$ and $-b\alpha + a\beta$ are not both zero. But if both coefficients were zero, the homogeneous system

$$\begin{aligned} du - cv &= 0 \\ -bu + av &= 0 \end{aligned}$$

would have the non-trivial solution $\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]$, which is impossible since

$$\begin{vmatrix} d & -c \\ -b & a \end{vmatrix} = ad - bc \neq 0.$$

Therefore $T(l)$ is a line, and its equation is

$$(d\alpha - c\beta) x + (-b\alpha + a\beta) y + (ad - bc)\gamma = 0.$$

<u>Case 2.</u> Assume $[T]$ is not invertible. This case is left as an exercise for the reader. ■

Exercise 17 *Let T be a linear operator $\mathbb{R}^2$ and let l be a line. Prove that if $[T]$ is not invertible, then $T(l)$ is either a point or a line.*

Proposition 18 *Let S and T be linear operators on $\mathbb{R}^n$. Then*

$$[T \circ S] = [T]\,[S]\,.$$

Proof. Note that

$$[T \circ S]\,[\mathbf{x}] = [(T \circ S)\,(\mathbf{x})] = [T\,(S\,(\mathbf{x}))] = [T]\,[(S\,(\mathbf{x}))] = [T]\,([S]\,[\mathbf{x}]) = ([T]\,[S])\,[\mathbf{x}]\,.$$

Therefore $[T \circ S] = [T]\,[S]\,.$ ■

A 2×2 elementary matrix E is the result of performing a single elementary row operation on the 2×2 identity matrix. Thus E has one of the following forms:

$$E_1 = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix},\ E_2 = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix},\ E_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ E_4 = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix},\ E_5 = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}.$$

Suppose that T is a linear operator on $\mathbb{R}^2$whose matrix representation $[T]$ in the standard basis is an elementary matrix E. When $k > 0$, multiplication by

- E_1 shears by a factor of k in the x direction;
- E_2 shears by a factor of k in the y direction;
- E_3 reflects in the line $y = x$;
- E_4 compresses by a factor of k in the x-direction when $0 < k < 1$;
- E_5 compresses by a factor of k in the y-direction when $0 < k < 1$;
- E_4 expands by a factor of k in the x-direction when $k > 1$;
- E_5 expands by a factor of k in the y-direction when $k > 1$.

When $k = -1$, multiplication by

- E_4 reflects in the y-axis;
- E_5 reflects in the x-axis.

When $k < 0$ and $k \neq -1$, write

$$E_4 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} |k| & 0 \\ 0 & 1 \end{bmatrix} \text{ and } E_5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & |k| \end{bmatrix}.$$

Then multiplication by

- E_4 compresses or expands by a factor of $|k|$ in the x-direction then reflects in the y-axis;
- E_5 compresses or expands by a factor of $|k|$ in the y-direction then reflects in the x-axis.

To summarize: *Multiplication by an elementary matrix represents*

- *a shear parallel to a coordinate axis*
- *a compression or expansion parallel to a coordinate axis*
- *a reflection in the line $y = x$*
- *a reflection in a coordinate axis*
- *a compression or an expansion parallel to a coordinate axis followed by a reflection in a coordinate axis.*

To determine the geometric effect of multiplication by a nonsingular matrix A, use multiplication by elementary matrices to row reduce A to I then solve for A :

$$E_k E_{k-1} \cdots E_1 A = I$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

Then multiplication by A is the sequence of the actions given by multiplication by E_k^{-1}, then by E_{k-1}^{-1}, then by E_{k-2}^{-1}, and so on.

Example 19 *To describe the geometric effect of multiplication by*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$

row reduce A to I as follows:

$$E_1 A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$

$$E_2 (E_1 A) = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$E_3 (E_2 E_1 A) = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solving for A we obtain

$$\begin{aligned} A = E_1^{-1} E_2^{-1} E_3^{-1} &= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Thus multiplication by A is equivalent to performing the following sequence of geometric operations:

- *shear by a factor of* 2 *in the x-direction;*
- *expand by a factor of* 2 *in the y-direction;*
- *reflect in the x-axis;*
- *shear by a factor of* 3 *in the y-direction.*

Example 20 *(Rotations as a product of elementary operators) Let $k \in \mathbb{Z}$. A rotation about the origin through an angle of $2k\pi$ is the identity map, whose matrix is the identity matrix. The matrix for rotation through an angle of $(2k+1)\pi$ is*

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which represents a reflection in the x-axis followed by a reflection in the y-axis. If $\theta \neq k\pi$, then the matrix for rotation through an angle θ is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \sin\theta \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \cos\theta & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\csc\theta \end{bmatrix} \begin{bmatrix} 1 & \cot\theta \\ 0 & 1 \end{bmatrix}.$$

This identifies a rotation through an angle of θ with the following sequence of geometric operations:

- *shear by a factor of* $\cot\theta$ *in the x-direction;*

- *expand by a factor of* $-\csc\theta$ *in the* y*-direction;*
- *shear by a factor of* $\cos\theta$ *in the* y*-direction;*
- *reflect in the line* $y = x$;
- *compress by a factor of* $\sin\theta$ *in the* y*-direction.*

This decomposition is *not* unique and depends on the sequence of elementary row operations we choose in the course of our row reduction.

9-11-2012

Change of Coordinates

Let $\mathcal{A} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be two ordered bases for a vector space V over a field F. Given $\mathbf{x} \in V$, express $\mathbf{x}$ as (unique) linear combinations of vectors in $\mathcal{A}$ or $\mathcal{B}$:

$$\mathbf{x} = a_1\mathbf{u}_1 + \cdots + a_n\mathbf{u}_n = b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n.$$

This provides two different ways to label the vector $\mathbf{x}$:

$$[\mathbf{x}]_{\mathcal{A}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad \text{and} \quad [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

Our goal is to use matrix multiplication to replace the system of coordinates given by $\mathcal{A}$ with the one given by $\mathcal{B}$, which amounts to representing the identity transformation $I : \mathbb{R}^n \to \mathbb{R}^n$ defined $I(\mathbf{x}) = \mathbf{x}$ as a matrix relative to the bases $\mathcal{A}$ and $\mathcal{B}$.

Think of $\mathcal{A}$ as the basis for the domain of the identity I and think of $\mathcal{B}$ as the basis for its range. Then I sends a vector $\mathbf{x}$ labeled in basis $\mathcal{A}$ to itself but relabeled in basis $\mathcal{B}$. The matrix for I in bases $\mathcal{A}$ and $\mathcal{B}$ is called the *transition matrix from $\mathcal{A}$ to $\mathcal{B}$ coordinates* and is given by

$$[I]_{\mathcal{B},\mathcal{A}} = [[I(\mathbf{u}_1)]_{\mathcal{B}} \mid \cdots \mid [I(\mathbf{u}_n)]_{\mathcal{B}}] = [[\mathbf{u}_1]_{\mathcal{B}} \mid \cdots \mid [\mathbf{u}_n]_{\mathcal{B}}].$$

This establishes the following relationship between vectors written in the bases $\mathcal{A}$ and $\mathcal{B}$:

$$[\mathbf{x}]_{\mathcal{B}} = [I]_{\mathcal{B},\mathcal{A}} [\mathbf{x}]_{\mathcal{A}}.$$

Since $[I]_{\mathcal{B},\mathcal{A}}$ is nonsingular, we multiply both sides by $[I]_{\mathcal{B},\mathcal{A}}^{-1}$ and get

$$[I]_{\mathcal{B},\mathcal{A}}^{-1} [\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{A}}.$$

Thus the transition matrix from $\mathcal{B}$ to $\mathcal{A}$ coordinates is the inverse and we have

$$[I]_{\mathcal{B},\mathcal{A}}^{-1} = [I]_{\mathcal{A},\mathcal{B}}.$$

When $\mathcal{A}$ is the standard basis, as it is in our next example, we usually suppress the subscript $\mathcal{A}$. But for purposes of demonstration, we'll display the subscript to help elucidate the general principle that applies to any basis $\mathcal{A}$.

Example 21 *Let $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis for $\mathbb{R}^2$ and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$, where $[\mathbf{v}_1]_{\mathcal{A}} = \left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$ and $[\mathbf{v}_2]_{\mathcal{A}} = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$. To compute the transition matrix $[I]_{\mathcal{B},\mathcal{A}} = [[\mathbf{e}_1]_{\mathcal{B}} \mid [\mathbf{e}_2]_{\mathcal{B}}]$ we solve the linear systems*

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = b_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} = b_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and obtain $[\mathbf{e}_1]_{\mathcal{B}} = \left[\begin{smallmatrix} -1/2 \\ 1/2 \end{smallmatrix}\right]$ and $[\mathbf{e}_1]_{\mathcal{B}} = \left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right]$. Then the transition matrix from $\mathcal{A}$ to $\mathcal{B}$-coordinates is

$$[I]_{\mathcal{B},\mathcal{A}} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix},$$

and the $\mathcal{B}$-coordinate matrix $[\mathbf{v}]_{\mathcal{B}}$ of the vector $[\mathbf{v}]_{\mathcal{A}} = \left[\begin{smallmatrix} 4 \\ -2 \end{smallmatrix}\right]$ is given by

$$\begin{aligned} [\mathbf{v}]_{\mathcal{B}} &= [I]_{\mathcal{B},\mathcal{A}} [\mathbf{v}]_{\mathcal{A}} \\ &= \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}. \end{aligned}$$

Furthermore, since

$$[I]_{\mathcal{A},\mathcal{B}} = [I]^{-1}_{\mathcal{B},\mathcal{A}} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

we can recover the $\mathcal{A}$-coordinates from $\mathcal{B}$-coordinates:

$$[\mathbf{v}]_{\mathcal{A}} = [I]_{\mathcal{A},\mathcal{B}}\,[\mathbf{v}]_{\mathcal{B}}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Note that the columns of $[I]_{\mathcal{A},\mathcal{B}}$ in the example above are exactly the coordinate matrices $[\mathbf{v}_1]_{\mathcal{A}} = \left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$ and $[\mathbf{v}_2]_{\mathcal{A}} = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$. This suggests the following general fact:

Theorem 22 *Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an ordered bases for $\mathbb{R}^n$. The transition matrix P from $\mathcal{B}$-coordinates to standard coordinates is*

$$P = [[\mathbf{v}_1] \mid \cdots \mid [\mathbf{v}_n]]\,.$$

Transition matrices between orthonormal bases are always *orthogonal* matrices. This is especially nice because orthogonal matrices A preserve the Eucildean inner product (hence distance and angle) and are easy to invert: $A^{-1} = A^T$. For this reason, we will use transition matrices between orthonormal bases when defining general rigid motions such as reflections and rotations.

Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an ordered bases for $\mathbb{R}^n$, and let P be the transition matrix from $\mathcal{B}$-coordinates to standard coordinates. Let T be a linear operator on $\mathbb{R}^n$ and consider its matrix representation $[T]_{\mathcal{B}}$. Then

$$[T]_{\mathcal{B}}\,P^{-1}\,[\mathbf{x}] = [T]_{\mathcal{B}}\,[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{B}} = P^{-1}\,[T(\mathbf{x})] = P^{-1}\,[T]\,[\mathbf{x}]\,.$$

Multiplying both sides by P gives

$$P\,[T]_{\mathcal{B}}\,P^{-1}\,[\mathbf{x}] = [T]\,[\mathbf{x}]\,.$$

Since this holds for all $\mathbf{x} \in \mathbb{R}^n$ we have

$$P\,[T]_{\mathcal{B}}\,P^{-1} = [T]$$

and we have recovered the matrix $[T]$ from $[T]_{\mathcal{B}}$ by conjugation. Now $\det(P^{-1})\det(P) = \det(P^{-1}P) = \det(I) = 1$ implies

$$\det[T] = \det(P^{-1}\,[T]_{\mathcal{B}}\,P) = \det(P^{-1})\det[T]_{\mathcal{B}}\det(P) = \det[T]_{\mathcal{B}}\,,$$

which proves:

Theorem 23 *Let $\mathcal{B}$ be an ordered basis for $\mathbb{R}^n$ and let T be a linear operator on $\mathbb{R}^n$. Then*

$$\det[T]_{\mathcal{B}} = \det[T]\,,$$

i.e., the determinant of the matrix for T is independent of the choice of basis.

It makes sense, therefore, to talk about the "determinant" of a linear operator.

Definition 24 *Let T be a linear operator on $\mathbb{R}^n$. The* ***determinant of*** *T is defined by*

$$\det(T) = \det[T]\,.$$

The operatorT is ***nonsingular*** *whenever* $\det(T) \neq 0$.

Example 25 *Let* $\mathcal{B} = \{\mathbf{u},\ \mathbf{v}\}$ *be the ordered basis for* $\mathbb{R}^2$ *with* $[\mathbf{u}] = \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$ *and* $[\mathbf{v}] = \left[\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right]$*. Define a linear operator* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *by*

$$T(\mathbf{u}) = 3\mathbf{u}{-}4\mathbf{v} \quad \text{and} \quad T(\mathbf{v}) = -\mathbf{u}{+}3\mathbf{v}.$$

Then

$$[T]_{\mathcal{B}} = \begin{bmatrix} 3 & -1 \\ -4 & 3 \end{bmatrix}$$

and the transition matrix P *from* $\mathcal{B}$ *to standard coordinates is*

$$P = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}.$$

Consequently,

$$P^{-1} = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}.$$

Therefore the matrix for T *in the standard basis is*

$$[T] = P\,[T]_{\mathcal{B}}\,P^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -4 & 3 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 5 \end{bmatrix}.$$

Observe that $\det\,[T] = \det\,[T]_{\mathcal{B}} = 5$.

Let T be a nonsingular linear operator on $\mathbb{R}^n$. Let $\mathbf{b} \in \mathbb{R}^n$ and consider the linear system

$$[T]\,\mathbf{x} = \mathbf{b}. \tag{2}$$

Since $[T]$ is invertible, equation (2) has the (unique) solution

$$\mathbf{x} = [T]^{-1}\,\mathbf{b},$$

which holds for all $\mathbf{b} \in \mathbb{R}^2$. Therefore T is surjective. Furthermore, if $[T]\,\mathbf{x} = [T]\,\mathbf{y}$, then multiplying both sides on the left by $[T]^{-1}$ gives $\mathbf{x} = \mathbf{y}$ and the map T is also injective. This proves:

Theorem 26 *Every nonsingular linear map is bijective.*

Definition 27 *Let* V *and* W *be vector spaces over a field* F*. A bijective linear map* $T : V \to W$ *is called an* ***isomorphism****. When* $T : V \to W$ *is an isomorphism, the spaces* V *and* W *are said to be* ***isomorphic****.*

9-12-12

Reflections in $\mathbb{R}^2$ and Rotations in $\mathbb{R}^3$

Reflections in $\mathbb{R}^2$.

Let $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis and let $\mathcal{B}' = \{\mathbf{v}_1, \mathbf{v}_2\}$ be an orthonormal basis. Note that the $\mathcal{B}'$-matrix for the reflection F in the axis of $\mathbf{v}_1$ is simply

$$[F]_{\mathcal{B}'} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus changing coordinates from $\mathcal{B}$ to $\mathcal{B}'$, reflecting in $\mathcal{B}'$ coordinates, and changing from $\mathcal{B}'$ back to $\mathcal{B}$, makes it easy to reflect in any line ℓ through the origin with angle of inclination θ. Construct the basis $\mathcal{B}'$ as follows: Let

$$[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \quad \text{and} \quad [\mathbf{v}_2]_{\mathcal{B}} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}.$$

Then ℓ is the axis of $\mathbf{v}_1$ and is perpendicular to $\mathbf{v}_2$. The orthogonal transition matrix from $\mathcal{B}'$ to $\mathcal{B}$ coordinates is

$$[I]_{\mathcal{B},\mathcal{B}'} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix};$$

its transpose is the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$ coordinates. Think of a point (x, y) in the plane as the terminal point of the vector $[\mathbf{x}]_{\mathcal{B}} = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$. To find the coordinates of its image $F(x, y) = (x', y')$ when reflected in line ℓ, perform the following computation:

$$[F(\mathbf{x})]_{\mathcal{B}} = [I]_{\mathcal{B},\mathcal{B}'} [F]_{\mathcal{B}'} [I]^T_{\mathcal{B},\mathcal{B}'} [\mathbf{x}]_{\mathcal{B}}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos 2\theta + y\sin 2\theta \\ x\sin 2\theta - y\cos 2\theta \end{bmatrix}.$$

Thus we have proved:

Theorem 28 *Let ℓ be a line through the origin of $\mathbb{R}^2$ with angle of inclination θ. The equations for the reflection in line ℓ are*

$$\begin{aligned} x' &= x\cos 2\theta + y\sin 2\theta \\ y' &= x\sin 2\theta - y\cos 2\theta. \end{aligned}$$

Example 29 *Let ℓ be the line with $\theta = \frac{\pi}{4}$. Then as one expects,*

$$\begin{aligned} x' &= y \\ y' &= x. \end{aligned}$$

Rotations in $\mathbb{R}^3$.

Rotations in space about an arbitrary axis ℓ through the origin can be described in a similar way. Let $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis and let $\mathcal{B}' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be an orthonormal basis. The $\mathcal{B}'$-matrix for the rotation R about the axis of $\mathbf{v}_1$ through angle θ is simply

$$[R]_{\mathcal{B}'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

Thus changing coordinates from $\mathcal{B}$ to $\mathcal{B}'$, rotating in $\mathcal{B}'$ coordinates, and changing from $\mathcal{B}'$ back to $\mathcal{B}$, makes it easy to rotate about ℓ. Choose a point (a,b,c) on ℓ with a and b not both zero, and let $\mathbf{v}_1$ be the vector

$$[\mathbf{v}_1]_{\mathcal{B}} = \frac{1}{\sqrt{a^2+b^2+c^2}}\begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Let

$$[\mathbf{v}_2]_{\mathcal{B}} = \frac{1}{\sqrt{a^2+b^2}}\begin{bmatrix} -b \\ a \\ 0 \end{bmatrix}$$

and let $\mathbf{v}_3 = \mathbf{v}_1 \times \mathbf{v}_2$. Then

$$[\mathbf{v}_3]_{\mathcal{B}} = \frac{1}{\sqrt{(a^2+b^2+c^2)(a^2+b^2)}}\begin{bmatrix} -ac \\ -bc \\ a^2+b^2 \end{bmatrix}$$

and $\mathcal{B}' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis. The transition matrix from $\mathcal{B}'$ to $\mathcal{B}$ coordinates is

$$[I]_{\mathcal{B},\mathcal{B}'} = [[\mathbf{v}_1]_{\mathcal{B}} \,|\, [\mathbf{v}_2]_{\mathcal{B}} \,|\, [\mathbf{v}_3]_{\mathcal{B}}]$$

$$= \left[\frac{1}{\sqrt{a^2+b^2+c^2}}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \frac{1}{\sqrt{a^2+b^2}}\begin{bmatrix} -b \\ a \\ 0 \end{bmatrix} \middle| \frac{1}{\sqrt{(a^2+b^2+c^2)(a^2+b^2)}}\begin{bmatrix} -ac \\ -bc \\ a^2+b^2 \end{bmatrix} \right].$$

Think of a point (x,y,z) in space as the terminal point of the vector $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. To find the coordinates of its image $R(x,y,z) = (x',y',z')$ when rotated about the line ℓ through angle θ, perform the following computation:

$$[R(\mathbf{x})]_{\mathcal{B}} = [I]_{\mathcal{B},\mathcal{B}'}\,[R]_{\mathcal{B}'}\,[I]^T_{\mathcal{B},\mathcal{B}'}\,[\mathbf{x}]_{\mathcal{B}}.$$

Then

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = [I]_{\mathcal{B},\mathcal{B}'}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}[I]^T_{\mathcal{B},\mathcal{B}'}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Example 30 *Let's rotate through an angle θ about the line ℓ through $(a,b,c) = (1,1,1)$. Using the transition matrix*

$$[I]_{\mathcal{B},\mathcal{B}'} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$$

we obtain

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = [I]_{\mathcal{B},\mathcal{B}'}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}[I]^T_{\mathcal{B},\mathcal{B}'}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}x + \frac{1}{3}y + \frac{1}{3}z + \frac{2}{3}x\cos\theta - \frac{1}{3}y\cos\theta - \frac{1}{3}z\cos\theta - \frac{1}{6}y(\sin\theta)\sqrt{2}\sqrt{6} + \frac{1}{6}z(\sin\theta)\sqrt{2}\sqrt{6} \\ \frac{1}{3}x + \frac{1}{3}y + \frac{1}{3}z - \frac{1}{3}x\cos\theta + \frac{2}{3}y\cos\theta - \frac{1}{3}z\cos\theta + \frac{1}{6}x(\sin\theta)\sqrt{2}\sqrt{6} - \frac{1}{6}z(\sin\theta)\sqrt{2}\sqrt{6} \\ \frac{1}{3}x + \frac{1}{3}y + \frac{1}{3}z - \frac{1}{3}x\cos\theta - \frac{1}{3}y\cos\theta + \frac{2}{3}z\cos\theta - \frac{1}{6}x(\sin\theta)\sqrt{2}\sqrt{6} + \frac{1}{6}y(\sin\theta)\sqrt{2}\sqrt{6} \end{bmatrix}.$$

To view an animation of this, click on the link "Rotations in 3-space" on the class web page and download the associated Mathematica notebook file. Load the file and click on [Evaluate] -> [Evaluate notebook]. Enjoy!

9-16-10

Inner Product Spaces and Orthogonal Transformations

Definition 31 *An F-**inner product space** is a vector space V over a field F equipped with a scalar-valued function $<\ ,\ >: V \times V \to F$, called an **inner product**, that satisfies following conditions for all $a, b, c \in V$ and $t \in F$:*

1. $\langle a, b\rangle = \langle b, a\rangle$
2. $\langle a+b, c\rangle = \langle a+c, b+c\rangle$
3. $\langle ta, b\rangle = \langle a, tb\rangle = t\langle a, b\rangle$
4. $\langle a, a\rangle \geq 0$ *and* $\langle a, a\rangle = 0$ *if and only if* $a = 0$.

Scalar valued functions on a vector space are called *forms*. Axiom (1) says that the form $\langle\ ,\ \rangle$ is symmetric. Axioms (2) and (3) tell us that $\langle\ ,\ \rangle$ is a *bilinear form*, i.e., for each fixed element $x \in V$, the functions $\langle x,\ \rangle : V \to \mathbb{R}$ and $\langle\ , x\rangle : V \to \mathbb{R}$ are linear. Axiom (4) says that $\langle\ ,\ \rangle$ is *positive definite* (never negative) and *non-degenerate* (the inner product of a non-zero vector with itself is never zero). In these terms, we say that *an inner product is a symmetric, positive definite, non-degenerate bilinear form on $V \times V$.*

Definition 32 *Let $(V, \langle\ ,\ \rangle)$ be an inner product space. The **norm** on V induced by $\langle\ ,\ \rangle$ is the form $\|\ \| : V \to \mathbb{R}$ defined by $\|a\| = \sqrt{\langle a, a\rangle}$. The **distance** d between two vectors $a, b \in V$ relative to $\langle\ ,\ \rangle$ is defined by $d(a, b) = \|a - b\|$.*

Definition 33 *The **Euclidean inner product** (often called the "dot product") of two vectors $\mathbf{x} = [x_1 \cdots x_n]^T$ and $\mathbf{y} = [y_1 \cdots y_n]^T$ in $\mathbb{R}^n$ is defined by*

$$\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^T\mathbf{y} = [x_1 \cdots x_n\ ]\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1y_1 + \cdots + x_ny_n.$$

We sometimes use the notation $\mathbf{x} \bullet \mathbf{y}$ to denote $\langle \mathbf{x}, \mathbf{y}\rangle$.

Note that if A is an $n \times n$ matrix, then

$$A\mathbf{x} \bullet \mathbf{y} = (A\mathbf{x})^T\mathbf{y} = \mathbf{x}^TA^T\mathbf{y} = \mathbf{x} \bullet A^T\mathbf{y}. \tag{3}$$

Theorem 34 (The Polarization Identity) *Let $\mathbf{x}, \mathbf{y}$ be elements of an inner product space $(V, \langle\ ,\ \rangle)$. Then*

$$\langle \mathbf{x}, \mathbf{y}\rangle = \frac{1}{4}\|\mathbf{x}+\mathbf{y}\|^2 - \frac{1}{4}\|\mathbf{x}-\mathbf{y}\|^2. \tag{4}$$

Proof.

$$\|\mathbf{x}+\mathbf{y}\|^2 = \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{x}\rangle + \mathbf{2}\langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle = \|x\|^2 + \mathbf{2}\langle \mathbf{x}, \mathbf{y}\rangle + \|\mathbf{y}\|^2$$

and

$$\|\mathbf{x}-\mathbf{y}\|^2 = \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{x}\rangle - \mathbf{2}\langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle = \|x\|^2 - \mathbf{2}\langle \mathbf{x}, \mathbf{y}\rangle + \|\mathbf{y}\|^2.$$

Therefore

$$\frac{1}{4}\|\mathbf{x}+\mathbf{y}\|^2 - \frac{1}{4}\|\mathbf{x}-\mathbf{y}\|^2 = \frac{1}{4}\left(\|x\|^2 + \mathbf{2}\langle \mathbf{x}, \mathbf{y}\rangle + \|\mathbf{y}\|^2\right) - \frac{1}{4}\left(\|x\|^2 - \mathbf{2}\langle \mathbf{x}, \mathbf{y}\rangle + \|\mathbf{y}\|^2\right) = \langle \mathbf{x}, \mathbf{y}\rangle.$$

■

Definition 35 *A square matrix A is **orthogonal** if $A^T = A^{-1}$.*

Theorem 36 *Let A be an $n \times n$ real matrix. The following statements are all equivalent:*

1. *A is an orthogonal matrix.*

2. $\|A\mathbf{x}\| = \|\mathbf{x}\|$ *for all* $\mathbf{x} \in \mathbb{R}^n$, *i.e., multiplication by* A *preserves Euclidean norm.*

3. $A\mathbf{x} \bullet A\mathbf{y} = \mathbf{x} \bullet \mathbf{y}$ *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, *i.e., multiplication by* A *preserves Euclidean inner product.*

Proof. (1) $\Rightarrow$ (2). Let $\mathbf{x} \in \mathbb{R}^n$. Using the fact that $A^T A = I$ and the identity in equation (3),

$$\|A\mathbf{x}\|^2 = A\mathbf{x} \bullet A\mathbf{x} = \mathbf{x} \bullet A^T A\mathbf{x} = \mathbf{x} \bullet \mathbf{x} = \|\mathbf{x}\|^2 .$$

(2) $\Rightarrow$ (3). Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. By the polarization identity (4) and assumption (2),

$$\begin{aligned} A\mathbf{x} \bullet A\mathbf{y} &= \frac{1}{4} \|A\mathbf{x}+A\mathbf{y}\|^2 - \frac{1}{4} \|A\mathbf{x}-A\mathbf{y}\|^2 = \frac{1}{4} \|A(\mathbf{x}+\mathbf{y})\|^2 - \frac{1}{4} \|A(\mathbf{x}-\mathbf{y})\|^2 \\ &= \frac{1}{4} \|\mathbf{x}+\mathbf{y}\|^2 - \frac{1}{4} \|\mathbf{x}-\mathbf{y}\|^2 = \mathbf{x} \bullet \mathbf{y}. \end{aligned}$$

(3) $\Rightarrow$ (1). Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. By assumption (3) and equation (3),

$$0 = A\mathbf{x} \bullet A\mathbf{y} - \mathbf{x} \bullet \mathbf{y} = \mathbf{x} \bullet A^T A\mathbf{y} - \mathbf{x} \bullet I\mathbf{y} = \mathbf{x} \bullet \left(A^T A\mathbf{y} - I\mathbf{y}\right) = \mathbf{x} \bullet \left(A^T A - I\right)\mathbf{y}.$$

Since this holds for all $\mathbf{x} \in \mathbb{R}^n$, it holds in particular when $\mathbf{x} = \left(A^T A - I\right)\mathbf{y}$.Thus

$$0 = \left(A^T A - I\right)\mathbf{y} \bullet \left(A^T A - I\right)\mathbf{y} = \left\|\left(A^T A - I\right)\mathbf{y}\right\|^2$$

in which case

$$\left(A^T A - I\right)\mathbf{y} = \mathbf{0}.$$

This is a homogeneous system of linear equations satisfied by all $\mathbf{y} \in \mathbb{R}^n$. In particular, this holds for $\mathbf{y} = \mathbf{e}_i$, in which case the i^{th} column of $\left(A^T A - I\right)$ is $\mathbf{0}$. Since this is true for all $i = 1, \ldots, n$, the matrix $A^T A - I = \mathbf{0}$ and A is orthogonal. ■

Definition 37 *A linear map* $L : \mathbb{R}^n \to \mathbb{R}^n$ *is an* ***orthogonal transformation*** *if and only if its matrix in the standard basis is an orthogonal matrix.*

Example 38 *Rotations about the origin and reflections in lines through the origin are orthogonal transformations of the plane.*

Unlike the determinant, orthogonality is somewhat basis dependent. Since the purpose of the Euclidean inner product is to measure Euclidean length and angle, it makes sense to study orthogonality relative to coordinate matrices of vectors in an *orthonormal basis,* i.e., a basis of pairwise orthogonal unit vectors. Consider $\mathbb{R}^n$ with its Euclidean inner product and an orthonormal basis

$$\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\},$$

where each $\mathbf{u}_i$ is a coordinate matrix in the standard basis. The associated transition matrix P from $\mathcal{B}$-coordinates to standard coordinates is

$$P = [\mathbf{u}_1 \mid \cdots \mid \mathbf{u}_n],$$

and by orthonormality of the $\mathbf{u}_i$'s we have

$$P^T P = \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix} [\mathbf{u}_1 \mid \cdots \mid \mathbf{u}_n] = \begin{bmatrix} \mathbf{u}_1^T\mathbf{u}_1 & \mathbf{u}_1^T\mathbf{u}_2 & \cdots & \mathbf{u}_1^T\mathbf{u}_n \\ \mathbf{u}_2^T\mathbf{u}_1 & \mathbf{u}_2^T\mathbf{u}_2 & & \mathbf{u}_2^T\mathbf{u}_n \\ \vdots & & \ddots & \vdots \\ \mathbf{u}_n^T\mathbf{u}_1 & \mathbf{u}_n^T\mathbf{u}_2 & \cdots & \mathbf{u}_n^T\mathbf{u}_n \end{bmatrix} = I.$$

Therefore P is orthogonal and so is is the transition matrix $P^{-1} = P^T$ from the standard basis to the orthonormal basis $\mathcal{B}$. But multiplication by P and P^{-1} preserve the Euclidean inner product by Theorem 36, so if L is an orthogonal transformation, then $[L]$ is orthogonal by definition and

$$[L]_{\mathcal{B}} [L]_{\mathcal{B}}^T = (P^T [L] P)(P^T [L] P)^T = P^T [L] P P^T [L]^T P = I,$$

which proves that $[L]_{\mathcal{B}}$ is orthogonal. In summary:

Theorem 39 *The matrix of an orthogonal transformation in any orthonormal basis is orthogonal.*

Corollary 40 *If $L : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal transformation, then*

$$\det(L) = \pm 1.$$

Proof. Recall that for all square matrices A and B, $\det(A) = \det\left(A^T\right)$ and $\det(AB) = \det A \det B$. Therefore

$$[\det(L)]^2 = \det([L])\det\left([L]^T\right) = \det\left([L]\,[L]^T\right) = \det(I) = 1.$$

■

Let $(V, \langle , \rangle)$ be an inner product space over a field F, and let W be a subspace of V.

Definition 41 *The **orthogonal complement** of W is the subset*

$$W^{\perp} = \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{w}\rangle = 0 \textit{ for all } \mathbf{w} \in W\}.$$

Note that $W^{\perp}$ is a subspace of V. If $\mathbf{v}, \mathbf{v}' \in W^{\perp}$ then for all $\mathbf{w} \in W$ we have $\langle \mathbf{v} + \mathbf{v}', \mathbf{w}\rangle = \langle \mathbf{v}, \mathbf{w}\rangle + \langle \mathbf{v}', \mathbf{w}\rangle = \mathbf{0} + \mathbf{0} = \mathbf{0}$ so that $\mathbf{v} + \mathbf{v}' \in W^{\perp}$. Furthermore, for all $t \in F$ we have $\langle t\mathbf{v}, \mathbf{w}\rangle = t\langle \mathbf{v}, \mathbf{w}\rangle = t\mathbf{0} = \mathbf{0}$ so that $t\mathbf{v} \in W^{\perp}$. Therefore $W^{\perp}$ is a subspace of V.

Also observe that $W \cap W^{\perp} = \{\mathbf{0}\}$. Of course $\mathbf{0} \in W \cap W^{\perp}$ since W and $W^{\perp}$ are subspaces of V. Conversely, if $\mathbf{u} \in W \cap W^{\perp}$, then $\langle \mathbf{u}, \mathbf{u}\rangle = 0$ since $\mathbf{u} \in W^{\perp}$ and $\mathbf{u} \in W$, and $\mathbf{u} = \mathbf{0}$ by definition of inner product. Therefore $W \cap W^{\perp} = \{\mathbf{0}\}$.

Exercise 42 *Given V and W as above, let $\mathbf{v} \in V$. Prove that there exist unique elements $\mathbf{w} \in W$ and $\mathbf{w}' \in W^{\perp}$ such that $\mathbf{v} = \mathbf{w} + \mathbf{w}'$.*

Exercise 43 *Given V as above, let $\mathbf{v} \in V$. Let A and B be arbitrary subspaces of V such that* $\operatorname{span}(A \cup B) = V$ *and $A \cap B = \{\mathbf{0}\}$. Then there exist unique elements $\mathbf{a} \in A$ and $\mathbf{b} \in B$ such that $\mathbf{v} = \mathbf{a} + \mathbf{b}$.*

This leads to the notion of "direct sum".

Definition 44 *Let W and W' be subspaces of a vector space V such that $W \cap W' = \{\mathbf{0}\}$. The **direct sum of** W **and** W' is the set*

$$W \oplus W' = \{\mathbf{w} + \mathbf{w}' \mid \mathbf{w} \in W \textit{ and } \mathbf{w}' \in W'\}.$$

Exercise 45 *Prove that $W \oplus W'$ is a subspace of V.*

Example 46 *Let $V = \mathbb{R}^n$ with its Euclidean inner product. Let $S = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\} \subset \mathbb{R}^n$, let $W = \operatorname{span}(S)$, and let*

$$A = \begin{bmatrix} \mathbf{w}_1^T \\ \vdots \\ \mathbf{w}_k^T \end{bmatrix}.$$

Then $\mathbf{x} \in W^{\perp}$ if and only if

$$\mathbf{w}_i^T \mathbf{x} = 0$$

for all i if and only if

$$A\mathbf{x} = \mathbf{0}.$$

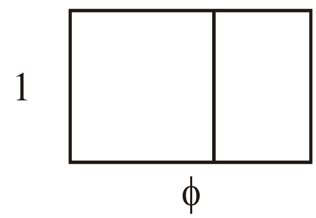
Thus $W^{\perp}$ is exactly the null space of A and $\mathbb{R}^n = W \oplus W^{\perp}$.

11-14-12

The Fibonacci Sequence and Golden Ratio:
An Application of Diagonalization

Consider a rectangle of width 1 and length ϕ with the following property: *If a unit square is removed from one end, the ratio length/width for the remaining rectangle is ϕ,* i.e.,

$$1/(\phi - 1) = \phi.$$



The ratio ϕ is called the *golden ratio.* Let's calculate ϕ. Cross-multiplying gives

$$\phi^2 - \phi - 1 = 0$$

so

$$\phi = \frac{1+\sqrt{5}}{2} \text{ and } \bar{\phi} = \frac{1-\sqrt{5}}{2}.$$

Let S denote the vector space of complex sequences. Addition and scalar multiplication are defined component-wise, i.e., $\{a_n\} + \{b_n\} = \{a_n + b_n\}$ and $s\{a_n\} = \{sa_n\}$. Let $L : S \to S$ be the lag operator defined by

$$L\{a_n\}_{n=0}^{\infty} = \{a_{n+1}\}_{n=0}^{\infty}.$$

Then $L\left(s\{a_n\}_{n=0}^{\infty} + t\{b_n\}_{n=0}^{\infty}\right) = L\{sa_n + tb_n\}_{n=0}^{\infty} = \{sa_{n+1} + tb_{n+1}\}_{n=0}^{\infty} = s\{a_{n+1}\}_{n=0}^{\infty} + t\{b_{n+1}\}_{n=0}^{\infty} = sL\{a_n\}_{n=0}^{\infty} + tL\{b_n\}_{n=0}^{\infty}$ and L is a linear operator on S. Let $L^2 = L \circ L$; then $L^2 - L - I$ acts on a sequence in the following way:

$$\begin{aligned}\left(L^2 - L - I\right)\{a_n\}_{n=0}^{\infty} &= \{a_{n+2}\}_{n=0}^{\infty} - \{a_{n+1}\}_{n=0}^{\infty} - \{a_n\}_{n=0}^{\infty} \\ &= \{a_{n+2} - a_{n+1} - a_n\}_{n=0}^{\infty}.\end{aligned}$$

Thus the kernel of $L^2 - L - I$ consists of all sequences $\{a_n\}_{n=0}^{\infty}$ such that $a_{n+2} - a_{n+1} - a_n = 0$ for all $n \geq 0$, i.e.,

$$E = N\left(L^2 - L - I\right) = \left\{\{a_n\}_{n=0}^{\infty} \mid a_{n+2} = a_{n+1} + a_n\right\}.$$

Definition 47 *An element of E is called a **Fibonacci-like sequence**.*

The classical *Fibonacci sequence*

$$\{f_n\}_{n=0}^{\infty} = \{1, 0, 1, 1, 2, 3, 5, 8, \ldots\}$$

is an element of E since $f_{n+2} = f_{n+1} + f_n$, and a Fibonacci-like sequence $\{a_n\}$ can be expressed in terms of $\{f_n\}$ in the following way:

$$\begin{array}{ccccc} a_0 & = & 1a_0 & + & 0a_1 \\ a_1 & = & 0a_0 & + & 1a_1 \\ a_2 & = & 1a_0 & + & 1a_1 \\ a_3 & = & 1a_0 & + & 2a_1 \\ a_4 & = & 2a_0 & + & 3a_1 \\ a_5 & = & 3a_0 & + & 5a_1 \\ \vdots & & \vdots & & \vdots \\ \{a_n\}_{n=0}^{\infty} & = & a_0\{f_n\}_{n=0}^{\infty} & + & a_1 L\{f_n\}_{n=0}^{\infty}. \end{array}$$

Thus every Fibonacci-like sequence is a linear combination of $\{f_n\}$ and $L\{f_n\}$. Since $\{f_n\}$ and $L\{f_n\}$ are linearly independent, E is a 2-dimensional subspace of S with basis $\mathcal{B}=\{\{f_n\}, L\{f_n\}\}$. Let's determine the matrix M for L in the basis $\mathcal{B}$. Since $L^2\{f_n\}=\{f_n\}+L\{f_n\}$ we have

$$M=[L]_{\mathcal{B}}=\left[[L\{f_n\}]_{\mathcal{B}} \mid \left[L^2\{f_n\}\right]_{\mathcal{B}}\right]=\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Proposition 48 $M^n=\begin{bmatrix} f_n & f_{n+1} \\ f_{n+1} & f_{n+2} \end{bmatrix}$ *for all* $n\geq 1$.

Proof. First note that the statement holds for $n=1$:

$$M^1=\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} f_1 & f_2 \\ f_2 & f_3 \end{bmatrix}.$$

Inductively, if the statement holds for all $k\leq n$, then

$$M^{n+1}=M^nM=\begin{bmatrix} f_n & f_{n+1} \\ f_{n+1} & f_{n+2} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} f_{n+1} & f_n+f_{n+1} \\ f_{n+2} & f_{n+1}+f_{n+2} \end{bmatrix}=\begin{bmatrix} f_{n+1} & f_{n+2} \\ f_{n+2} & f_{n+3} \end{bmatrix}.$$

■

Corollary 49

$$f_{n+1}=\begin{bmatrix} 1 & 0 \end{bmatrix}M^n\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Now to compute the powers of M explicitly, we will diagonalize. The characteristic equation of M is

$$\det(M-\lambda I)=\det\begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}=\lambda^2-\lambda-1=0,$$

so quite remarkably, the eigenvalues of M are ϕ and $\bar{\phi}$. We need two linearly independent eigenvectors:

$\underline{\lambda=\phi}$:

$$\begin{bmatrix} -\phi & 1 \\ 1 & 1-\phi \end{bmatrix}\xrightarrow{\text{row reduce}}\begin{bmatrix} 1 & 1-\phi \\ 0 & 0 \end{bmatrix},$$

and a corresponding eigenvector is

$$\begin{bmatrix} \phi-1 \\ 1 \end{bmatrix}.$$

$\underline{\lambda=\bar{\phi}}$:

$$\begin{bmatrix} -\bar{\phi} & 1 \\ 1 & 1-\bar{\phi} \end{bmatrix}\xrightarrow{\text{row reduce}}\begin{bmatrix} 1 & 1-\bar{\phi} \\ 0 & 0 \end{bmatrix},$$

and a corresponding eigenvector is

$$\begin{bmatrix} \bar{\phi}-1 \\ 1 \end{bmatrix}.$$

Now form the matrix

$$P=\begin{bmatrix} \phi-1 & \bar{\phi}-1 \\ 1 & 1 \end{bmatrix}$$

and note that $\det P=\phi-\bar{\phi}=\sqrt{5}$. Thus

$$P^{-1}=\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 1-\bar{\phi} \\ -1 & \phi-1 \end{bmatrix}$$

and

$$P^{-1}MP=\begin{bmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{bmatrix}\text{ or equivalently, } M=P\begin{bmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{bmatrix}P^{-1}.$$

Hence

$$M^n = \left(P \begin{bmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{bmatrix} P^{-1}\right)^n = \begin{bmatrix} \phi - 1 & \bar{\phi} - 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \phi^n & 0 \\ 0 & \bar{\phi}^n \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 - \bar{\phi} \\ -1 & \phi - 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} * & \phi^n - \bar{\phi}^n \\ \phi^n - \bar{\phi}^n & * \end{bmatrix}.$$

By Corollary 49 we have

$$f_{n+1} = \begin{bmatrix} 1 & 0 \end{bmatrix} M^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} * & \phi^n - \bar{\phi}^n \\ \phi^n - \bar{\phi}^n & * \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\phi^n - \bar{\phi}^n}{\sqrt{5}},$$

and we have proved:

Theorem 50 *Let* $\{f_n\}_{n=0}^{\infty}$ *be the Fibonacci sequence* $\{1, 0, 1, 1, 2, 3, 5, ...\}$. *Then*

$$f_{n+1} = \frac{\phi^n - \bar{\phi}^n}{\sqrt{5}}.$$

Example 51 *Note that* $\phi\bar{\phi} = -1$ *and* $\phi + \bar{\phi} = 1$. *Thus*

$$\begin{aligned} f_5 = \frac{\phi^4 - \bar{\phi}^4}{\sqrt{5}} &= \frac{(\phi - \bar{\phi})\left(\phi^3 + \phi^2\bar{\phi} + \phi\bar{\phi}^2 + \bar{\phi}^3\right)}{(\phi - \bar{\phi})} \\ &= \phi^3 - \phi - \bar{\phi} + \bar{\phi}^3 = \phi^3 + \bar{\phi}^3 - 1 \\ &= (\phi + \bar{\phi})\left(\phi^2 - \phi\bar{\phi} + \bar{\phi}^2\right) - 1 \\ &= \phi^2 + \bar{\phi}^2 = \frac{6 + 2\sqrt{5} + 6 - 2\sqrt{5}}{4} \\ &= 3. \end{aligned}$$

We conclude with a proof of the following remarkable fact:

Theorem 52 $\lim\limits_{n\to\infty} \frac{f_{n+1}}{f_n} = \phi$.

Proof. By Theorem 50 we have

$$\frac{f_{n+1}}{f_n} = \left(\frac{\phi^n - \bar{\phi}^n}{\sqrt{5}}\right)\left(\frac{\sqrt{5}}{\phi^{n-1} - \bar{\phi}^{n-1}}\right) = \frac{\phi^n - \bar{\phi}^n}{\phi^{n-1} - \bar{\phi}^{n-1}}.$$

To simplify notation, let $a = 1 + \sqrt{5}$ and $b = 1 - \sqrt{5}$. Then $\phi^n = a^n/2^n$ and $\bar{\phi}^n = b^n/2^n$, and the expression above reduces to

$$\frac{f_{n+1}}{f_n} = \frac{1}{2}\frac{a^n - b^n}{a^{n-1} - b^{n-1}} = \frac{1}{2}\frac{(a-b)\left(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}\right)}{(a-b)\left(a^{n-2} + a^{n-3}b + \cdots + ab^{n-3} + b^{n-2}\right)}$$

$$= \frac{1}{2}\frac{a^{n-1}\left(1 + b/a + \cdots + b^{n-2}/a^{n-2} + b^{n-1}/a^{n-1}\right)}{a^{n-2}\left(1 + b/a + \cdots + b^{n-3}/a^{n-2} + b^{n-2}/a^{n-2}\right)} = \frac{a}{2}\frac{\sum_{k=0}^{n-1}(b/a)^k}{\sum_{k=0}^{n-2}(b/a)^k}.$$

Now $b/a = \left(\sqrt{5} - 3\right)/2 \approx -0.38197$, so the geometric series $\sum_{n=0}^{\infty}(b/a)^n$ converges and we have

$$\lim_{n\to\infty}\frac{f_{n+1}}{f_n} = \frac{a}{2}\frac{\lim\limits_{n\to\infty}\sum_{k=0}^{n-1}(b/a)^k}{\lim\limits_{n\to\infty}\sum_{k=0}^{n-2}(b/a)^k} = \frac{a}{2} = \phi.$$

■

9-30-2010

Orthogonal Diagonalization

Problem 53 *Given a real $n \times n$ matrix A, under what conditions does there exist an orthogonal matrix P that diagonalizes A?*

Proposition 54 *An orthogonally diagonalizable real matrix A has the following properties:*

1. *A is symmetric.*
2. *A has an orthonormal set of eigenvectors.*

Proof. Since A is orthogonally diagonalizable, there exists an orthogonal matrix P and a diagonal matrix D such that

$$P^T A P = D. \tag{5}$$

(1) Solving for A we have $A = PDP^T$. Thus $A^T = \left(PDP^T\right)^T = \left(P^T\right)^T D^T P^T = PDP^T = A$, and A is symmetric.
(2) Multiplying both sides of (5) on the left by P gives $AP = PD$. Let $P = [\mathbf{p}_1 \mid \cdots \mid \mathbf{p}_n] = [p_{ij}]$. Then

$$AP = A[\mathbf{p}_1 \mid \cdots \mid \mathbf{p}_n] = [A\mathbf{p}_1 \mid \cdots \mid A\mathbf{p}_n];$$

on the other hand,

$$PD = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \vdots \\ p_{n1} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} = \begin{bmatrix} p_{11}\lambda_1 & \cdots & p_{1n}\lambda_n \\ \vdots & & \vdots \\ p_{n1}\lambda_1 & \cdots & p_{nn}\lambda_n \end{bmatrix} = [\lambda_1 \mathbf{p}_1 | \cdots | \lambda_n \mathbf{p}_n].$$

Therefore $A\mathbf{p}_i = \lambda_i \mathbf{p}_i$ for all i, and each $\mathbf{p}_i$ is an eigenvector for A. Furthermore, the $\mathbf{p}_i$'s are orthonormal by assumption. ■

Each of these properties is also sufficient.

Proposition 55 *A matrix with an orthonormal set of eigenvectors is orthogonally diagonalizable.*

Proof. Suppose that A has an orthonormal set of eigenvectors $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$. Then $A\mathbf{p}_i = \lambda_i \mathbf{p}_i$ for each i, and the matrix $P = [\mathbf{p}_1 \mid \cdots \mid \mathbf{p}_n]$ diagonalizes A. Since $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$ is an orthonormal set, we have

$$P^T P = \begin{bmatrix} \mathbf{p}_1^T \\ \vdots \\ \mathbf{p}_n^T \end{bmatrix} [\mathbf{p}_1 \mid \cdots \mid \mathbf{p}_n] = \begin{bmatrix} \mathbf{p}_1 \cdot \mathbf{p}_1 & \cdots & \mathbf{p}_1 \cdot \mathbf{p}_n \\ \vdots & & \vdots \\ \mathbf{p}_n \cdot \mathbf{p}_1 & \cdots & \mathbf{p}_n \cdot \mathbf{p}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} = I.$$

Therefore P is orthogonal matrix. ■

The proof that a real symmetric matrix is orthogonally diagonalizable requires two preliminary results.

Proposition 56 *A real symmetric matrix has real eigenvalues.*

Proof. Let A be an $n \times n$ real symmetric matrix and suppose that $A\mathbf{x} = \lambda \mathbf{x}$ with $\mathbf{x} \neq \mathbf{0}$ and possibly complex. To isolate λ, we first note that

$$\bar{\mathbf{x}}^T (A\mathbf{x}) = \bar{\mathbf{x}}^T (\lambda \mathbf{x}) = \lambda \bar{\mathbf{x}}^T \mathbf{x}.$$

On the other hand,

$$\bar{\mathbf{x}}^T (A\mathbf{x}) = (A\mathbf{x})^T \bar{\mathbf{x}}$$

(since $\mathbf{u}^T\mathbf{v} = \mathbf{v}^T\mathbf{u}$ for complex vectors $\mathbf{u}$ and $\mathbf{v}$). Therefore

$$\lambda\bar{\mathbf{x}}^T\mathbf{x} = (A\mathbf{x})^T\,\bar{\mathbf{x}} = \mathbf{x}^T A^T \bar{\mathbf{x}} = \mathbf{x}^T A \bar{\mathbf{x}},$$

where the last equality follows from the symmetry of A. Now $\bar{A} = A$ since A is real, hence

$$\lambda\bar{\mathbf{x}}^T\mathbf{x} = \mathbf{x}^T A\bar{\mathbf{x}} = \mathbf{x}^T\left(\overline{A\mathbf{x}}\right) = \mathbf{x}^T\left(\overline{\lambda\mathbf{x}}\right) = \mathbf{x}^T\left(\bar{\lambda}\bar{\mathbf{x}}\right) = \bar{\lambda}\mathbf{x}^T\bar{\mathbf{x}}.$$

Since $\bar{\mathbf{x}}^T\mathbf{x} = \mathbf{x}^T\bar{\mathbf{x}} \neq \mathbf{0}$ we have

$$\mathbf{0} = \lambda\bar{\mathbf{x}}^T\mathbf{x} - \bar{\lambda}\mathbf{x}^T\bar{\mathbf{x}} = \left(\lambda - \bar{\lambda}\right)\bar{\mathbf{x}}^T\mathbf{x}.$$

Therefore $\lambda = \bar{\lambda}$ so that $\lambda \in \mathbb{R}$. ■

We also need the following special case of Schur's Triangularization Theorem (see Theorem 151). We defer the proof.

Theorem 57 *If a real matrix A has real eigenvalues, there is an orthogonal matrix Q and an upper triangular matrix T such that $Q^T AQ = T$.*

Theorem 58 *A real symmetric matrix is orthogonally diagonalizable.*

Proof. Let A be a real symmetric matrix. Then Proposition 56 implies that the eigenvalues of A are real and Theorem 57 implies that A is upper triangularizable, i.e., there is an orthogonal matrix Q and an upper triangular matrix M such that $Q^T AQ = M$. Applying the transpose and the fact that A is symmetric we obtain

$$M^T = \left(Q^T AQ\right)^T = Q^T A^T \left(Q^T\right)^T = Q^T AQ = M.$$

Thus M is a symmetric upper triangular matrix and is consequently a diagonal matrix. ■

We collect these results and summarize them in the following theorem:

Theorem 59 *Let A be a real $n \times n$ matrix. The following are equivalent:*

1. *A is symmetric.*
2. *A has an orthonormal set of eigenvectors.*
3. *A is orthogonally diagonalizable.*

11-15-12

Spectral Decomposition and Quadratic Forms

Recall that all of the terms in a homogeneous polynomial have the same degree. For example, $xy + z^2$ is homogeneous, while $xy + z$ is not. Note that if $p(x_1, \dots, x_n)$ is homogeneous of degree k, then $p(ax_1, \dots, ax_n) = a^k p(x_1, \dots, x_n)$.

Definition 60 *A* ***form*** *is a homogeneous polynomial function $p : \mathbb{R}^n \to \mathbb{R}$.*

Example 61 *Here are examples of linear and quadratic forms:*

$$l(x, y, z) = 2x - y + 3z$$

$$q(x, y) = 2x^2 + 3xy - 4y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Notice that the 2×2 matrix in Example 61 is symmetric; the entries along the main diagonal are the coefficients of x^2 and y^2 and the entries off the main diagonal are half the coefficient of xy. In general, a quadratic form in variables $x_1, \dots, x_n$ is uniquely encoded by an $n \times n$ symmetric matrix $A = (a_{ij})$ whose diagonal entry a_{ii} is the coefficient of x_i^2 and whose off-diagonal entry a_{ij} is half the coefficient of $x_i x_j$. Thus a quadratic form $q(\mathbf{x})$ in n variables can be expressed uniquely in terms of an $n \times n$ symmetric matrix A as

$$q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}.$$

Note that the Euclidean inner product $\langle \mathbf{a}, \mathbf{b} \rangle$ can be expressed in terms of matrix multiplication as

$$\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a}^T \mathbf{b}.$$

Thus $q(\mathbf{x})$ can also be expressed as

$$q(\mathbf{x}) = \langle A\mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{x}, A\mathbf{x} \rangle.$$

<u>**Question #1:**</u> *If $A = (a_{ij})$ is a real symmetric non-diagonal matrix, the quadratic form $q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ has non-zero cross-terms of the form $2a_{ij}x_i x_j$ for some $i \neq j$. Is there change variables $\mathbf{x} = P\mathbf{y}$ so that $\mathbf{x}^T A \mathbf{x} = (P\mathbf{y})^T A (P\mathbf{y}) = \mathbf{y}^T (P^T A P) \mathbf{y}$ has no cross-terms?*

The answer to Question #1 is the content of our next theorem:

Theorem 62 (Principle Axis Theorem) *Let $q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ be a quadratic form in variables $x_1, \dots, x_n$. Let P be an orthogonal matrix that orthogonally diagonalizes A. If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A and $y_1, \dots, y_n$ are new variables such that $\mathbf{x} = P\mathbf{y}$, then*

$$\mathbf{x}^T A \mathbf{x} = \mathbf{y}^T (P^T A P) \mathbf{y} = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2.$$

Recall the following procedure to orthogonally diagonalize A:

1. Find a basis for each eigenspace of A.
2. Apply Gram-Schmidt and obtain an orthonormal basis for each eigenspace.
3. Form the matrix P whose columns are the basis vectors constructed in step 2.

Example 63 *Consider the matrix $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$ whose eigenvalues are $\lambda_1 = 8$, $\lambda_2 = 2$ and $\lambda_3 = 2$. Canonical bases for the eigenspaces are*

$$\lambda_1 = 8 : \left\{ \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \lambda_2 = 2 : \left\{ \mathbf{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Note that $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle = \langle \mathbf{x}_1, \mathbf{x}_3 \rangle = 0$. *Applying Gram-Schmidt gives*

$$\left\{ \mathbf{v}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \ \mathbf{v}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \ \mathbf{v}_3 = \frac{\mathbf{x}_3 - \langle \mathbf{x}_3, \mathbf{v}_2 \rangle \mathbf{v}_2}{\|\mathbf{x}_3 - \langle \mathbf{x}_3, \mathbf{v}_2 \rangle \mathbf{v}_2\|} = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \right\}.$$

Thus

$$P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix} \ \text{ and } \ P^T AP = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Example 64 *Let* $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$. *By our calculations in Example 63,*

$$2x_1^2 + 4x_2^2 + 4x_3^2 + 4x_1x_2 + 4x_1x_3 + 4x_2x_3 = \mathbf{x}^T A\mathbf{x} = \mathbf{y}^T \left(P^T AP\right) \mathbf{y} = 8y_1^2 + 2y_2^2 + 2y_3^2.$$

Question #2: *If* $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ *is a quadratic form in two or three variables and* c *is a constant, is there a nice way to describe the level set defined by* $q(\mathbf{x}) = c$*?*

Theorem 65 *If* $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ *is a quadratic form in two variables and* c *is a constant, the level curve defined by* $q(\mathbf{x}) = c$ *is a conic. If* $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ *is a quadratic form in three variables and* c *is a constant, the level surface defined by* $q(\mathbf{x}) = c$ *is a quadric.*

Example 66 *Refer to Example 70, and let* $P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$; *then* $P^T AP = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ *and*

$$2x_1x_2 = \mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T \left(P^T AP\right) \mathbf{y} = \mathbf{y}^T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y} = y_1^2 - y_2^2.$$

Thus the level curve defined by $2x_1x_2 = 1$ *is the hyperbola* $y_1^2 - y_2^2 = 1$. *Similarly, our calculations in Example 64 imply that the level surface defined by*

$$2x_1^2 + 4x_2^2 + 4x_3^2 + 4x_1x_2 + 4x_1x_3 + 4x_2x_3 = 1$$

is the ellipsoid $8y_1^2 + 2y_2^2 + 2y_3^2 = 1$.

Let S^{n-1} denote the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$, i.e.,

$$S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| = 1\}.$$

Since S^{n-1} is a closed and bounded subset of $\mathbb{R}^n$, the Extreme Value Theorem tells us that continuous functions on S^{n-1} attain their maximum and minimum values at some point of S^{n-1}.

Question #3: *What are the maximum and minimum values of a quadratic form* $q(\mathbf{x}) = x^T Ax$ *as* $\mathbf{x}$ *ranges over* S^{n-1}*? At which points* $\mathbf{x} \in S^{n-1}$ *are the maximum and minimum values of* q *attained?*

The answer is a consequence of the Spectral Decomposition Theorem.

Theorem 67 *(Spectral Decomposition) Let* A *be an* $n \times n$ *a real symmetric matrix with eigenvalues* $\lambda_1, \ldots, \lambda_n$. *Let* $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ *be an orthonormal basis for* $\mathbb{R}^n$ *such that* $A\mathbf{u}_i = \lambda_i \mathbf{u}_i$. *Then*

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T.$$

Proof. Let $C = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T$. Then for each i,

$$\begin{aligned}(A - C)\mathbf{u}_i &= A\mathbf{u}_i - \lambda_1 \mathbf{u}_1 \left(\mathbf{u}_1^T \mathbf{u}_i\right) + \cdots + \lambda_i \mathbf{u}_i \left(\mathbf{u}_i^T \mathbf{u}_i\right) + \cdots + \lambda_n \mathbf{u}_n \left(\mathbf{u}_n^T \mathbf{u}_i\right) \\ &= A\mathbf{u}_i - \lambda_i \mathbf{u}_i = \mathbf{0}.\end{aligned}$$

Furthermore, for each i, let $\mathbf{e}_i = [0 \cdots 1 \cdots 0]^T$ with 1 in the i^{th} position and write $\mathbf{e}_i = \left(\mathbf{e}_i^T \mathbf{u}_1\right) \mathbf{u}_1 + \cdots + \left(\mathbf{e}_i^T \mathbf{u}_n\right) \mathbf{u}_n$. Then the i^{th} column of $A - C$ is

$$\begin{aligned}(A - C)\mathbf{e}_i &= (A - C)\left[\left(\mathbf{e}_i^T \mathbf{u}_1\right) \mathbf{u}_1 + \cdots + \left(\mathbf{e}_i^T \mathbf{u}_n\right) \mathbf{u}_n\right] \\ &= \left(\mathbf{e}_i^T \mathbf{u}_1\right)(A - C)\mathbf{u}_1 + \cdots + \left(\mathbf{e}_i^T \mathbf{u}_n\right)(A - C)\mathbf{u}_n = \mathbf{0},\end{aligned}$$

by our previous calculation. It follows that $A - C$ is the zero matrix, and $A = C$. ∎

Theorem 68 *Let A be an $n \times n$ a real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. As $\mathbf{x}$ ranges over S^{n-1} we have:*

1. $\lambda_1 \geq \mathbf{x}^T A\mathbf{x} \geq \lambda_n$.
2. *If* $A\mathbf{x} = \lambda_i \mathbf{x}$, *then* $\mathbf{x}^T A\mathbf{x} = \lambda_i$.

Proof. (1) Choose an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ such that $A\mathbf{u}_i = \lambda_i \mathbf{u}_i$ and write $\mathbf{x} = \left(\mathbf{x}^T\mathbf{u}_1\right)\mathbf{u}_1 + \cdots + \left(\mathbf{x}^T\mathbf{u}_n\right)\mathbf{u}_n$. To simplify notation, let $b_i = \mathbf{x}^T\mathbf{u}_i$; then

$$\begin{aligned}
\mathbf{x}^T\mathbf{x} &= (b_1\mathbf{u}_1 + \cdots + b_n\mathbf{u}_n)^T (b_1\mathbf{u}_1 + \cdots + b_n\mathbf{u}_n) \\
&= \left(b_1\mathbf{u}_1^T + \cdots + b_n\mathbf{u}_n^T\right)(b_1\mathbf{u}_1 + \cdots + b_n\mathbf{u}_n) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} b_i b_j \mathbf{u}_i^T\mathbf{u}_j = \sum_{i=1}^{n} b_i^2 \mathbf{u}_i^T\mathbf{u}_i \\
&= b_1^2 + \cdots + b_n^2.
\end{aligned}$$

Since $\mathbf{x}^T\mathbf{x} = \|\mathbf{x}\|^2 = 1$ we have

$$\left(\mathbf{x}^T\mathbf{u}_1\right)^2 + \cdots + \left(\mathbf{x}^T\mathbf{u}_n\right)^2 = \mathbf{x}^T\mathbf{x} = 1.$$

Consider the spectral decomposition

$$A = \lambda_1\mathbf{u}_1\mathbf{u}_1^T + \cdots + \lambda_n\mathbf{u}_n\mathbf{u}_n^T.$$

Using the fact that $\mathbf{u}_i^T\mathbf{x} = \mathbf{x}^T\mathbf{u}_i$, we have

$$\begin{aligned}
\mathbf{x}^T A\mathbf{x} &= \mathbf{x}^T\left(\lambda_1\mathbf{u}_1\mathbf{u}_1^T + \cdots + \lambda_n\mathbf{u}_n\mathbf{u}_n^T\right)\mathbf{x} \\
&= \lambda_1\mathbf{x}^T\mathbf{u}_1\mathbf{u}_1^T\mathbf{x} + \cdots + \lambda_n\mathbf{x}^T\mathbf{u}_n\mathbf{u}_n^T\mathbf{x} \\
&= \lambda_1\left(\mathbf{x}^T\mathbf{u}_1\right)^2 + \cdots + \lambda_n\left(\mathbf{x}^T\mathbf{u}_n\right)^2.
\end{aligned}$$

Now replace each λ_i with the smallest eigenvalue λ_n; then

$$\mathbf{x}^T A\mathbf{x} \geq \lambda_n\left(\mathbf{x}^T\mathbf{u}_1\right)^2 + \cdots + \lambda_n\left(\mathbf{x}^T\mathbf{u}_n\right)^2 = \lambda_n\left[\left(\mathbf{x}^T\mathbf{u}_1\right)^2 + \cdots + \left(\mathbf{x}^T\mathbf{u}_n\right)^2\right] = \lambda_n.$$

Similarly, replace each λ_i with the largest eigenvalue λ_1; then

$$\mathbf{x}^T A\mathbf{x} \leq \lambda_1\left(\mathbf{x}^T\mathbf{u}_1\right)^2 + \cdots + \lambda_1\left(\mathbf{x}^T\mathbf{u}_n\right)^2 = \lambda_1\left[\left(\mathbf{x}^T\mathbf{u}_1\right)^2 + \cdots + \left(\mathbf{x}^T\mathbf{u}_n\right)^2\right] = \lambda_1.$$

(2) $\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T(\lambda_i\mathbf{x}) = \lambda_i\mathbf{x}^T\mathbf{x} = \lambda_i$. ∎

Corollary 69 *As $\mathbf{x}$ ranges over S^{n-1}, the quadratic form $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ attains its maximum λ_1 at $\pm\mathbf{u}_1$ and its minimum λ_n at $\pm\mathbf{u}_n$.*

Note that the absolute extrema of q when restricted to S^{n-1} are terminal points of pair-wise orthogonal unit vectors in $\mathbb{R}^n$.

Example 70 *Consider the quadratic form*

$$q(x,y) = 2xy = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Since the eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are $\lambda_1 = 1$ and $\lambda_2 = -1$, the maximum and minimum values of q on the unit circle S^1 are 1 and -1. Furthermore, the maximum value 1 is attained at $\pm\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, which are the unit eigenvectors associated with λ_1, and the minimum value -1 is attained at $\pm\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, which are the unit eigenvectors associated with λ_2.

Definition 71 *A quadratic form* $\mathbf{x}^T A\mathbf{x}$ *is* ***positive definite*** *if and only if* $\mathbf{x}^T A\mathbf{x} > 0$ *for all* $\mathbf{x} \neq \mathbf{0}$ *and* $\mathbf{x}^T A\mathbf{x} = 0$ *iff* $\mathbf{x} = \mathbf{0}$. *A symmetric matrix* A *is* ***positive definite*** *if and only if* $\mathbf{x}^T A\mathbf{x}$ *is positive definite. A quadratic form* $\mathbf{x}^T A\mathbf{x}$ *is* ***negative definite*** *if and only if* $\mathbf{x}^T A\mathbf{x} < 0$ *for all* $\mathbf{x} \neq \mathbf{0}$ *and* $\mathbf{x}^T A\mathbf{x} = 0$ *iff* $\mathbf{x} = \mathbf{0}$. *A symmetric matrix* A *is* ***negative definite*** *if and only if* $\mathbf{x}^T A\mathbf{x}$ *is negative definite.*

Example 72 *The Euclidean inner product is a positive definite quadratic form since*

$$x_1^2 + \cdots + x_n^2 = \langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^T\mathbf{x} = \mathbf{x}^T I\mathbf{x}.$$

Question #4: *When is a quadratic form positive definite?*

Theorem 73 *A symmetric matrix* A *is positive definite iff all of the eigenvalues of* A *are positive.*

Proof. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of A. If A is positive definite, consider an eigenvalue λ_i and a corresponding eigenvector $\mathbf{x}$. Then

$$0 < \mathbf{x}^T A\mathbf{x} = \mathbf{x}^T (\lambda_i \mathbf{x}) = \lambda_i \mathbf{x}^T\mathbf{x} = \lambda_i \|\mathbf{x}\|^2,$$

where $\|\mathbf{x}\|$ is the Euclidean norm. But $\|\mathbf{x}\|^2 > 0$ since $\mathbf{x} \neq \mathbf{0}$, therefore $\lambda_i > 0$. Conversely, if $\lambda_n > 0$ and $\mathbf{x} \neq \mathbf{0}$, let $\mathbf{y} = \mathbf{x}/\|\mathbf{x}\|$. Then by Theorem 68,

$$0 < \lambda_n \leq \mathbf{y}^T A\mathbf{y} = \left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right)^T A\frac{\mathbf{x}}{\|\mathbf{x}\|} = \frac{1}{\|\mathbf{x}\|^2}\mathbf{x}^T A\mathbf{x}.$$

Multiplying both sides by $\|\mathbf{x}\|^2$ gives $\mathbf{x}^T A\mathbf{x} > \mathbf{0}$. ■

Example 74 *The eigenvalues of the matrix* $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ *are* $\lambda_1 = 6$ *and* $\lambda_2 = 2$. *Therefore* A *is positive definite by Theorem 73. Indeed, if* $\mathbf{x} \neq \mathbf{0}$ *and* θ *is the angle between* $\mathbf{x}$ *and* $\mathbf{e}_1$, *let* $x = \|\mathbf{x}\| \cos\theta$ *and* $y = \|\mathbf{x}\| \sin\theta$. *Then*

$$\begin{aligned} \mathbf{x}^T A\mathbf{x} &= 4x^2 + 4xy + 4y^2 \\ &= 4\|\mathbf{x}\|^2 \left(\cos^2\theta + \cos\theta\sin\theta + \sin^2\theta\right) \\ &= 4\|\mathbf{x}\|^2 \left(1 + \tfrac{1}{2}\sin 2\theta\right) > 0. \end{aligned}$$

Definition 75 *For* $1 \leq k \leq n$, *the* k^{th} ***principal submatrix*** *of an* $n \times n$ *matrix* $A = [a_{ij}]$ *is*

$$\begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix}.$$

Theorem 76 *A symmetric matrix* A *is positive definite iff every principal subdeterminant of* A *is positive.*

We omit the proof.

Example 77 *The principal subdeterminants of the matrix* $A = \begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & 4 \\ -3 & 4 & 9 \end{bmatrix}$ *are:*

$$\det[2] = 2, \quad \det\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 3, \quad \text{and} \quad \det A = 1.$$

Therefore A *is positive definite by Theorem 76.*

Exercise 78 *Since the quadratic form in Example 77 is positive definite, the quadric given by* $\mathbf{x}^T A\mathbf{x} = 1$ *is an ellipsoid. Perform an orthogonal change of variables to eliminate the cross-terms and express this ellipsoid in the standard form* $\frac{y_1^2}{a^2} + \frac{y_2^2}{b^2} + \frac{y_3^2}{c^2} = 1$.

Let us apply these results to prove the Second Derivative Test for differentiable functions of two variables.

Theorem 79 *(Second Derivative Test) Let f be differentiable with continuous mixed partials in some open disk containing $(0,0)$, and suppose that $\nabla f(0,0) = 0$ and $f_{xx}(0,0) f_{yy}(0,0) - [f_{xy}(0,0)]^2 > 0$.*

1. *If $f_{xx}(0,0) > 0$, then $f(0,0)$ is a local minimum.*
2. *If $f_{xx}(0,0) < 0$, then $f(0,0)$ is a local maximum.*

Proof. Given a unit vector $\mathbf{u} = (u, v)$, restrict f to the line through the origin parametrized by $x = tu$, $y = tv$, $t \in \mathbb{R}$ and consider the real valued function of one variable

$$g(t) = f(x, y) = f(tu, tv).$$

Then

$$g'(t) = f_x(x, y) u + f_y(x, y) v,$$

by the chain rule, and

$$g'(0) = f_x(0,0) u + f_y(0,0) v = 0,$$

i.e., 0 is a critical number. Furthermore,

$$\begin{aligned} g''(t) &= \frac{\partial}{\partial x}[f_x(x,y) u] u + \frac{\partial}{\partial y}[f_x(x,y) u] v \\ &\quad + \frac{\partial}{\partial x}[f_y(x,y) v] u + \frac{\partial}{\partial y}[f_y(x,y) v] v \\ &= f_{xx}(x,y) u^2 + 2 f_{xy}(x,y) uv + f_{yy}(x,y) v^2 \end{aligned}$$

and

$$g''(0) = f_{xx}(0,0) u^2 + 2 f_{xy}(0,0) uv + f_{yy}(0,0) v^2.$$

If $g''(0) > 0$, then $g(0)$ is a local minimum by the single variable Second Derivative Test; and if $g''(0) < 0$, then $g(0)$ is a local maximum. Now $g''(0)$ varies as $\mathbf{u}$ ranges over the circle S^1. Thus $f(0,0)$ is a local minimum if $g''(0) > 0$ for every unit vector $\mathbf{u}$; and $f(0,0)$ is a local maximum if $g''(0) < 0$ for every unit vector $\mathbf{u}$. I claim that the quadratic form $q(u,v) = g''(0)$ is positive definite when $f_{xx}(0,0) > 0$, in which case $f(0,0)$ is a local minimum, and negative definite when $f_{xx}(0,0) < 0$, in which case $f(0,0)$ is a local maximum.

Let $A = f_{xx}(0,0)$, $B = f_{xy}(0,0)$, $C = f_{yy}(0,0)$ and write

$$q(u,v) = Au^2 + 2Buv + Cv^2 = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{u}^T S \mathbf{u}.$$

To compute the eigenvalues of S, solve the characteristic equation

$$\begin{aligned} 0 &= \det(S - \lambda I) = (A - \lambda)(C - \lambda) - B^2 \\ &= \lambda^2 - (A + C)\lambda + \left(AC - B^2\right) \\ &= \lambda^2 - (A + C)\lambda + D. \end{aligned}$$

Since $D = AC - B^2 > 0$ by assumption, we have

$$\lambda = \frac{1}{2}\left[A + C \pm \sqrt{(A + C)^2 - 4D}\right].$$

Since the eigenvalues of S are real, we have $(A + C)^2 \geq 4D$.

<u>*Case 1.*</u> Suppose $A = f_{xx}(0,0) > 0$. Then $AC > B^2 \geq 0$ implies $C > 0$ so that

$$\lambda_1 = \frac{1}{2}\left[A + C + \sqrt{(A + C)^2 - 4D}\right] > 0.$$

On the other hand, $D > 0$ implies $(A+C)^2 > (A+C)^2 - 4D$. Taking square roots of both sides gives

$$A + C > \sqrt{(A+C)^2 - 4D}$$

or equivalently,

$$A + C - \sqrt{(A+C)^2 - 4D} > 0.$$

Thus

$$\lambda_2 = \frac{1}{2}\left[A + C - \sqrt{(A+C)^2 - 4D}\right] > 0.$$

Since λ_1 and λ_2 are both positive, $q(u,v) = \mathbf{u}^T S \mathbf{u}$ is positive definite.

Case 2. Suppose $A = f_{xx}(0,0) < 0$. Then $AC > B^2 \geq 0$ implies $C < 0$ so that

$$\lambda_2 = \frac{1}{2}\left[A + C - \sqrt{(A+C)^2 - 4D}\right] < 0.$$

On the other hand, $D > 0$ implies $(A+C)^2 - 4D < (A+C)^2$ and taking square roots of both sides gives

$$\sqrt{(A+C)^2 - 4D} < |A+C| = -(A+C)$$

or

$$A + C + \sqrt{(A+C)^2 - 4D} < 0.$$

Therefore

$$\lambda_1 = \frac{1}{2}\left[A + C + \sqrt{(A+C)^2 - 4D}\right] < 0.$$

Since λ_1 and λ_2 are both negative, $g''(0) = \mathbf{u}^T S \mathbf{u}$ is negative definite. ■

Definition 80 *A quadratic form $q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ is **non-degenerate** if all eigenvalues of A are non-zero.*

Definition 81 *The **signature** of a non-degenerate quadratic form $q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$, denoted by $\operatorname{sig}(A)$, is the number of positive eigenvalues of A minus the number of negative eigenvalues of A.*

Theorem 82 *Let $q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ be a non-degenerate quadratic form in two variables.*

1. *If $\operatorname{sig}(A) = 2$, then $\mathbf{x}^T A \mathbf{x} = 1$ is an ellipse.*
2. *If $\operatorname{sig}(A) = 1$, then $\mathbf{x}^T A \mathbf{x} = 1$ is an hyperbola.*

Theorem 83 *Let $\mathbf{x}^T A \mathbf{x}$ be a non-degenerate quadratic form in three variables.*

1. *If $\operatorname{sig}(A) = 3$, then $\mathbf{x}^T A \mathbf{x} = 1$ is an ellipsoid.*
2. *If $\operatorname{sig}(A) = 2$, then $\mathbf{x}^T A \mathbf{x} = 1$ is an hyperboloid of one sheet.*
3. *If $\operatorname{sig}(A) = 1$, then $\mathbf{x}^T A \mathbf{x} = 1$ is an hyperboloid of two sheets.*

11-10-2010

Space-Time: A Euclidean Pseudo-Inner Product Space

Recall that an inner product on a vector space V is a symmetric, bilinear, positive definite quadratic form on V. In contrast, a pseudo inner product is indefinite, and as this lecture demonstrates, gives rise to many surprising, interesting and counter-intuitive situations.

Definition 84 *A* ***Euclidean pseudo-inner product*** *on $\mathbb{R}^n$ is a symmetric, bilinear, indefinite quadratic form $\langle -,- \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by*

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = x_1y_1 + \cdots + x_iy_i - x_{i+1}y_{i-1} - \cdots - x_ny_n$$

for some $1 \leq i < n$.

Minkowski Space

For simplicity, we consider 2-dimensional space-time, or *Minkowski space,* which is the Euclidean pseudo-inner product space

$$\mathbb{R}^2_1 = \{(t, x) : t, x \in \mathbb{R}\}$$

with pseudo-inner product defined by

$$\langle (t_1, x_1), (t_2, x_2) \rangle = t_1t_2 - x_1x_2.$$

The *Minkowski norm*

$$\|(t, x)\| = \sqrt{t^2 - x^2}$$

ranges over all non-negative real and positive imaginary values.

Curves of constant Minkowski norm a satisfy the equation

$$t^2 - x^2 = a^2. \tag{6}$$

The parameter a determines three families of such curves:

- When $a = 0$, (6) defines the *light cone* $x = \pm t$.
- When $a \in \mathbb{R}^+$, (6) defines a *real hyperbolic circle of radius* a, which is the hyperbola $t^2 - x^2 = a^2$ inside the light cone.
- When $a = ib \in i\mathbb{R}^+$, (6) defines an *imaginary hyperbolic circle of radius* ib, which is the hyperbola $x^2 - t^2 = b^2$ outside the light cone.

- *Isotropic vectors* have zero Minkowski norm and live on the light-cone.
- *Time-like vectors* have positive real Minkowski norm and live inside the light-cone.
- *Space-like vectors* have positive imaginary Minkowski norm and live outside the light-cone.

Let $C : (t(u), x(u))$, $a \leq u \leq b$, be a parametrized curve in $\mathbb{R}^2_1$ and define the *hyperbolic arc length s of C* to be

$$s = \int_a^b \|(t'(u), x'(u))\| \, du = \int_a^b \sqrt{(t')^2 - (x')^2} du.$$

Hyperbolic Circles

Recall that Euclidean angle θ measures the arc length along the unit circle in $\mathbb{R}^2$ from $(1, 0)$ to $(\cos\theta, \sin\theta)$. Note that the area of the sector subtending the arc from $(\cos\theta, -\sin\theta)$ to $(\cos\theta, \sin\theta)$ is also θ. On the other hand, the arc length from $(1, 0)$ to $(\cosh\theta, \sinh\theta)$ along the real hyperbolic circle $t^2 - x^2 = 1$ is θi, but we'd like it to be θ. Thankfully, the area of the real hyperbolic sector subtending the hyperbolic arc from $(\cosh\theta, -\sinh\theta)$ to $(\cosh\theta, \sinh\theta)$ is exactly θ. So let's redefine Euclidean angle θ to be the area of the sector subtending the arc from $(\cos\theta, -\sin\theta)$ to $(\cos\theta, \sin\theta)$; then analogously, define the *hyperbolic angle* θ to be the area of the real hyperbolic sector subtending the real hyperbolic arc from $(\cosh\theta, -\sinh\theta)$ to $(\cosh\theta, \sinh\theta)$. Note that $(\cos\theta, \sin\theta)$ and $(-\cos\theta, -\sin\theta)$ are antipodes on the unit circle, and likewise $(\cosh\theta, \sinh\theta)$ and $(-\cosh\theta, -\sinh\theta)$ are antipodes on the real hyperbolic circle $t^2 - x^2 = 1$.

Parametrize the imaginary hyperbolic circle $C : x^2 - t^2 = b^2$ by $t = b\sinh\theta$, $x = b\cosh\theta$. Then the arc length function along C is

$$s(\theta) = \int_0^\theta \|(b\cosh u, b\sinh u)\| \, du = \sqrt{b^2}\int_0^\theta du = b\theta$$

so that $\theta = s/b$. Now substituting for θ in terms of s reparametrizes C by arc length and gives the position function

$$\mathbf{r}(s) = b\left(\sinh\left(\frac{s}{b}\right), \cosh\left(\frac{s}{b}\right)\right)$$

with velocity

$$\mathbf{v}(s) = \left(\cosh\left(\frac{s}{b}\right), \sinh\left(\frac{s}{b}\right)\right)$$

and constant speed

$$\|\mathbf{v}(s)\| = \sqrt{\cosh^2\left(\frac{s}{b}\right) - \sinh^2\left(\frac{s}{b}\right)} = 1.$$

Thus the unit tangent vector field $\mathbf{T}(s)$ along C is simply the velocity vector field $\mathbf{v}(s)$ along C, and the curvature κ of C is the magnitude of $\mathbf{T}'(s)$, i.e., the instantaneous rate at which $\mathbf{T}$ changes direction. Thus

$$\mathbf{T}'(s) = \mathbf{v}'(s) = \frac{1}{b}\left(\sinh\left(\frac{s}{b}\right), \cosh\left(\frac{s}{b}\right)\right)$$

and the curvature

$$\kappa = \|\mathbf{T}'(s)\| = \frac{1}{b}\sqrt{\sinh^2\left(\frac{s}{b}\right) - \cosh^2\left(\frac{s}{b}\right)} = \frac{i}{b} = -\frac{1}{bi}$$

is the *negative reciprocal of the imaginary radius.* Note that unlike Euclidean circular motion, in which the acceleration and position have opposite directions, the acceleration and position of imaginary hyperbolic circular motion *have the same direction:*

$$\mathbf{b}(s) = \mathbf{v}'(s) = \frac{1}{b^2}\mathbf{r}(s).$$

Nevertheless, hyperbolic and Euclidean circles have similar properties.

Exercises

1. Show that the length of the arc along the real hyperbolic unit circle from $(1, 0)$ to $(\cosh\theta, \sinh\theta)$ is θi.

2. Show that the area of the real hyperbolic unit sector with angle 2θ, i.e., the plane region bounded by the lines $x = \pm t\tanh\theta$ and $t^2 - x^2 = 1$, is exactly θ. (Your integration will require a hyperbolic trigonometric substitution.)

3. Given $a > 0$, parametrize the real hyperbolic circle $C : t^2 - x^2 = a^2$ by $t = a\cosh\theta$, $x = a\sinh\theta$. Reparametrize by arc length and show that the curvature $\kappa = \frac{1}{a}$ and that acceleration and position have opposite directions.

The Poincaré Group

Recall that an isometry (fixing the origin) is a norm preserving linear transformation. Plane Euclidean isometries are rotations ρ_θ about the origin through angle θ and reflections σ_θ in lines through the origin with inclination θ. Euclidean rotations fix circles centered at the origin and send lines through the origin to lines through the origin; reflections fix their reflecting lines point-wise. Rotations ρ_θ are represented by orthogonal matrices with determinant $+1$:

$$[\rho_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Reflections σ_θ are represented by orthogonal matrices with determinant -1:

$$[\sigma_\theta] = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{bmatrix};$$

and in particular,

$$[\sigma_0] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Euclidean isometries are represented by elements of the *orthogonal group* $O(2)$, which is the union of two disjoint components

$$[\rho_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ and } [\rho_\theta][\sigma_0] = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

The component

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \cos\theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

parametrizes the circle

$$C_1 : u_1^2 + u_2^2 = 2$$

in the 2-plane spanned by

$$\mathcal{B}_1 = \left\{ \mathbf{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \right\}.$$

Note that the trivial rotation

$$[\rho_0] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \sqrt{2}\mathbf{u}_1 + 0\mathbf{u}_2$$

is the point $\left(\sqrt{2}, 0\right)$ on this circle in the basis $\mathcal{B}_1$. Similarly, the component

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = \cos\theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

parametrize the circle

$$C_2 : u_3^2 + u_4^2 = 2$$

in the 2-plane spanned by

$$\mathcal{B}_2 = \left\{ \mathbf{u}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}, \ \mathbf{u}_4 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \right\}.$$

Since $\mathcal{B}_1 \cup \mathcal{B}_2$ is linearly independent in $\mathbb{R}^{2\times 2}$, $C_1 \cap C_2 = \varnothing$ and $C_1 \cup C_2 = O(2)$.

The situation in Minkowski space is similar but a bit more complicated. Here the isometries are hyperbolic rotations and hyperbolic reflections. The group of all such transformations, called the *Poincaré group* $O(1,1)$, has four connected components, which appear as the branches of two hyperbolas in $\mathbb{R}^{2\times 2}$. A *hyperbolic rotation* R_θ *through angle* θ is represented by the matrix

$$[R_\theta] = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$$

and is given in coordinates by

$$R_\theta(t, x) = (t\cosh\theta + x\sinh\theta, t\sinh\theta + x\cosh\theta).$$

Note that R_θ fixes hyperbolic circles: If $(\bar{t}, \bar{x}) = R_\theta(t, x)$, then

$$(\bar{t})^2 - (\bar{x})^2 = (t\cosh\theta + x\sinh\theta)^2 - (t\sinh\theta + x\cosh\theta)^2 = t^2 - x^2.$$

Furthermore, R_θ sends lines through the origin to lines through the origin since

$$R_\theta\,(a,0) = a\,(\cosh\theta, \sinh\theta)\,.$$

Denote reflection in the t-axis by

$$S_0\,(t,x) = (t,-x)$$

and reflection in the x-axis by

$$S_\infty\,(t,x) = (-t,x)\,.$$

Then S_0 and S_∞ fix hyperbolic circles since

$$t^2 - x^2 = t^2 - (-x)^2 = (-t)^2 - x^2.$$

Let l_θ be a line through the origin with inclination $\theta \neq \pm\pi/4$, and let R_θ be the hyperbolic rotation that rotates l_θ onto the t-axis if $-\pi/4 < \theta < \pi/4$, and rotates l_θ onto the x-axis if $\pi/4 < \theta < 3\pi/4$. The *hyperbolic reflection in line* l is the composition

$$S_m = \begin{cases} R_\theta^{-1} S_0 R_\theta, & \text{if } -\pi/4 < \theta < \pi/4 \\ R_\theta^{-1} S_\infty R_\theta, & \text{if } \pi/4 < \theta < 3\pi/4. \end{cases}$$

Hyperbolic reflections fix hyperbolic circles and fix their reflecting lines point-wise. Somewhat surprisingly perhaps, there are no hyperbolic reflections in the lines $x = \pm t$ (see Exercise 7 below). The reflections S_0 and S_∞ are represented by the matrices

$$[S_0] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad [S_\infty] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Minkowski isometries are represented by elements of the Poincaré group $O\,(1,1)$, which is the union of four mutually disjoint components:

$$[R_\theta] = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}, \qquad [R_\theta]\,[S_0]\,[S_\infty] = \begin{bmatrix} -\cosh\theta & -\sinh\theta \\ -\sinh\theta & -\cosh\theta \end{bmatrix},$$

$$[R_\theta]\,[S_0] = \begin{bmatrix} \cosh\theta & -\sinh\theta \\ \sinh\theta & -\cosh\theta \end{bmatrix}, \qquad [R_\theta]\,[S_\infty] = \begin{bmatrix} -\cosh\theta & \sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix}.$$

The components

$$\begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} = \cosh\theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sinh\theta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} -\cosh\theta & -\sinh\theta \\ -\sinh\theta & -\cosh\theta \end{bmatrix} = -\cosh\theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \sinh\theta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

form the two branches of the hyperbola

$$H_1 : u_1^2 - u_4^2 = 2$$

in the 2-plane spanned by

$$\mathcal{B}_1' = \left\{ \mathbf{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \ \mathbf{u}_4 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \right\}.$$

The trivial hyperbolic rotation

$$[R_0] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \sqrt{2}\mathbf{u}_1 + 0\mathbf{u}_4$$

is the point $\left(\sqrt{2}, 0\right)$ on this hyperbola in the basis $\mathcal{B}_1$. The components

$$\begin{bmatrix} \cosh\theta & -\sinh\theta \\ \sinh\theta & -\cosh\theta \end{bmatrix} = \cosh\theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \sinh\theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} -\cosh\theta & \sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix} = -\cosh\theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \sinh\theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

form the two branches of the hyperbola

$$H_2 : u_3^2 - u_2^2 = 2$$

in the 2-plane spanned by

$$\mathcal{B}_2' = \left\{ \mathbf{u}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix},\ \mathbf{u}_2 = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \right\}.$$

Since $\mathcal{B}_1' \cup \mathcal{B}_2'$ is linearly independent in $\mathbb{R}^{2\times 2}$, $H_1 \cap H_2 = \varnothing$ and $H_1 \cup H_2 = O(1,1)$.

Exercises

4. Find the matrices $\left[R_\theta^{-1}\right]$, $\left[R_\theta^{-1} S_0 R_\theta\right]$, and $\left[R_\theta^{-1} S_\infty R_\theta\right]$.
5. Prove that a hyperbolic reflection fixes its reflecting line point-wise.
6. Prove that S_0 and S_∞ are the only hyperbolic reflections that are also Euclidean reflections.
7. Prove that a Minkowski norm preserving linear transformation that fixes the line $x = t$ point-wise is the identity transformation. Prove the analogous statement for the line $x = -t$.

Special Relativity

The speed of light $c \approx 3 \times 10^8$ m/sec.

- An *event* is a point (t, x) in space-time.
- The *world-line* of a particle P is a parametrized curve $\mathbf{r}(t) = (ct, x(t))$.
- The *relative velocity* of P along its world line is $\mathbf{v}(t) = (c, x'(t))$.
- The *ordinary velocity* of P is $x'(t)$.
- The *relative speed* of P along its world line is $\|\mathbf{v}\| = \sqrt{c^2 - (x')^2}$.
- The *ordinary speed* of P is $|x'|$.

Physical Assumption 1: *The ordinary speed of a particle cannot exceed the speed of light.*

Hence $(x')^2 \le c^2$ and

$$\|\mathbf{v}\|^2 = c^2 - (x')^2 \ge 0.$$

Thus vectors tangent to the world-line of a particle in motion are either time-like or isotropic.

Physical Assumption 2: *A particle traveling at the speed of light has zero mass.*

- The world-line of a particle at rest is a horizontal line inside the light cone.
- The world-line of a particle with non-zero mass is a curve inside the light cone.
- The world-line of a particle with non-zero mass and constant speed is a line inside the light cone.
- The world-line of a photon is a line on the light cone.

Consider the world-line $\mathbf{r}(t) = (ct, x(t))$ of a particle P with non-zero mass and constant ordinary velocity v in the positive x-direction. The relative velocity of P is (c, v) and its relative speed is $\sqrt{c^2 - v^2}$. The arc length function s for the world-line of P is

$$s(t) = \int_0^t \|(c, v)\| \, du = \sqrt{c^2 - v^2} \int_0^t du = t\sqrt{c^2 - v^2}; \tag{7}$$

hence

$$t = \frac{s}{\sqrt{c^2 - v^2}} = \frac{s/c}{\sqrt{1 - v^2/c^2}}.$$

The *proper elapsed time* of P is the quantity

$$\frac{s}{c} = t\sqrt{1 - v^2/c^2}.$$

If P is at rest, for example, its proper elapsed time is t.

Lorentz Transformations

"Lorentz transformations" are special elements of the Poincaré group that change coordinates from reference frame $\bar{K}(c\bar{t}, \bar{x})$ to reference frame $K(ct, x)$ or vice versa as $\bar{K}$ moves along a straight line with constant velocity relative to K.

Physical Assumption 3: *The speed of light c is the same in every frame of reference.*

Assume that $\bar{K}$ moves in the positive x direction in K with constant speed v. A *Lorentz transformation* is a hyperbolic change of coordinates

$$\begin{bmatrix} ct \\ x \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} c\bar{t} \\ \bar{x} \end{bmatrix}. \tag{8}$$

Note that

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

lies in the component of $I \in O(1,1)$ since $A \to I$ as $v \to 0$. Hence A is a hyperbolic rotation

$$R_\theta = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$$

and equation (8) becomes

$$\begin{bmatrix} ct \\ x \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} c\bar{t} \\ \bar{x} \end{bmatrix} \tag{9}$$

Example. Let $\theta = \ln 2$; then $\cosh\theta = \frac{5}{4}$ and $\sinh\theta = \frac{3}{4}$. Thus

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix}.$$

This particular hyperbolic rotation moves the point $(4,0)$ "counterclockwise" along the hyperbola $t^2 - x^2 = 16$ to the point $(5,3)$, and the point $(-3,5)$ "clockwise" along the hyperbola $x^2 - t^2 = 16$ to the point $(0,4)$. The flows along these two hyperbolas asymptotically approach the light cone $x = t$ in the second quadrant (see Figure 1).

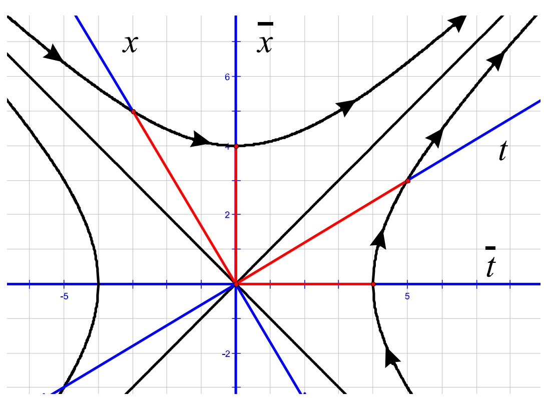


Figure 1. Hyperbolic rotations fix hyperbolas.

Lorentz transformations change coordinates of

- *time-like vectors (inside the light cone) from* $\bar{K}$*-coordinates to* K*-coordinates and*
- *space-like vectors (outside the light cone) from* K*-coordinates to* $\bar{K}$*-coordinates.*

Let's investigate the motion of a particle P with non-zero mass positioned at the origin $\bar{O}$ in the moving frame $\bar{K}$ as it moves in the positive x-direction in frame K with constant speed v. Since P is at rest in frame $\bar{K}$, its world line in $\bar{K}$ is the parametrized curve $(c\bar{t}, 0)$ contained in the $\bar{t}$ axis. But when viewed from frame K, its world line has positive slope inside the light cone and is parameterized by $(ct, x) = (c\bar{t}\cosh\theta, c\bar{t}\sinh\theta)$ via equation (9). Now dividing second components by first components gives

$$\tanh\theta = \frac{x}{ct} = \frac{vt}{ct} = \frac{v}{c}. \tag{10}$$

Now using the fact that $\cosh\theta > 0$, solve for $\cosh\theta$ in the identity

$$1 = \cosh^2\theta - \sinh^2\theta = \cosh^2\theta\left(1 - \tanh^2\theta\right)$$

and obtain

$$\cosh\theta = \frac{1}{\sqrt{1 - \tanh^2\theta}} = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Combining this with equation (10) gives

$$\sinh\theta = \frac{v/c}{\sqrt{1 - v^2/c^2}},$$

and substituting in (9) we obtain

$$\begin{aligned} t &= \frac{1}{\sqrt{1 - v^2/c^2}}\left(\bar{t} + \left(v/c^2\right)\bar{x}\right) \\ x &= \frac{1}{\sqrt{1 - v^2/c^2}}\left(v\bar{t} + \bar{x}\right). \end{aligned}$$

In matrix form this is

$$\begin{bmatrix} t \\ x \end{bmatrix} = \frac{1}{\sqrt{1 - v^2/c^2}}\begin{bmatrix} 1 & v/c^2 \\ v & 1 \end{bmatrix}\begin{bmatrix} \bar{t} \\ \bar{x} \end{bmatrix}. \tag{11}$$

Physical Implications:

Suppose that velocity $v << c$, i.e., v is small relative to the speed of light. Then v/c is negligible and the Lorentz transformations in (11) reduce to

$$\begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix}\begin{bmatrix} \bar{t} \\ \bar{x} \end{bmatrix}$$

or equivalently

$$t = \bar{t} \text{ and } x = v\bar{t} + \bar{x}.$$

These are the *Galilean transformations* of classical physics in which relativistic effects are not apparent. However, when relative speeds v are near c, the Lorentz transformations produce some surprising and dramatic relativistic effects.

Length Contraction:

A spaceship flying through space along a line with constant speed v flies by the international space station. At instant $\bar{t}$ in the moving frame $\bar{K}(c\bar{t}, \bar{x})$ of the spaceship, the ship's captain observes that the endpoints of the space station are positioned at $\bar{x}_1$ and $\bar{x}_2$ on the $\bar{x}$-axis; thus its ordinary length measured by the ship's captain is $\Delta\bar{x} = \bar{x}_2 - \bar{x}_1$. Thinking of these measurements as events, their $\bar{K}$-coordinates are $(\bar{t}, \bar{x}_1)$ and $(\bar{t}, \bar{x}_2)$, and we can use equation (11) to change coordinates and calculate the ordinary length $\Delta x = x_2 - x_1$ in the fixed reference frame $K\,(ct, x)$ of the space station. According to (11), the relationship between the lengths $\Delta\bar{x}$ and Δx at instant $\bar{t}$ is

$$\Delta x = x_2 - x_1 = \frac{v\bar{t} + \bar{x}_2}{\sqrt{1 - v^2/c^2}} - \frac{v\bar{t} + \bar{x}_1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - v^2/c^2}}\Delta\bar{x},$$

or equivalently,

$$\Delta\bar{x} = \sqrt{1 - v^2/c^2}\,\Delta x.$$

Since $\sqrt{1 - v^2/c^2} < 1$, ordinary length in frame K appears to contract when viewed from from $\bar{K}$. For example, if $v = .73c$, then

$$\sqrt{1 - .73^2} \approx \sqrt{.47} \approx .69;$$

if $\Delta x = \sqrt{34} \approx 5.83$, then

$$\Delta\bar{x} \approx (.69)\,(5.83) \approx 4.$$

So the ordinary length of the space station measured by the spaceship captain appears to be about 31% less than the ordinary length measured by the space station manager.

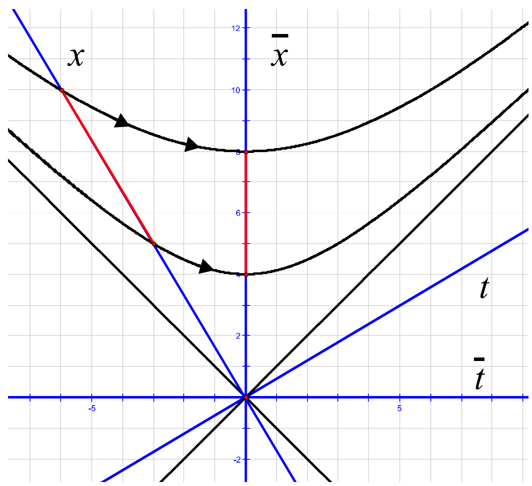


Figure 2. Length contracts as imaginary hyperbolic rotation angle increases.

In summary, *to an observer in a reference frame moving along a straight line with constant speed v relative to a fixed reference frame, the ordinary length of an object at rest in the fixed frame appears to be shorter than it does to an observer in the fixed frame by a factor of* $\sqrt{1 - v^2/c^2}$. And indeed, $\Delta\bar{x} \to 0$ as $v \to c$. This phenomenon is called the *Lorentz length contraction.*

Time Dilation:

Now suppose a clock on board the spaceship is positioned at the origin $\bar{O}$ in the moving frame $\bar{K}$ of the spaceship. As the spaceship passes the space station, the captain takes two clock readings $\bar{t}_1$ and $\bar{t}_2$ and determines the elapsed time to be $\Delta\bar{t} = \bar{t}_2 - \bar{t}_1$. Thinking of these two readings as events, their $\bar{K}$-coordinates are $(\bar{t}_1, 0)$ and $(\bar{t}_2, 0)$, and the relationship between the elapsed time $\Delta\bar{t}$ measured in the moving frame and the elapsed time Δt in the fixed frame given by (11) is

$$\Delta t = \frac{\bar{t}_2}{\sqrt{1 - v^2/c^2}} - \frac{\bar{t}_1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - v^2/c^2}}\Delta\bar{t}.$$

Since $\frac{1}{\sqrt{1 - v^2/c^2}} > 1$, elapsed time in frame $\bar{K}$ appears to dilate when viewed from K. For example, if $v = .73c$, then

$$\frac{1}{\sqrt{1 - .73^2}} \approx \frac{1}{\sqrt{.47}} \approx 1.46;$$

if $\Delta\bar{t} = 4$, then

$$\Delta t \approx (1.46)\,(4) = 5.84.$$

So as far as the space station manager can tell, space station clocks appear to run about 46% faster than clocks on board the passing spaceship.

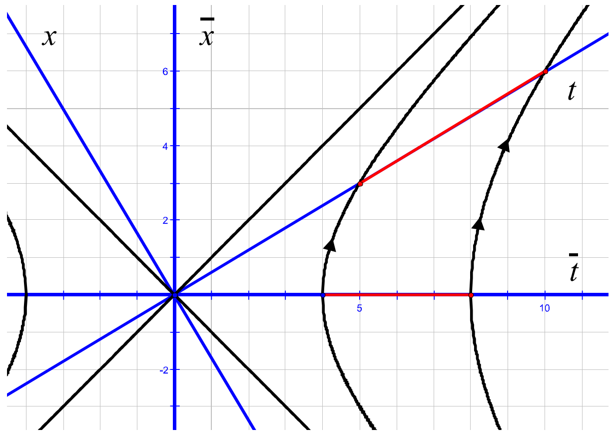


Figure 3. Time dilates as real hyperbolic angle increases.

In summary, *to an observer in a fixed reference frame, the elapsed time measured in a reference frame moving along a straight line with constant speed* v *appears to dilate by a factor of* $1/\sqrt{1 - v^2/c^2}$. And indeed, $\Delta t \to \infty$ as $v \to c$. This phenomenon is called the *Lorentz time dilation*.

Moral: Live fast; live long (relatively speaking...)!

Exercises

8. Consider a particle P positioned at the origin $\bar{O}$ in a frame $\bar{K}$ moving relative to a fixed frame K in the positive x-direction. Prove that the world line of P in K lies inside the light cone.

9. Compute the factors of length contraction and time dilation when $v = .9c$ and $v = .99c$.

11-18-2010

The Dimension Theorem

Let V and W be vector spaces over a field F, and let $T : V \to W$ be a linear map.

Definition 85 *The* ***kernel*** *of T is the set* $\ker(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$. *The* ***range*** *of T is the set* $\operatorname{range}(T) = \{\mathbf{w} \in W \mid \mathbf{w} = T(\mathbf{v}),$ *for some* $\mathbf{v} \in V\}$.

Exercise 86 *Prove that* $\ker(T)$ *is a subspace of V and* $\operatorname{range}(T)$ *is a subspace of W.*

Given an $m \times n$ matrix $A = [A_1 \mid \cdots \mid A_n]$, consider the matrix transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T_A(\mathbf{x}) = A\mathbf{x}$. Then $\ker(T_A)$ is the solution space of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$, called the *nullspace* of A, and is denoted by $N(A)$. If $\mathbf{y} \in \operatorname{range}(T_A)$, choose $\mathbf{x} = [x_1 \ \cdots \ x_n]^T \in \mathbb{R}^n$ such that $T_A(\mathbf{x}) = \mathbf{y}$. Then

$$\mathbf{y} = A\mathbf{x} = A_1x_1 + A_2x_2 + \cdots + A_nx_n,$$

and $\operatorname{range}(T_A)$ is the *column space* of A, which is denoted by $R(A)$.

Definition 87 *Given an $m \times n$ matrix A, the* ***nullity*** *of A is the dimension of* $\ker(T_A)$ *and the* ***rank*** *of A is the dimension of* $\operatorname{range}(T_A)$.

Thus the nullity of A is the dimension of $N(A)$ and and the rank of A is the dimension of $R(A)$.

Theorem 88 (The Dimension Theorem) *Given a matrix A, let n be the number of columns. Then*

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n.$$

Proof. The dimension k of the column space of A is the number of leading ones in any row-echelon form of A, hence $\operatorname{rank}(A) = k$. On the other hand, $\operatorname{nullity}(A)$ is the number of independent parameters in the solution space of $A\mathbf{x} = \mathbf{0}$, i.e., the number of columns in any row-echelon form of A that do not contain a leading 1. Therefore $\operatorname{nullity}(A) = n - k = n - \operatorname{rank}(A)$. ∎

10-2-12

Similarity Invariants

In this lecture we study some properties that are shared by similar matrices.

Definition 89 *Let A and B be $n \times n$ matrices. Then B* ***is similar to*** *A if there is an invertible $n \times n$ matrix P such that $B = P^{-1}AP$. When B is similar to A we write $B \sim A$.*

Example 90 *If A is diagonalizable, there is a diagonal matrix D similar to A.*

Exercise 91 *Prove that similarity is an equivalence relation on the set $M_n(\mathbb{R})$ of real $n \times n$ matrices.*

Some of important properties shared by similar matrices are the determinant, trace, rank, nullity, and eigenvalues.

Proposition 92 *Similar matrices have the same determinant.*

Proof. Suppose $B = P^{-1}AP$. Then

$$\begin{aligned}\det(B) &= \det\left(P^{-1}AP\right) = \det\left(P^{-1}\right)\det(A)\det(P) \\ &= \det\left(P^{-1}\right)\det(P)\det(A) = \det\left(P^{-1}P\right)\det(A) \\ &= \det(I)\det(A) = \det(A).\end{aligned}$$

■

Proposition 93 *Similar matrices have the same trace.*

Proof. First note that for all matrices $G = (g_{ij})$ and $H = (h_{ij})$ we have

$$\begin{aligned}\operatorname{tr}(GH) &= \operatorname{tr}\begin{bmatrix} \sum_{j=1}^n g_{1j}h_{j1} & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & \sum_{j=1}^n g_{nj}h_{jn} \end{bmatrix} \\ &= \sum_{i=1}^n \sum_{j=1}^n g_{ij}h_{ji} = \sum_{j=1}^n \sum_{i=1}^n h_{ji}g_{ij} = \operatorname{tr}(HG).\end{aligned}$$

Now if $B = P^{-1}AP$, then $\operatorname{tr}(B) = \operatorname{tr}\left[\left(P^{-1}A\right)P\right] = \operatorname{tr}\left[P\left(P^{-1}A\right)\right] = \operatorname{tr}(A)$. ■

Proposition 94 *Similar matrices have the same nullity.*

Proof. Suppose $B = P^{-1}AP$ and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis for the null space $N(B)$. Then $B\mathbf{v}_i = P^{-1}AP\mathbf{v}_i = \mathbf{0}$ for each i, and $A(P\mathbf{v}_i) = \mathbf{0}$. Hence $P\mathbf{v}_i \in N(A)$ for all i. I claim that $S = \{P\mathbf{v}_1, \ldots, P\mathbf{v}_k\}$ is a basis for $N(A)$. To show S spans $N(A)$, let $\mathbf{u} \in N(A)$. Then $\mathbf{0} = A\mathbf{u} = PBP^{-1}\mathbf{u}$, and multiplying both sides by P^{-1} gives $B\left(P^{-1}\mathbf{u}\right) = \mathbf{0}$ so that $P^{-1}\mathbf{u} \in N(B)$. Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ spans $N(B)$, there exist $c_i \in \mathbb{R}$ such that $P^{-1}\mathbf{u} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$. Then multiplying both sides by P gives $\mathbf{u} = P(c_1\mathbf{v}_1) + \cdots + P(c_k\mathbf{v}_k) = c_1(P\mathbf{v}_1) + \cdots + c_k(P\mathbf{v}_k)$. Therefore S spans. To show S is linear independent, suppose $c_1(P\mathbf{v}_1) + \cdots + c_k(P\mathbf{v}_k) = 0$. Multiplying both sides by P^{-1} gives $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$. Then $c_i = 0$ for all i since $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent, and it follows that A and B have the same nullity k. ■

Corollary 95 *Similar matrices have the same rank.*

Proof. By the Dimension Theorem, $\operatorname{rank}(B) = n - \operatorname{nullity}(B) = n - \operatorname{nullity}(A) = \operatorname{rank}(A)$. ■

Proposition 96 *Similar matrices have the same eigenvalues.*

Proof. Suppose $B = P^{-1}AP$. Let λ be an eigenvalue for A and let $\mathbf{x}$ be a corresponding eigenvector. Then $A\mathbf{x} = \lambda\mathbf{x}$ so that $PBP^{-1}\mathbf{x} = \lambda\mathbf{x}$. Multiplying both sides by P^{-1} gives $B\left(P^{-1}\mathbf{x}\right) = \lambda P^{-1}\mathbf{x}$. But $P^{-1}\mathbf{x} \neq \mathbf{0}$ since P is invertible, so λ is an eigenvalue of B. The converse is similar and left as an exercise for the reader.

■

Note that if λ is an eigenvalue of a nonsingular matrix A and $\mathbf{x}$ is a corresponding eigenvector, then $A\mathbf{x} \neq \mathbf{0}$ since the nullity of A is 0. But $A\mathbf{x} = \lambda\mathbf{x} \neq \mathbf{0}$ implies $\lambda \neq 0$. Therefore the eigenvalues of a nonsingular matrix A are non-zero.

Row-reduction to Hessenberg Form

Let A be an $n \times n$ matrix.

1. Let $A = (a_{ij})$. If $a_{31} = a_{41} = \cdots = a_{n1} = 0$, set $A_1 = A$ and go to step 2. Otherwise, find a matrix $A_1 = \left(a'_{ij}\right)$ similar to A such that $a'_{31} = a'_{41} = \cdots = a'_{n1} = 0$ as follows:

   (a) If $a_{21} \neq 0$, set $B_1 = A$ and go to step 1b. Otherwise, choose row $k > 2$ such that $a_{k1} \neq 0$. Let P_1 be the permutation matrix obtained from I by interchanging rows 2 and k. Then $P_1 A$ is the matrix obtained from A by interchanging rows 2 and k. Note that $P_1^{-1} = P_1$. Multiplying $P_1 A$ by P_1 on the right interchanges columns 2 and k. Let

   $$B_1 = P_1 A P_1.$$

   (b) Given $B_1 = (b_{ij})$ with $b_{21} \neq 0$, use row 2 to eliminate all non-zero entries below b_{21}. Let Q_1 be the matrix obtained by performing the same sequence of row operations on the identity matrix I that you performed on B_1. Then $Q_1 B$ is the matrix obtained from B_1 by eliminating all non-zero entries below b_{21}. Note that $Q_1^{-1} = 2I - Q_1$. Multiplying $Q_1 B_1$ by Q_1^{-1} on the right performs the column operations on $Q_1 B_1$ analogous to the row operations that determined Q_1. Let

   $$A_1 = Q_1 B_1 Q_1^{-1}.$$

2. Let $A_1 = (a_{ij})$. If $a_{42} = a_{52} = \cdots = a_{n2} = 0$, set $A_2 = A_1$ and go to step 3. Otherwise, find a matrix $A_2 = \left(a'_{ij}\right)$ similar to A_1 such that $a'_{42} = a'_{52} = \cdots = a'_{n2} = 0$ as follows:

   (a) If $a_{32} \neq 0$, set $B_2 = A_1$ and go to step 2b. Otherwise, choose row $k > 3$ such that $a_{k2} \neq 0$. Let P_2 be the permutation matrix obtained from I by interchanging rows 3 and k. Then $P_2 A_1$ is the matrix obtained from A_1 by interchanging rows 3 and k. Multiplying $P_2 A$ by P_2 on the right interchanges columns 3 and k. Let

   $$B_2 = P_2 A_1 P_2.$$

   (b) Given $B_2 = (b_{ij})$ with $b_{32} \neq 0$, use row 3 to eliminate all non-zero entries below b_{32}. Let Q_2 be the matrix obtained by performing the same sequence of row operations on the identity matrix I that you performed on B_2. Then $Q_2 B_2$ is the matrix obtained from B_2 by eliminating all non-zero entries below b_{21}. Multiplying $Q_2 B_2$ by Q_2^{-1} on the right performs the column operations on $Q_2 B_2$ analogous to the row operations that determined Q_2. Let

   $$A_2 = Q_2 B_2 Q_2^{-1}.$$

3. Continuing in this manner, eliminate all non-zero entries below the subdiagonal in each successive column until Hessenberg form is obtained.

10-25-2012

Definition 97 *An $n \times n$ Hessenberg matrix H is* ***unreduced*** *if all entries along the subdiagonal are non-zero; otherwise H is* ***reduced.***

The objective of this lecture is to understand the mechanics of Krylov's Method, which produces the characteristic polynomial of an unreduced $n \times n$ Hessenberg matrix H. A proof that Krylov's Method produces the characteristic polynomial of a general unreduced $n \times n$ Hessenberg matrix is a consequence of the Cayley-Hamilton Theorem (to come). From now on we'll use Anton's definition of the characteristic polynomial:

Definition 98 *Let A be an $n \times n$ matrix. The* ***characteristic polynomial of*** *A is defined $p_A(t) = \det(tI - A)$.*

Definition 99 *Let H be an unreduced $n \times n$ Hessenberg matrix and let $\mathbf{e}_1 = [1\ 0\ \cdots\ 0]^T \in \mathbb{R}^n$. The* ***Hessenberg chain of*** *H is the sequence*

$$\begin{aligned}
\mathbf{w}_0 &= \mathbf{e}_1 \\
\mathbf{w}_1 &= H\mathbf{w}_0 \\
\mathbf{w}_2 &= H\mathbf{w}_1 \\
&\vdots \\
\mathbf{w}_n &= H\mathbf{w}_{n-1}.
\end{aligned}$$

Proposition 100 *If $\{\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is the Hessenberg chain of an unreduced $n \times n$ Hessenberg matrix H, the subset $\{\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_{n-1}\}$ is linearly independent.*

Proof. Let $H = (h_{ij})$ be an unreduced $n \times n$ Hessenberg matrix. The subset $\{\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_{n-1}\}$ of the Hessenberg chain is linearly independent if and only if the matrix $[\mathbf{w}_0 \mid \mathbf{w}_1 \mid \cdots \mid \mathbf{w}_{n-1}]$ is upper triangular with non-zero diagonal entries. First,

$$\mathbf{w}_0 = \mathbf{e}_1 = [1\ 0\ \cdots\ 0]^T.$$

Second,

$$\mathbf{w}_1 = H\mathbf{w}_0 = \begin{bmatrix} * & h_{21} & 0 & \cdots \end{bmatrix}^T.$$

Third,

$$\mathbf{w}_2 = H\mathbf{w}_1 = \begin{bmatrix} * & * & \cdots \\ h_{21} & * & \cdots \\ 0 & h_{32} & \cdots \\ \vdots & 0 & \ddots \\ & \vdots & \ddots \end{bmatrix} \begin{bmatrix} * \\ h_{21} \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} * \\ * \\ h_{32}h_{21} \\ 0 \\ \vdots \end{bmatrix}.$$

Fourth,

$$\mathbf{w}_3 = H\mathbf{w}_2 = \begin{bmatrix} * & * & * & \cdots \\ h_{21} & * & * & \cdots \\ 0 & h_{32} & * & \cdots \\ \vdots & 0 & h_{43} & \cdots \\ & \vdots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} * \\ * \\ h_{32}h_{21} \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} * \\ * \\ * \\ h_{43}h_{32}h_{21} \\ 0 \\ \vdots \end{bmatrix}.$$

Inductively, assume that

$$\mathbf{w}_{i-1} = \begin{bmatrix} * & \cdots & * & h_{i,i-1} \cdots h_{32}h_{21} & 0 & \cdots & 0 \end{bmatrix}^T,$$

where $h_{i,i-1}\cdots h_{32}h_{21}$ is the i^{th} entry and the last $n-i$ entries are zero. To determine the $(i+1)^{st}$ entry of $\mathbf{w}_i = H\mathbf{w}_{i-1}$, first note that only the first i entries of $\mathbf{w}_{i-1}$ can contribute to the $(i+1)^{st}$ entry of $H\mathbf{w}_{i-1}$. On the other hand, since the first $i-1$ entries of the $(i+1)^{st}$ row of H are zero, the only entries of the $(i+1)^{st}$ row of H that can contribute to the $(i+1)^{st}$ entry of $H\mathbf{w}_{i-1}$ are $h_{i+1,j}$ with $j \geq i$. Therefore the $(i+1)^{st}$ entry of $H\mathbf{w}_{i-1}$ is the product of the i^{th} entry of the $(i+1)^{st}$ row of H with the i^{th} entry of $\mathbf{w}_{i-1}$, i.e.,

$$\underbrace{\begin{bmatrix} 0 & \cdots & 0 & h_{i+1,i} & * & \cdots & * \end{bmatrix}}_{(i+1)^{st} \text{ row of } H} \begin{bmatrix} * \\ \vdots \\ * \\ h_{i,i-1}\cdots h_{32}h_{21} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = h_{i+1,i}h_{i,i-1}\cdots h_{32}h_{21}.$$

Furthermore, if $i+2 \leq j \leq n$, the first i entries of the j^{th} row of H are zero and

$$\left[j^{th} \text{ row of } H\right] \mathbf{w}_{i-1} = 0.$$

Therefore

$$\mathbf{w}_i = \begin{bmatrix} * & \cdots & * & h_{i,i-1}\cdots h_{32}h_{21} & 0 & \cdots & 0 \end{bmatrix}^T,$$

completing the induction. Since H is unreduced, $h_{i,i-1} \neq 0$ for $2 \leq i \leq n$ and the entry $h_{i,i-1}\cdots h_{32}h_{21} \neq 0$.

■

A Hessenberg chain resembles a "Jordan chain". When L is a nilpotent matrix of index k and $L^{k-1}\mathbf{x} \neq 0$, the sequence $\{\mathbf{x}, L\mathbf{x}, L^2\mathbf{x}, \ldots, L^{k-1}\mathbf{x}\}$ is called *the Jordan chain on* $\mathbf{x}$. Jordan chains play a key role in the construction of the Jordan canonical form of L.

Algorithm 101 (Krylov's Method) *Let H be an unreduced $n \times n$ Hessenberg matrix. To compute the characteristic polynomial $p_H(t)$:*

1. *Build the Hessenberg chain $\{\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_n\}$ of H.*
2. *Solve the linear system $a_0\mathbf{w}_0 + a_1\mathbf{w}_1 + \cdots + a_{n-1}\mathbf{w}_{n-1} = -\mathbf{w}_n$.*
3. *Conclude that $p_H(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$.*

Example 102 *Find the characteristic polynomial of*

$$H = \begin{bmatrix} 5 & -2 \\ 6 & -2 \end{bmatrix}.$$

Then by definition,

$$p_H(t) = \det(tI - H) = \det \begin{bmatrix} t-5 & 2 \\ -6 & t+2 \end{bmatrix} = t^2 - 3t + 2. \tag{12}$$

To apply Krylov's Method, build the Hessenberg chain:

$$\mathbf{w}_0 = \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \mathbf{w}_1 = H\mathbf{w}_0 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}; \quad \mathbf{w}_2 = H\mathbf{w}_1 = \begin{bmatrix} 5 & -2 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 13 \\ 18 \end{bmatrix}$$

Solve the linear system $a_0\mathbf{w}_0 + a_1\mathbf{w}_1 = -\mathbf{w}_2$:

$$\left[\begin{array}{cc|c} 1 & 5 & -13 \\ 0 & 6 & -18 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right].$$

Then

$$a_1 = -3, \; a_0 = 2,$$

and

$$p_H(t) = t^2 - 3t + 2,$$

which agrees with our calculations in (12).

Example 103 *Find the characteristic polynomial of*

$$H = \begin{bmatrix} 2 & 2 & -1 \\ -1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}.$$

Then by definition,

$$p_H(t) = \det(tI - H) = \det \begin{bmatrix} t-2 & -2 & 1 \\ 1 & t+1 & -1 \\ 0 & -2 & t-1 \end{bmatrix} \tag{13}$$

$$= (t-2)\left[(t+1)(t-1) - 2\right] - (-2t+4) = (t-2)\left(t^2-3\right) + 2t - 4 = t^3 - 2t^2 - t + 2.$$

To apply Krylov's Method, build the Hessenberg chain:

$$\mathbf{w}_0 = \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad \mathbf{w}_1 = H\mathbf{w}_0 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}; \quad \mathbf{w}_2 = H\mathbf{w}_1 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}; \quad \mathbf{w}_3 = H\mathbf{w}_2 = \begin{bmatrix} 4 \\ -3 \\ -4 \end{bmatrix}.$$

Solve the linear system $a_0\mathbf{w}_0 + a_1\mathbf{w}_1 + a_2\mathbf{w}_2 = -\mathbf{w}_3$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & -4 \\ 0 & -1 & -1 & 3 \\ 0 & 0 & -2 & 4 \end{array}\right] \xrightarrow{row\ reduce} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Then

$$a_2 = -2, \; a_1 = -1, \; a_0 = 2,$$

and

$$p_H(t) = t^3 - 2t^2 - t + 2,$$

which agrees with our calculations in (13).

Example 104 *Find the characteristic polynomial of*

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & 0 & 2 & 2 \end{bmatrix}.$$

To apply Krylov's Method, build the Hessenberg chain:

$$\mathbf{w}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \quad \mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}; \quad \mathbf{w}_2 = \begin{bmatrix} 3 \\ 2 \\ -2 \\ 0 \end{bmatrix}; \quad \mathbf{w}_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \\ -4 \end{bmatrix}; \quad \mathbf{w}_4 = \begin{bmatrix} 5 \\ 4 \\ 0 \\ -4 \end{bmatrix}.$$

Solve the linear system $a_0\mathbf{w}_0 + a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + a_3\mathbf{w}_3 = -\mathbf{w}_4$:

$$\left[\begin{array}{cccc|c} 1 & 1 & 3 & 3 & -5 \\ 0 & 2 & 2 & 4 & -4 \\ 0 & 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & -4 & 4 \end{array}\right] \xrightarrow{row\ reduce} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

Then

$$a_3 = -1, \; a_2 = -1, \; a_1 = 1, \; a_0 = 0,$$

and

$$p_H(t) = t^4 - t^3 - t^2 + t.$$

Characteristic Polynomials of Reduced Hessenberg Matrices

Although Krylov's Method is ineffective when H is a reduced Hessenberg matrix, there is always a block decomposition of H in which unreduced Hessenberg blocks appear along the (block) diagonal, and the characteristic polynomial of H is the product of the characteristic polynomials of these unreduced Hessenberg blocks.

Proposition 105 *Let M be a square matrix with block decomposition of the form*

$$M = \left[\begin{array}{c|c} A^{m\times m} & B^{m\times n} \\ \hline \mathbf{0}^{n\times m} & C^{n\times n} \end{array}\right].$$

Then $\det M = \det A \det C$.

Proof. If $m = n = 1$, then M is a 2×2 matrix and $\det \left[\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right] = AC = \det A \det C$. Inductively, let $k \geq 2$ and assume that the statement holds for all such decompositions of $k\times k$ matrices M. Consider a $(k+1)\times(k+1)$ matrix with block decomposition

$$M = \left[\begin{array}{c|ccc} A & & B & \\ \hline & c_{11} & \cdots & c_{1r} \\ \mathbf{0} & \vdots & \ddots & \vdots \\ & c_{r1} & \cdots & c_{rr} \end{array}\right].$$

Let B_i be the matrix obtained from B by deleting its i^{th} column. Let C_i be the matrix obtained from C by deleting its i^{th} column and last row. Cofactor expanding along the last row of M gives

$$\det M = \pm c_{r1} \det \begin{bmatrix} A & B_1 \\ \mathbf{0} & C_1 \end{bmatrix} \pm c_{r2} \det \begin{bmatrix} A & B_2 \\ \mathbf{0} & C_2 \end{bmatrix} \pm \cdots \pm c_{rr} \det \begin{bmatrix} A & B_r \\ \mathbf{0} & C_r \end{bmatrix}.$$

Since $\begin{bmatrix} A & B_i \\ \mathbf{0} & C_i \end{bmatrix}$ is a $k\times k$ matrix for each i, we have $\det \begin{bmatrix} A & B_i \\ \mathbf{0} & C_i \end{bmatrix} = \det A \det C_i$ for each i by the induction hypothesis. Therefore

$$\begin{aligned} \det M &= \pm c_{r1} \det A \det C_1 \pm c_{r2} \det A \det C_2 \pm \cdots \pm c_{rr} \det A \det C_r \\ &= \det A \left(\pm c_{r1} \det C_1 \pm c_{r2} \det C_2 \pm \cdots \pm c_{rr} \det C_r\right) = \det A \det C. \end{aligned}$$

■

Proposition 105 extends immediately to block matrices with k blocks along the diagonal. Let

$$M = \left[\begin{array}{c|c|c|c} A_1 & * & \ddots & * \\ \hline \mathbf{0} & A_2 & \ddots & * \\ \hline \vdots & \ddots & \ddots & * \\ \hline \mathbf{0} & \mathbf{0} & \cdots & A_k \end{array}\right]$$

and think of $C = \left[\begin{array}{c|c|c} A_2 & \ddots & * \\ \hline \ddots & \ddots & * \\ \hline \mathbf{0} & \cdots & A_k \end{array}\right]$ as a single block. Then $\det M = \det A_1 \det C$. Inductively, $\det M = \det A_1 \cdots \det A_k$.

Example 106

$$\begin{aligned} \det \begin{bmatrix} a & b & * & * \\ c & d & * & * \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{bmatrix} &= a \det \begin{bmatrix} d & * & * \\ 0 & e & f \\ 0 & g & h \end{bmatrix} - c \det \begin{bmatrix} b & * & * \\ 0 & e & f \\ 0 & g & h \end{bmatrix} = ad\,(eh - fg) - bc\,(eh - fg) \\ &= adef - adfg - bceh + bcfg = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \det \begin{bmatrix} e & f \\ g & h \end{bmatrix}. \end{aligned}$$

We multiply block matrices the same way we multiply matrices with scalar entries. For example,

$$\left[\begin{array}{cc|cc} a & b & c & d \\ e & f & g & h \\ \hline i & j & k & l \\ m & n & p & q \end{array}\right]\left[\begin{array}{cc|cc} A & B & C & D \\ E & F & G & H \\ \hline I & J & K & L \\ M & N & P & Q \end{array}\right] =$$

$$= \left[\begin{array}{c|c} \begin{bmatrix} a & b \\ e & f \end{bmatrix}\begin{bmatrix} A & B \\ E & F \end{bmatrix} + \begin{bmatrix} c & d \\ g & h \end{bmatrix}\begin{bmatrix} I & J \\ M & N \end{bmatrix} & \begin{bmatrix} a & b \\ e & f \end{bmatrix}\begin{bmatrix} C & D \\ G & H \end{bmatrix} + \begin{bmatrix} c & d \\ g & h \end{bmatrix}\begin{bmatrix} K & L \\ P & Q \end{bmatrix} \\ \hline \begin{bmatrix} i & j \\ m & n \end{bmatrix}\begin{bmatrix} A & B \\ E & F \end{bmatrix} + \begin{bmatrix} k & l \\ p & q \end{bmatrix}\begin{bmatrix} I & J \\ M & N \end{bmatrix} & \begin{bmatrix} i & j \\ m & n \end{bmatrix}\begin{bmatrix} C & D \\ G & H \end{bmatrix} + \begin{bmatrix} k & l \\ p & q \end{bmatrix}\begin{bmatrix} K & L \\ P & Q \end{bmatrix} \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} aA+bE+cI+dM & aB+bF+cJ+dN & aC+bG+cK+dP & aD+bH+cL+dQ \\ eA+fE+gI+hM & eB+fF+gJ+hN & eC+fG+gK+hP & eD+fH+gL+hQ \\ \hline iA+jE+kI+lM & iB+jF+kJ+lN & iC+jG+kK+lP & iD+jH+kL+lQ \\ mA+nE+pI+qM & mB+nF+pJ+qN & mC+nG+pK+qP & mD+nH+pL+qQ \end{array}\right]$$

Theorem 107 *Let T be a block triangular matrix of the form*

$$T = \left[\begin{array}{c|c} A & B \\ \hline \mathbf{0} & C \end{array}\right].$$

Then λ is an eigenvalue of T if and only if λ is an eigenvalue of either A or C.

Proof. Let λ is an eigenvalue of T, and let

$$\mathbf{x} = \left[\begin{array}{c} \mathbf{u} \\ \hline \mathbf{v} \end{array}\right]$$

be a corresponding eigenvector partitioned so that block matrix multiplication on the left by T is defined. Then on the one hand,

$$\left[\begin{array}{c|c} A & B \\ \hline \mathbf{0} & C \end{array}\right]\left[\begin{array}{c} \mathbf{u} \\ \hline \mathbf{v} \end{array}\right] = \left[\begin{array}{c} A\mathbf{u}+B\mathbf{v} \\ \hline C\mathbf{v} \end{array}\right],$$

and on the other,

$$\left[\begin{array}{c|c} A & B \\ \hline \mathbf{0} & C \end{array}\right]\left[\begin{array}{c} \mathbf{u} \\ \hline \mathbf{v} \end{array}\right] = \lambda\left[\begin{array}{c} \mathbf{u} \\ \hline \mathbf{v} \end{array}\right] = \left[\begin{array}{c} \lambda\mathbf{u} \\ \hline \lambda\mathbf{v} \end{array}\right].$$

Therefore the following equations hold:

$$\begin{aligned} A\mathbf{u}+B\mathbf{v} &= \lambda\mathbf{u} \\ C\mathbf{v} &= \lambda\mathbf{v}. \end{aligned}$$

If $\mathbf{v} = \mathbf{0}$, then $\mathbf{u} \neq \mathbf{0}$ and the first equation reduces to $A\mathbf{u} = \lambda\mathbf{u}$, in which case λ is an eigenvalue of A.If $\mathbf{v} \neq \mathbf{0}$, the second equation implies that λ is an eigenvalue of C.

Conversely, if λ is an eigenvalue of A, choose a corresponding eigenvector $\mathbf{u}_1$. Then

$$\left[\begin{array}{c|c} A & B \\ \hline \mathbf{0} & C \end{array}\right]\left[\begin{array}{c} \mathbf{u}_1 \\ \hline \mathbf{0} \end{array}\right] = \left[\begin{array}{c} A\mathbf{u}_1 \\ \hline \mathbf{0} \end{array}\right] = \left[\begin{array}{c} \lambda\mathbf{u}_1 \\ \hline \mathbf{0} \end{array}\right] = \lambda\left[\begin{array}{c} \mathbf{u}_1 \\ \hline \mathbf{0} \end{array}\right]$$

and λ is an eigenvalue of T. If λ is not an eigenvalue of A but is an eigenvalue of C, choose a corresponding eigenvector $\mathbf{v}_1$. Then for all vectors $\mathbf{u}$ we have

$$\left[\begin{array}{c|c} A & B \\ \hline \mathbf{0} & C \end{array}\right]\left[\begin{array}{c} \mathbf{u} \\ \hline \mathbf{v}_1 \end{array}\right] = \left[\begin{array}{c} A\mathbf{u}+B\mathbf{v}_1 \\ \hline C\mathbf{v}_1 \end{array}\right] = \left[\begin{array}{c} A\mathbf{u}+B\mathbf{v}_1 \\ \hline \lambda\mathbf{v}_1 \end{array}\right].$$

Thus to show that λ is an eigenvalue of T, we must find a vector $\mathbf{u}_1$ such that $A\mathbf{u}_1+B\mathbf{v}_1 = \lambda\mathbf{u}_1$, or equivalently, $B\mathbf{v}_1 = \lambda\mathbf{u}_1 - A\mathbf{u}_1 = (\lambda I - A)\,\mathbf{u}_1$. But since λ is not an eigenvalue of A, $\det(\lambda I - A) \neq 0$ and $\lambda I - A$ is invertible. So choose $\mathbf{u}_1 = (\lambda I - A)^{-1} B\mathbf{v}_1$. Then $A\mathbf{u}_1+B\mathbf{v}_1 = \lambda\mathbf{u}_1$ and

$$\left[\begin{array}{c|c} A & B \\ \hline \mathbf{0} & C \end{array}\right]\left[\begin{array}{c} \mathbf{u}_1 \\ \hline \mathbf{v}_1 \end{array}\right] = \left[\begin{array}{c} A\mathbf{u}_1 + B\mathbf{v}_1 \\ \hline C\mathbf{v}_1 \end{array}\right] = \left[\begin{array}{c} \lambda\mathbf{u}_1 \\ \hline \lambda\mathbf{v}_1 \end{array}\right] = \lambda\left[\begin{array}{c} \mathbf{u}_1 \\ \hline \mathbf{v}_1 \end{array}\right],$$

in which case λ is an eigenvalue of T. ■

Example 108 *Consider the following reduced Hessenberg matrix H and the indicated block decomposition with unreduced Hessenberg matrices along the diagonal:*

$$H = \left[\begin{array}{cc|ccc|cc} 2 & 3 & 1 & 6 & -1 & 3 & 8 \\ 5 & 7 & 2 & 8 & 2 & 2 & 1 \\ \hline 0 & 0 & 4 & 1 & 3 & -5 & 2 \\ 0 & 0 & 6 & 1 & 2 & 4 & 3 \\ 0 & 0 & 0 & 4 & 1 & 2 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 6 & 5 \\ 0 & 0 & 0 & 0 & 0 & 7 & 3 \end{array}\right].$$

By induction, the eigenvalues of H are the eigenvalues of the unreduced Hessenberg blocks along the diagonal.

11-1-2012

Householder's Method – Hessenberg Form via Orthogonal Transformations

In this lecture we use "Householder transformations" to reduce a general $n \times n$ matrix to Hessenberg form. This technique is advantageous because it preserves symmetry, i.e., if A is symmetric and H is a Hessenberg matrix obtained via Householder transformations, then H is symmetric. Note that a symmetric Hessenberg matrix is *tridiagonal,* i.e., all non-zero entries lie on the subdiagonal, the main diagonal, and the superdiagonal. We begin with some geometry.

Let $\mathbf{u}$ be a non-zero vector in $\mathbb{C}^n$ with its Euclidean inner product, and recall that if $\mathbf{v} \in \mathbb{C}^n$, then

$$\mathbf{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u}^*\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}.$$

Consider the matrix

$$P = I - \frac{1}{\|\mathbf{u}\|^2}\mathbf{u}\mathbf{u}^*.$$

Proposition 109 *Multiplication by P is orthogonal projection onto $\mathbf{u}^\perp$, i.e., if $\mathbf{x} \in \mathbb{C}^n$, then*

$$P\mathbf{x} = \mathbf{proj}_{\mathbf{u}^\perp}\mathbf{x}.$$

Furthermore, the null space $N(P) = \operatorname{span}\{\mathbf{u}\}$.

Proof. For all $\mathbf{x} \in \mathbb{C}^n$ we have $P\mathbf{x} = \mathbf{x} - \mathbf{proj}_{\mathbf{u}}\mathbf{x} = \mathbf{proj}_{\mathbf{u}^\perp}\mathbf{x}$. Furthermore, $\mathbf{x} \in N(P)$ if and only if $P\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} - \frac{\mathbf{u}^*\mathbf{x}}{\|\mathbf{u}\|^2}\mathbf{u} = \mathbf{0}$ if and only if $\mathbf{x} = \frac{\mathbf{u}^*\mathbf{x}}{\|\mathbf{u}\|^2}\mathbf{u} = \mathbf{proj}_{\mathbf{u}}\mathbf{x}$ if and only if $\mathbf{x} = t\mathbf{u}$ for some $t \in \mathbb{C}$ if and only if $\mathbf{x} \in \operatorname{span}\{\mathbf{u}\}$. ■

Definition 110 *Let $\mathbf{x}, \mathbf{y}, \mathbf{u} \in \mathbb{C}^n$ with $\mathbf{u} \neq 0$. Then $\mathbf{y}$ **is the reflection of** $\mathbf{x}$ **in the subspace** $\mathbf{u}^\perp$ if and only if*

$$\mathbf{x} - \mathbf{y} = 2\ \mathbf{proj}_{\mathbf{u}}\mathbf{x}.$$

Now consider the matrix

$$Q = I - \frac{2}{\|\mathbf{u}\|^2}\mathbf{u}\mathbf{u}^*.$$

Proposition 111 *Multiplication by Q is reflection in the subspace $\mathbf{u}^\perp$, i.e., if $\mathbf{x} \in \mathbb{C}^n$, then*

$$\mathbf{x} - Q\mathbf{x} = 2\ \mathbf{proj}_{\mathbf{u}}\mathbf{x}.$$

Proof. For all $\mathbf{x} \in \mathbb{C}^n$ we have $Q\mathbf{x} = \mathbf{x} - 2\frac{\mathbf{u}^*\mathbf{x}}{\|\mathbf{u}\|^2}\mathbf{u} = \mathbf{x} - 2\ \mathbf{proj}_{\mathbf{u}}\mathbf{x}$. Thus $\mathbf{x} - Q\mathbf{x} = 2\ \mathbf{proj}_{\mathbf{u}}\mathbf{x}$. ■

Definition 112 *The **Householder matrix associated with** $\mathbf{u}$ is the $n \times n$ matrix*

$$Q = I - \frac{2}{\|\mathbf{u}\|^2}\mathbf{u}\mathbf{u}^*.$$

*The **Householder transformation associated with** $\mathbf{u}$ is the matrix transformation $T_Q(\mathbf{x}) = Q\mathbf{x}$.*

Theorem 113 *Real Householder matrices are symmetric and orthogonal.*

Proof. Let $\mathbf{u}$ be a non-zero vector in $\mathbb{R}^n$ and let Q the Householder matrix associated with $\mathbf{u}$. Then

$$Q^T = \left(I - \frac{2}{\|\mathbf{u}\|^2}\mathbf{u}\mathbf{u}^T\right)^T = I - \frac{2}{\|\mathbf{u}\|^2}\left(\mathbf{u}\mathbf{u}^T\right)^T = I - \frac{2}{\|\mathbf{u}\|^2}\left(\mathbf{u}^T\right)^T\mathbf{u}^T = Q$$

and Q is symmetric. Furthermore,

$$Q^2 = \left(I - \frac{2}{\|\mathbf{u}\|^2}\mathbf{u}\mathbf{u}^T\right)^2 = I - \frac{4}{\|\mathbf{u}\|^2}\mathbf{u}\mathbf{u}^T + \frac{4}{\|\mathbf{u}\|^4}\mathbf{u}\mathbf{u}^T\mathbf{u}\mathbf{u}^T = I - \frac{4}{\|\mathbf{u}\|^2}\mathbf{u}\mathbf{u}^T + \frac{4}{\|\mathbf{u}\|^4}\|\mathbf{u}\|^2\mathbf{u}\mathbf{u}^T = I$$

and Q is orthogonal. ■

Exercise 114 *Generalize Theorem 113 for complex Householder matrices: Prove that a complex Householder matrix Q is Hermitian and unitary.*

For the remainder of this lecture we'll restrict our attention to real Householder transformations acting on $\mathbb{R}^n$. In practice, we rarely need to compute Q explicitly because $Q\mathbf{x}$ is simply a difference of vectors in $\mathbb{R}^n$:

$$Q\mathbf{x} = \mathbf{x} - \underbrace{\left(\frac{2}{\|\mathbf{u}\|^2} \mathbf{u}^T \mathbf{x} \right)}_{\text{scalar}} \mathbf{u}.$$

More generally, if $A = [\mathbf{a}_1 \mid \cdots \mid \mathbf{a}_p]$ is an $n \times p$ matrix, and

$$\mathbf{w}_i = \left(\frac{2}{\|\mathbf{u}\|^2} \mathbf{u}^T \mathbf{a}_i \right) \mathbf{u},$$

then

$$QA = [Q\mathbf{a}_1 \mid \cdots \mid Q\mathbf{a}_p] = [\mathbf{a}_1 - \mathbf{w}_1 \mid \cdots \mid \mathbf{a}_p - \mathbf{w}_p].$$

Example 115 *Let*

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 4 \\ 3 \end{bmatrix} \quad \text{and} \quad A = [\mathbf{a}_1 \mid \mathbf{a}_2] = \begin{bmatrix} 1 & 6 \\ 2 & 0 \\ 1 & 5 \\ -2 & 3 \end{bmatrix}.$$

To compute $Q\mathbf{x}$, first evaluate

$$\mathbf{w} = \left(\frac{2}{\|\mathbf{u}\|^2} \mathbf{u}^T \mathbf{x} \right) \mathbf{u} = 2\mathbf{u}.$$

Then

$$Q\mathbf{x} = \mathbf{x} - \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 4 \\ 1 \end{bmatrix}$$

To compute $QA = [Q\mathbf{a}_1 | Q\mathbf{a}_2]$, first evaluate

$$\mathbf{w}_1 = \left(\frac{2}{\|\mathbf{u}\|^2} \mathbf{u}^T \mathbf{a}_1 \right) \mathbf{u} = \mathbf{u} \quad \text{and} \quad \mathbf{w}_2 = \left(\frac{2}{\|\mathbf{u}\|^2} \mathbf{u}^T \mathbf{a}_2 \right) \mathbf{u} = 3\mathbf{u}.$$

Then

$$\begin{aligned} QA &= [Q\mathbf{a}_1 \mid Q\mathbf{a}_2] = [\mathbf{a}_1 - \mathbf{w}_1 \mid \mathbf{a}_2 - \mathbf{w}_2] = [\mathbf{a}_1 - \mathbf{u} \mid \mathbf{a}_2 - 3\mathbf{u}] \\ &= \left[\begin{bmatrix} 1 \\ 2 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 6 \\ 0 \\ 5 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \\ 0 \\ 3 \end{bmatrix} \right] = \begin{bmatrix} 0 & 3 \\ 0 & -6 \\ 1 & 5 \\ -3 & 0 \end{bmatrix}. \end{aligned}$$

<u>**General Problem**</u>**:**

Given $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ and $1 \le k \le n$, *find a Householder matrix Q such that* $Q\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_{k-1} \\ s_k \\ 0 \\ \vdots \end{bmatrix}$.

Solution:

$$Let \quad s_k = \begin{cases} \sqrt{v_k^2 + \cdots + v_n^2}, & v_k < 0 \\ -\sqrt{v_k^2 + \cdots + v_n^2}, & v_k \geq 0 \end{cases} \quad and \quad \mathbf{u} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ v_k - s_k \\ v_{k+1} \\ \vdots \\ v_n \end{bmatrix}. \quad Then\ Q\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_{k-1} \\ s_k \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{v} - \mathbf{u}.$$

Proof. First note that $s_k^2 = v_k^2 + \cdots + v_n^2$. Then

$$\mathbf{u}^T\mathbf{v} = (v_k - s_k)\, v_k + v_{k+1}^2 + \cdots + v_n^2 = s_k^2 - s_k v_k$$

and

$$\|\mathbf{u}\|^2 = (v_k - s_k)^2 + v_{k+1}^2 + \cdots + v_n^2 = s_k^2 - 2s_k v_k + s_k^2 = 2\left(s_k^2 - s_k v_k\right).$$

Therefore

$$\frac{2\mathbf{u}^T\mathbf{v}}{\|\mathbf{u}\|^2} = \frac{2\left(s_k^2 - s_k v_k\right)}{2\left(s_k^2 - s_k v_k\right)} = 1$$

and

$$Q\mathbf{v} = \mathbf{v} - \mathbf{u}.$$

■

Example 116 *Let*

$$\mathbf{v} = \begin{bmatrix} 1 \\ 12 \\ 3 \\ 4 \end{bmatrix}.$$

a. Let $k = 2$. Then

$$s_2 = -\sqrt{144 + 9 + 16} = -13$$

and

$$\mathbf{u} = \begin{bmatrix} 0 \\ 12 - (-13) \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 25 \\ 3 \\ 4 \end{bmatrix}.$$

Let Q_1 be the Householder matrix associated with $\mathbf{u}$. Then

$$Q_1\mathbf{v} = \mathbf{v} - \mathbf{u} = \begin{bmatrix} 1 \\ -13 \\ 0 \\ 0 \end{bmatrix}.$$

b. Let $k = 3$. Then

$$s_3 = -\sqrt{9 + 16} = -5$$

and

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 3 - (-5) \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 8 \\ 4 \end{bmatrix}.$$

Let Q_2 be the Householder matrix associated with $\mathbf{u}$. Then

$$Q_2\mathbf{v} = \mathbf{v} - \mathbf{u} = \begin{bmatrix} 1 \\ 12 \\ -5 \\ 0 \end{bmatrix}.$$

Let

$$\mathbf{x}=\begin{bmatrix}4\\2\\5\\5\end{bmatrix}.$$

To compute $Q_2\mathbf{x}$, *first evaluate*

$$\mathbf{w}=\left(\frac{2}{\|\mathbf{u}\|^2}\mathbf{u}^T\mathbf{x}\right)\mathbf{u}=\left(\frac{2}{80}\cdot 60\right)\mathbf{u}=\frac{3}{2}\begin{bmatrix}0\\0\\8\\4\end{bmatrix}=\begin{bmatrix}0\\0\\12\\6\end{bmatrix}.$$

Then

$$Q_2\mathbf{x}=\mathbf{x}-\mathbf{w}=\begin{bmatrix}4\\2\\5\\5\end{bmatrix}-\begin{bmatrix}0\\0\\12\\6\end{bmatrix}=\begin{bmatrix}4\\2\\-7\\-1\end{bmatrix}$$

and

$$Q_2\left[\mathbf{v}\mid\mathbf{x}\right]=\begin{bmatrix}1&4\\12&2\\-5&-7\\0&-1\end{bmatrix}.$$

We now have the tools we need to present Householder's Method for orthogonally reducing a matrix A to Hessenberg form.

Algorithm 117 (Householder's Method)

Given

$$A_0=[\mathbf{a}_1\mid\mathbf{a}_2\mid\cdots\mid\mathbf{a}_n]=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&\ddots&\vdots\\a_{n1}&a_{n2}&\cdots&a_{nn}\end{bmatrix}.$$

1. To reduce column 1:

   (a) Compute $s_2=\pm\sqrt{a_{21}^2+a_{31}^2+\cdots+a_{n1}^2}$ and set

   $$\mathbf{u}_1=\begin{bmatrix}0\\a_{21}-s_2\\a_{31}\\\vdots\\a_{n1}\end{bmatrix}.$$

   (b) Let Q_1 be the Householder matrix associated with $\mathbf{u}_1$. For each $i=2,3,\ldots,n$, calculate

   $$\mathbf{b}_i=Q_1\mathbf{a}_i=\mathbf{a}_i-\left(\frac{2}{\|\mathbf{u}_1\|^2}\mathbf{u}_1^T\mathbf{a}_i\right)\mathbf{u}_1;$$

   then

   $$Q_1A_0=\left[\begin{array}{c|c|c|c}a_{11}&&&\\s_2&&&\\0&\mathbf{b}_2&\cdots&\mathbf{b}_n\\\vdots&&&\\0&&&\end{array}\right].$$

(c) Calculate $Q_1 A_0 Q_1$ as follows:

i. Form the matrix
$$C_1 = [\mathbf{c}_1 \mid \mathbf{c}_2 \mid \cdots \mid \mathbf{c}_n] = (Q_1 A_0)^T .$$

ii. For each $i = 1, 2, \ldots, n$, calculate
$$\mathbf{w}_i = \left(\frac{2}{\|\mathbf{u}_1\|^2} \mathbf{u}_1^T \mathbf{c}_i \right) \mathbf{u}_1;$$

then
$$\begin{aligned} Q_1 C_1 &= [Q_1 \mid Q_1 \mathbf{c}_2 \mid \cdots \mid Q_1 \mathbf{c}_n] \\ &= [\mathbf{c}_1 - \mathbf{w}_1 \mid \mathbf{c}_2 - \mathbf{w}_2 \mid \cdots \mid \mathbf{c}_n - \mathbf{w}_n] = Q_1 (Q_1 A_0)^T . \end{aligned}$$

iii. Compute the transpose and obtain
$$(Q_1 C_1)^T = \left[Q_1 (Q_1 A_0)^T \right]^T = \left(Q_1 A_0^T Q_1 \right)^T = Q_1 A_0 Q_1 .$$

(d) To prepare for the next iteration, set
$$A_1 = Q_1 A_0 Q_1 = [\mathbf{a}_1 \mid \mathbf{a}_2 \mid \cdots \mid \mathbf{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ s_2 & a_{22} & \cdots & a_{2n} \\ 0 & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{bmatrix} .$$

2. To reduce column 2.

(a) Compute $s_3 = \pm\sqrt{a_{32}^2 + a_{42}^2 + \cdots + a_{n2}^2}$ and set
$$\mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ a_{32} - s_3 \\ a_{42} \\ \vdots \\ a_{n2} \end{bmatrix} .$$

(b) Let Q_2 be the Householder matrix associated with $\mathbf{u}_2$. For each $i = 3, 4, \ldots, n$, calculate
$$\mathbf{b}_i = Q_2 \mathbf{a}_i = \mathbf{a}_i - \left(\frac{2}{\|u_2\|^2} u_2^T \mathbf{a}_i \right) \mathbf{u}_2;$$

then
$$Q_2 A_1 = \left[\begin{array}{c|c|c|c|c} a_{11} & a_{12} & & & \\ s_2 & a_{22} & & & \\ 0 & s_3 & & & \\ 0 & 0 & \mathbf{b}_3 & \cdots & \mathbf{b}_n \\ \vdots & \vdots & & & \\ 0 & 0 & & & \end{array} \right] .$$

(c) Calculate $Q_2 A_1 Q_2$ as follows:

i. Form the matrix
$$C_2 = [\mathbf{c}_1 \mid \mathbf{c}_2 \mid \cdots \mid \mathbf{c}_n] = (Q_2 A_1)^T .$$

ii. For $i = 1, 2, \ldots, n$, calculate

$$\mathbf{w}_i = \left(\frac{2}{\|\mathbf{u}_2\|^2} \mathbf{u}_2^T \mathbf{c}_i \right) \mathbf{u}_2$$

and obtain

$$\begin{aligned} Q_2 C_2 &= [Q_2 \mathbf{c}_1 \mid Q_2 \mathbf{c}_2 \mid \cdots \mid Q_2 \mathbf{c}_n] \\ &= [\mathbf{c}_1 - \mathbf{w}_1 \mid \mathbf{c}_2 - \mathbf{w}_2 \mid \cdots \mid \mathbf{c}_n - \mathbf{w}_n] = Q_2 \left(Q_2 A_0\right)^T. \end{aligned}$$

iii. Compute the transpose and obtain

$$\left[Q_2 \left(Q_2 A_1\right)^T \right]^T = Q_2 A_1 Q_2.$$

(d) To prepare for the next iteration, set

$$A_2 = Q_2 A_1 Q_2 = [\mathbf{a}_1 \mid \mathbf{a}_2 \mid \cdots \mid \mathbf{a}_n] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ s_2 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & s_3 & a_{33} & \cdots & a_{3n} \\ 0 & 0 & a_{43} & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & a_{n3} & \cdots & a_{nn} \end{bmatrix}.$$

3. Repeat this process until it terminates after $n-2$ steps and produces the Hessenberg matrix

$$H = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n-2} & a_{1,n-1} & a_{1n} \\ s_2 & a_{22} & \cdots & a_{2,n-2} & a_{2,n-1} & a_{2n} \\ 0 & s_3 & \cdots & a_{3,n-2} & a_{3,n-1} & a_{3n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & s_{n-1} & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & \cdots & 0 & a_{n,n-1} & a_{nn} \end{bmatrix} \sim A_0.$$

Example 118 *Let us orthogonally reduce the following matrix to Hessenberg form:*

$$A = [\mathbf{a}_1 \mid \mathbf{a}_2 \mid \mathbf{a}_3 \mid \mathbf{a}_4] = \begin{bmatrix} 1 & 2 & 4 & 2 \\ 3 & 3 & -4 & 2 \\ 0 & 3 & 9 & -1 \\ 0 & -4 & -2 & 8 \end{bmatrix}.$$

To apply Householder's Method and eliminate the -4 in column 2, set $k = 3$ then compute

$$s_3 = -\sqrt{3^3 + (-4)^2} = -5$$

and

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 3 - (-5) \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 8 \\ -4 \end{bmatrix}.$$

Let Q be the Householder matrix associated with $\mathbf{u}$. Evaluate

$$\mathbf{w}_3 = \left(\frac{2}{\|\mathbf{u}\|^2} \mathbf{u}^T \mathbf{a}_3 \right) \mathbf{u} = \left(\frac{2}{80} \cdot 80 \right) \begin{bmatrix} 0 \\ 0 \\ 8 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 16 \\ -8 \end{bmatrix}$$

and

$$\mathbf{w}_4 = \left(\frac{2}{\|\mathbf{u}\|^2} \mathbf{u}^T \mathbf{a}_4 \right) \mathbf{u} = \left(\frac{2}{80} \cdot (-40) \right) \begin{bmatrix} 0 \\ 0 \\ 8 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -8 \\ 4 \end{bmatrix}.$$

Then

$$Q\mathbf{a}_3 = \mathbf{a}_3 - \mathbf{w}_3 = \begin{bmatrix} 4 \\ -4 \\ 9 \\ -2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 16 \\ -8 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -7 \\ 6 \end{bmatrix}$$

and

$$Q\mathbf{a}_4 = \mathbf{a}_4 - \mathbf{w}_3 = \begin{bmatrix} 2 \\ 2 \\ -1 \\ 8 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -8 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 7 \\ 4 \end{bmatrix}$$

so that

$$Q\left[\mathbf{a}_1 \mid \mathbf{a}_2 \mid \mathbf{a}_3 \mid \mathbf{a}_4\right] = \begin{bmatrix} 1 & 2 & 4 & 2 \\ 3 & 3 & -4 & 2 \\ 0 & -5 & -7 & 7 \\ 0 & 0 & 6 & 4 \end{bmatrix}.$$

To calculate QAQ, form the matrix

$$C = \left[\mathbf{c}_1 \mid \mathbf{c}_2 \mid \mathbf{c}_3 \mid \mathbf{c}_4\right] = (QA)^T = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 2 & 3 & -5 & 0 \\ 4 & -4 & -7 & 6 \\ 2 & 2 & 7 & 4 \end{bmatrix}$$

and compute $QC = \left[Q\mathbf{c}_1 \mid Q\mathbf{c}_2 \mid Q\mathbf{c}_3 \mid Q\mathbf{c}_4\right]$:

$$\mathbf{w}_1 = \left(\frac{2}{\|\mathbf{u}\|^2}\mathbf{u}^T\mathbf{c}_1\right)\mathbf{u} = \left(\frac{2}{80}\cdot 24\right)\begin{bmatrix} 0 \\ 0 \\ 8 \\ -4 \end{bmatrix} = \frac{3}{5}\begin{bmatrix} 0 \\ 0 \\ 8 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{24}{5} \\ -\frac{12}{5} \end{bmatrix}$$

$$\mathbf{w}_2 = \left(\frac{2}{\|\mathbf{u}\|^2}\mathbf{u}^T\mathbf{c}_2\right)\mathbf{u} = \left(\frac{2}{80}\cdot (-40)\right)\begin{bmatrix} 0 \\ 0 \\ 8 \\ -4 \end{bmatrix} = -\begin{bmatrix} 0 \\ 0 \\ 8 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -8 \\ 4 \end{bmatrix}$$

$$\mathbf{w}_3 = \left(\frac{2}{\|\mathbf{u}\|^2}\mathbf{u}^T\mathbf{c}_3\right)\mathbf{u} = \left(\frac{2}{80}\cdot (-84)\right)\begin{bmatrix} 0 \\ 0 \\ 8 \\ -4 \end{bmatrix} = -\frac{21}{10}\begin{bmatrix} 0 \\ 0 \\ 8 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{84}{5} \\ \frac{42}{5} \end{bmatrix}$$

$$\mathbf{w}_4 = \left(\frac{2}{\|\mathbf{u}\|^2}\mathbf{u}^T\mathbf{c}_4\right)\mathbf{u} = \left(\frac{2}{80}\cdot 32\right)\begin{bmatrix} 0 \\ 0 \\ 8 \\ -4 \end{bmatrix} = \frac{4}{5}\begin{bmatrix} 0 \\ 0 \\ 8 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{32}{5} \\ -\frac{16}{5} \end{bmatrix}$$

$$\mathbf{c}_1 - \mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \frac{24}{5} \\ -\frac{12}{5} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -\frac{4}{5} \\ \frac{22}{5} \end{bmatrix} \qquad \mathbf{c}_2 - \mathbf{w}_2 = \begin{bmatrix} 3 \\ 3 \\ -4 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -8 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 4 \\ -2 \end{bmatrix}$$

$$\mathbf{c}_3 - \mathbf{w}_3 = \begin{bmatrix} 0 \\ -5 \\ -7 \\ 7 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -\frac{84}{5} \\ \frac{42}{5} \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ \frac{49}{5} \\ -\frac{7}{5} \end{bmatrix} \qquad \mathbf{c}_4 - \mathbf{w}_4 = \begin{bmatrix} 0 \\ 0 \\ 6 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \frac{32}{5} \\ -\frac{16}{5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{2}{5} \\ \frac{36}{5} \end{bmatrix}.$$

Therefore

$$QC = \left[\mathbf{c}_1 - \mathbf{w}_1 \mid \mathbf{c}_2 - \mathbf{w}_2 \mid \mathbf{c}_3 - \mathbf{w}_3 \mid \mathbf{c}_4 - \mathbf{w}_4\right] = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 2 & 3 & -5 & 0 \\ -\frac{4}{5} & 4 & \frac{49}{5} & -\frac{2}{5} \\ \frac{22}{5} & -2 & -\frac{7}{5} & \frac{36}{5} \end{bmatrix}$$

and

$$H = QAQ = (QC)^T = \begin{bmatrix} 1 & 2 & -\frac{4}{5} & \frac{22}{5} \\ 3 & 3 & 4 & -2 \\ 0 & -5 & \frac{49}{5} & -\frac{7}{5} \\ 0 & 0 & -\frac{2}{5} & \frac{36}{5} \end{bmatrix}.$$

11-1-12

Two Applications of Householder Transformations: Least Squares Solutions and Approximate Eigenvalues

First, we revisit the problem of finding the least squares solution of an inconsistent system of linear equations. Consider an $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$ with $m \geq n$, and recall that a least squares solution $\mathbf{x}^*$ minimizes the quantity $\|\mathbf{b} - A\mathbf{x}\|$ as $\mathbf{x}$ ranges over $\mathbb{R}^n$; thus

$$\|\mathbf{b} - A\mathbf{x}^*\| \leq \|\mathbf{b} - A\mathbf{x}\| \tag{14}$$

for all $\mathbf{x} \in \mathbb{R}^n$. The vector $\mathbf{b} - A\mathbf{x}^*$ is the component of $\mathbf{b}$ orthogonal to the column space $R(A)$, so of course, if $\mathbf{b} \in R(A)$, the component of $\mathbf{b}$ orthogonal to $R(A)$ is $\mathbf{0}$ and $A\mathbf{x}^* = \mathbf{b}$.

Let Q be an orthogonal matrix, and recall that multiplication by Q preserves the Euclidean norm: $\|Q\mathbf{x}\| = \|\mathbf{x}\|$. Hence inequality (14) implies

$$\|Q\mathbf{b} - QA\mathbf{x}^*\| = \|Q(\mathbf{b} - A\mathbf{x}^*)\| = \|\mathbf{b} - A\mathbf{x}^*\| \leq \|\mathbf{b} - A\mathbf{x}\| = \|Q(\mathbf{b} - A\mathbf{x})\| = \|Q\mathbf{b} - QA\mathbf{x}\|$$

for all $\mathbf{x} \in \mathbb{R}^n$. Conversely, if $\|Q\mathbf{b} - QA\mathbf{x}^*\| \leq \|Q\mathbf{b} - QA\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$, then

$$\|\mathbf{b} - A\mathbf{x}^*\| = \|Q\mathbf{b} - QA\mathbf{x}^*\| \leq \|Q\mathbf{b} - QA\mathbf{x}\| = \|\mathbf{b} - A\mathbf{x}\|,$$

which proves:

Proposition 119 *Let $A\mathbf{x} = \mathbf{b}$ be an $m \times n$ linear system with $m \geq n$ and let Q be an orthogonal matrix. Then $\mathbf{x}^*$ is a least squares solution of $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{x}^*$ is a least squares solution of $QA\mathbf{x} = Q\mathbf{b}$.*

In particular, Proposition 119 applies when Q is a product of Householder matrices and QA has the following especially nice form:

Definition 120 *An $m \times n$ matrix with $m \geq n$ has **upper trapezoidal form** if the first n rows form an $n \times n$ upper triangular matrix and the remaining rows are all zero:*

$$\begin{bmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem 121 *If A is an $m \times n$ matrix with $m \geq n$, there is an $m \times m$ orthogonal matrix Q such that QA has upper trapezoidal form.*

Proof. Given as $m \times n$ matrix $A = (a_{ij})$ with $m \geq n$, let

$$s_1 = \pm\sqrt{a_{11}^2 + \cdots + a_{m1}^2}, \quad \mathbf{u}_1 = \begin{bmatrix} a_{11} - s_1 \\ a_{12} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \text{and} \quad Q_1 = I - \frac{2}{\|\mathbf{u}_1\|^2}\mathbf{u}_1\mathbf{u}_1^T.$$

Then

$$Q_1 A = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ 0 & b_{23} & \cdots & b_{3n} \\ \vdots & \vdots & & \vdots \\ 0 & b_{mn} & \cdots & b_{nn} \end{bmatrix}.$$

Let

$$s_2 = \pm\sqrt{b_{22}^2 + \cdots + b_{m2}^2}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ b_{22} - s_2 \\ b_{3,2} \\ \vdots \\ b_{m,2} \end{bmatrix} \quad \text{and} \quad Q_2 = I - \frac{2}{\|\mathbf{u}_2\|^2}\mathbf{u}_2\mathbf{u}_2^T.$$

Then

$$Q_2Q_1A = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ 0 & c_{22} & c_{23} & \cdots & c_{2n} \\ 0 & 0 & c_{24} & \cdots & c_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & c_{2m} & \cdots & c_{nn} \end{bmatrix}.$$

Continuing this process until it terminates after n steps and produces the $m \times n$ matrix $Q_n \cdots Q_1 A$ with upper trapezoidal form. But $Q = Q_n \cdots Q_1$ is an orthogonal matrix, and QA has upper trapezoidal form. ■

Example 122 *Let us apply the procedure in the proof of Theorem 121 to the matrix*

$$A = [\mathbf{a}_1 \mid \mathbf{a}_2] = \begin{bmatrix} 1 & -\frac{2}{3} \\ -1 & 3 \\ 0 & -2 \\ -1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Compute

$$s_1 = -\sqrt{1+1+1+1} = -2 \quad \text{and} \quad \mathbf{u}_1 = \begin{bmatrix} 1-(-2) \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Then $\|\mathbf{u}_1\|^2 = 12$ *and*

$$Q_1 = I - \frac{1}{6}\mathbf{u}_1\mathbf{u}_1^T.$$

Since $\mathbf{u}_1^T\mathbf{a}_1 = 6$ *and* $\mathbf{u}_1^T\mathbf{a}_2 = -6$ *we have*

$$\mathbf{w}_1 = \left(\frac{1}{6}\mathbf{u}_1^T\mathbf{a}_1\right)\mathbf{u}_1 = \mathbf{u}_1 \quad \text{and} \quad \mathbf{w}_2 = \left(\frac{1}{6}\mathbf{u}_1^T\mathbf{a}_2\right)\mathbf{u}_1 = -\mathbf{u}_1$$

so that

$$Q_1A = [Q_1\mathbf{a}_1 \mid Q_1\mathbf{a}_2] = [\mathbf{a}_1 - \mathbf{u}_1 \mid \mathbf{a}_2 + \mathbf{u}_1] = \begin{bmatrix} -2 & \frac{7}{3} \\ 0 & 2 \\ 0 & -2 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let $Q_1A = [\mathbf{c}_1 \mid \mathbf{c}_2]$. *Compute*

$$s_2 = -\sqrt{4+4+1} = -3 \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 2-(-3) \\ -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

Then $\|\mathbf{u}_2\|^2 = 30$ *and*

$$S_2 = I - \frac{1}{15}\mathbf{u}_2\mathbf{u}_2^T.$$

Since $\mathbf{u}_2^T\mathbf{c}_2 = 15$ *we have*

$$\mathbf{w}_2 = \left(\frac{1}{15}\mathbf{u}_2^T\mathbf{c}_2\right)\mathbf{u}_2 = \mathbf{u}_2$$

and

$$Q_2Q_1A = [Q_2\mathbf{c}_1 \mid Q_2\mathbf{c}_2] = [\mathbf{c}_1 \mid \mathbf{c}_2 - \mathbf{u}_2] = \begin{bmatrix} -2 & \frac{7}{3} \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $Q = Q_2Q_1$ *is an orthogonal matrix and* QA *has upper trapezoidal form.*

Continuing our discussion of the $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$ with $m \geq n$, let Q be an orthogonal matrix such that QA has upper trapezoidal form

$$QA = \begin{bmatrix} T \\ \mathbf{0} \end{bmatrix},$$

where T is an $n \times n$ upper triangular matrix and $\mathbf{0}$ is the $(m-n) \times n$ zero matrix. Then

$$QA\mathbf{x} = Q\mathbf{b}$$

can be expressed in block matrix form as

$$\begin{bmatrix} T \\ \mathbf{0} \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix},$$

where $\mathbf{c}$ is an $n \times 1$ and $\mathbf{d}$ is an $(m-n) \times 1$. Thus

$$QA\mathbf{x} - Q\mathbf{b} = \begin{bmatrix} T \\ \mathbf{0} \end{bmatrix} \mathbf{x} - \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix} = \begin{bmatrix} T\mathbf{x} - \mathbf{c} \\ -\mathbf{d} \end{bmatrix}$$

and

$$\|QA\mathbf{x} - Q\mathbf{b}\|^2 = \begin{bmatrix} T\mathbf{x} - \mathbf{c} & -\mathbf{d} \end{bmatrix} \begin{bmatrix} T\mathbf{x} - \mathbf{c} \\ -\mathbf{d} \end{bmatrix} = \|T\mathbf{x} - \mathbf{c}\|^2 + \|\mathbf{d}\|^2 \geq \|\mathbf{d}\|^2.$$

Thus $\mathbf{x}^*$ minimizes $\|QA\mathbf{x} - Q\mathbf{b}\|$ if and only if $\mathbf{x}^*$ minimizes $\|T\mathbf{x} - \mathbf{c}\|$. Combining this with the statement in Proposition 119 we have:

Proposition 123 *Let $A\mathbf{x} = \mathbf{b}$ be an $m \times n$ linear system with $m \geq n$. Let Q be an orthogonal matrix such that QA has upper trapezoidal form, and express $QA\mathbf{x} = Q\mathbf{b}$ in block matrix form as*

$$\begin{bmatrix} T \\ \mathbf{0} \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix},$$

where T is an $n \times n$ upper triangular matrix and $\mathbf{c}$ is a $n \times 1$ matrix. Then $\mathbf{x}^*$ is a least squares solution of $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{x}^*$ is a least squares solution of $T\mathbf{x} = \mathbf{c}$.

Theorem 124 *Let A be an $m \times n$ matrix with $m \geq n$, let $\mathbf{b} \in \mathbb{R}^m$, and let Q be an $n \times n$ orthogonal matrix such that*

$$QA = \begin{bmatrix} T \\ \mathbf{0} \end{bmatrix} \quad \text{and} \quad Q\mathbf{b} = \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix},$$

where T is an $n \times n$ upper triangular matrix and $\mathbf{c}$ is a $n \times 1$ matrix. If $\operatorname{rank}(A) = n$, then T is invertible and $\mathbf{x}^* = T^{-1}\mathbf{c}$ is the unique

1. *solution of $T\mathbf{x} = \mathbf{c}$.*
2. *least squares solution of $A\mathbf{x} = \mathbf{b}$.*

Proof. (1) Let $A = [\mathbf{a}_1 \mid \cdots \mid \mathbf{a}_n]$. Since $\operatorname{rank}(A) = n$, the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is linearly independent. If $\{Q\mathbf{a}_1, \ldots, Q\mathbf{a}_n\}$ is linearly independent, then

$$QA = [Q\mathbf{a}_1 \mid \cdots \mid Q\mathbf{a}_n] = \begin{bmatrix} T \\ \mathbf{0} \end{bmatrix}$$

has linearly independent columns and so does T, in which case T is invertible and the system $T\mathbf{x} = \mathbf{c}$ has the unique solution $\mathbf{x}^* = T^{-1}\mathbf{c}$. So to complete the proof, assume that $b_1 Q\mathbf{a}_1 + \cdots + b_n Q\mathbf{a}_n = \mathbf{0}$, where $b_i \in \mathbb{R}$, then $Q(b_1\mathbf{a}_1 + \cdots + b_n\mathbf{a}_n) = \mathbf{0}$. Multiplying both sides on the left by the orthogonal matrix Q^T gives $b_1\mathbf{a}_1 + \cdots + b_n\mathbf{a}_n = Q^T\mathbf{0} = \mathbf{0}$. But linear independence of $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ implies $b_i = 0$ for all i. Therefore $\{Q\mathbf{a}_1, \ldots, Q\mathbf{a}_n\}$ is linearly independent.

(2) The fact that $\mathbf{x}^*$ is the unique solution of $T\mathbf{x} = \mathbf{c}$ implies $\|T\mathbf{x}^* - \mathbf{c}\| = 0$ and $\|T\mathbf{x} - \mathbf{c}\| > 0$ for all $\mathbf{x} \neq \mathbf{x}^*$. Therefore $\mathbf{x}^*$ is the *unique least squares solution* of $T\mathbf{x} = \mathbf{c}$. But $\mathbf{x}^*$ is a least squares solution of $T\mathbf{x} = \mathbf{c}$ if and only if $\mathbf{x}^*$ is a least squares solution of $A\mathbf{x} = \mathbf{b}$, by Proposition 123. Consequently, $\mathbf{x}^*$ is the unique least squares solution of $A\mathbf{x} = \mathbf{b}$ (distinct least squares solutions $\mathbf{x}'$ and $\mathbf{x}^*$ of $A\mathbf{x} = \mathbf{b}$ would be least squares solutions of $T\mathbf{x} = \mathbf{c}$). ■

Example 125 *Find the equation of the regression line $y = a + bx$ that best fits the four points $(0,1)$, $(2,0)$, $(3,1)$, and $(3,2)$.*

The y-intercept a and slope b of the regression line are the components of the least squares solution $\mathbf{x}^* = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ of the inconsistent system $A\mathbf{x} = \mathbf{b}$, where

$$A = [\mathbf{a}_1 \mid \mathbf{a}_2] = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

Eliminate the second through fourth entries in the first column:

$$s_1 = -\sqrt{4} = -2;\ \mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix};\ \|\mathbf{u}_1\|^2 = 12;\ \mathbf{u}_1^T\mathbf{a}_2 = 8;\ Q_1\mathbf{a}_2 = \mathbf{a}_2 - \frac{1}{6}(8)\,\mathbf{u}_1 = \begin{bmatrix} -4 \\ 2/3 \\ 5/3 \\ 5/3 \end{bmatrix};\ \mathbf{u}_1^T\mathbf{b} = \mathbf{6};$$

then

$$Q_1A = \begin{bmatrix} -2 & -4 \\ 0 & 2/3 \\ 0 & 5/3 \\ 0 & 5/3 \end{bmatrix} \quad \text{and} \quad Q_1\mathbf{b} = \mathbf{b} - \mathbf{u}_1 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Eliminate the third and fourth entries in the second column:

$$s_2 = -\sqrt{\frac{4}{9} + \frac{25}{9} + \frac{25}{9}} = -\sqrt{6};\quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ \frac{2}{3} + \sqrt{6} \\ 5/3 \\ 5/3 \end{bmatrix};\quad \|\mathbf{u}_2\|^2 = 12 + \frac{4}{3}\sqrt{6};\quad \mathbf{u}_2^T(Q_1\mathbf{b}) = 1 - \sqrt{6};$$

then

$$Q_2(Q_1A) = \begin{bmatrix} -2 & -4 \\ 0 & -\sqrt{6} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Q_2(Q_1\mathbf{b}) = \mathbf{b} - \left(\frac{3 - 2\sqrt{6}}{10}\right)\mathbf{u}_2 = \begin{bmatrix} -2 \\ -\frac{1}{6}\sqrt{6} \\ -1 + \frac{2}{3}\sqrt{6} \\ \frac{2}{3}\sqrt{6} \end{bmatrix}.$$

Let $Q = Q_2Q_1$ and write $QA\mathbf{x} = Q\mathbf{b}$ in block matrix form as $\begin{bmatrix} T \\ \mathbf{0} \end{bmatrix}\mathbf{x} = \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix}$.

Then $\mathbf{x}^*$ is the unique solution of $T\mathbf{x} = \mathbf{c}$: $\left[\begin{array}{cc|c} -2 & -4 & -2 \\ 0 & -\sqrt{6} & -\frac{1}{6}\sqrt{6} \end{array}\right] \xrightarrow{\text{Row reduce}} \left[\begin{array}{cc|c} 1 & 0 & 2/3 \\ 0 & 1 & 1/6 \end{array}\right].$

The equation of the regression line is $y = \frac{2}{3} + \frac{1}{6}x$.

Definition 126 *Let A be an $m \times n$ matrix of rank n. A QR-**factorization of** A is a factorization $A = QR$ with the following properties:*

1. *Q is an $m \times n$ matrix whose columns form an orthonormal set in $\mathbb{R}^m$.*
2. *R is an $n \times n$ nonsingular upper-triangular matrix.*
3. *$RQ \sim A$.*

Given an $m \times n$ matrix with rank n and $m \geq n$, we apply Householder transformations to find a QR-factorization of A use the fact that $RQ \sim A$ to numerically approximate the eigenvalues of A (the *QR-algorithm*).

Theorem 127 *Every $m \times n$ matrix A with $m \geq n$ and rank n has a QR-factorization.*

Proof. By Theorem 121, there is an orthogonal $m \times m$ matrix S such that

$$SA = \begin{bmatrix} T \\ \mathbf{0} \end{bmatrix}, \tag{15}$$

where T is an $n \times n$ upper triangular matrix. Furthermore, since $\text{rank}(A) = n$, we know that T is nonsingular by Theorem 124. Now multiply both sides of equation (15) by the orthogonal matrix $S^T = [\mathbf{s}_1 \mid \cdots \mid \mathbf{s}_m]$, whose columns form an orthonormal basis for $\mathbb{R}^m$. Then

$$A = [\mathbf{s}_1 \mid \cdots \mid \mathbf{s}_n \mid \mathbf{s}_{n+1} \mid \cdots \mid \mathbf{s}_m] \begin{bmatrix} T \\ \mathbf{0} \end{bmatrix}. \tag{16}$$

Now when computing this matrix product, note that each entry of $\mathbf{s}_{n+1}, \ldots, \mathbf{s}_m$ is multiplied by zero and contributes nothing. Hence

$$A = [\mathbf{s}_1 \mid \cdots \mid \mathbf{s}_n] T$$

is a factorization of A different from (16), but equally valid. Set $Q = [\mathbf{s}_1 \mid \cdots \mid \mathbf{s}_n]$ and $R = T$, then $A = QR$. Finally, note that $R = Q^T A$ and $RQ = Q^T AQ$. Therefore $RQ \sim A$ and $A = QR$ is a QR-factorization. ■

Example 128 *Let's compute the QR-factorization of*

$$A = [\mathbf{a}_1 \mid \mathbf{a}_2 \mid \mathbf{a}_3] = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & -1 \\ 4 & 8 & 6 \end{bmatrix}.$$

First,

$$s_1 = -\sqrt{9+16} = -5 \quad \textit{and} \quad \mathbf{u}_1 = \begin{bmatrix} 3-(-5) \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 4 \end{bmatrix}.$$

Since $\|\mathbf{u}_1\|^2 = 80$ and $\mathbf{u}_1^T \mathbf{a}_2 = \mathbf{u}_1^T \mathbf{a}_3 = 40$, the Householder matrix associated with $\mathbf{u}_1$ is

$$S_1 = I - \frac{1}{40} \mathbf{u}_1 \mathbf{u}_1^T$$

and

$$S_1 \mathbf{a}_2 = \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix} - \begin{bmatrix} 8 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \\ 4 \end{bmatrix}$$

$$S_1 \mathbf{a}_3 = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -6 \\ -1 \\ 2 \end{bmatrix}.$$

Thus

$$S_1 A = [\mathbf{b}_1 \mid \mathbf{b}_2 \mid \mathbf{b}_3] = \begin{bmatrix} -5 & -7 & -6 \\ 0 & 3 & -1 \\ 0 & 4 & 2 \end{bmatrix}.$$

Second,

$$s_2 = -\sqrt{9+16} = -5 \quad \textit{and} \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 3-(-5) \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix}.$$

Since $\|\mathbf{u}_2\|^2 = 80$ and $\mathbf{u}_2^T \mathbf{b}_3 = 0$, the Householder matrix associated with $\mathbf{u}_2$ is

$$S_2 = I - \frac{1}{40} \mathbf{u}_2 \mathbf{u}_2^T$$

and

$$S_2\mathbf{b}_3 = \begin{bmatrix} -6 \\ -1 \\ 2 \end{bmatrix}.$$

Thus

$$R = S_2S_1A = \begin{bmatrix} -5 & -7 & -6 \\ 0 & -5 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Set $Q = S_2S_1$ and obtain the QR-factorization

$$A = QR.$$

If A is an $n \times n$ matrix and $A = QR$ is a QR-factorization, we can use the fact that $RQ \sim A$ to approximate the eigenvalues of A in the following way:

The QR-Algorithm.

Choose $\varepsilon > 0$
Let $A_1 := A$
Let $k := 0$
Repeat
  Let $k := k + 1$
    Compute the QR-factorization $A_k = Q_kR_k$
  Let $A_{k+1} := R_kQ_k$
Until *sub-diagonal entries of* A_{k+1} *have absolute value less than* ε.

Example 129 *Let's use the QR-Algorithm to estimate the eigenvalues of*

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & -1 \\ 4 & 8 & 5 \end{bmatrix}.$$

The QR-factorization found in Example 128 is $A = QR$, where

$$Q = \left(I - \frac{1}{40}\mathbf{u}_1\mathbf{u}_1^T\right)\left(I - \frac{1}{40}\mathbf{u}_2\mathbf{u}_2^T\right) = I - \frac{1}{40}\mathbf{u}_2\mathbf{u}_2^T - \frac{1}{40}\mathbf{u}_1\mathbf{u}_1^T + \frac{1}{100}\mathbf{u}_1\mathbf{u}_2^T$$

$$= I - \frac{1}{40}\mathbf{u}_1\mathbf{u}_1^T + \frac{1}{20}\left(\frac{1}{5}\mathbf{u}_1 - \frac{1}{2}\mathbf{u}_2\right)\mathbf{u}_2^T = I - \frac{1}{10}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\mathbf{u}_1^T + \frac{1}{50}\begin{bmatrix} 4 \\ -10 \\ -3 \end{bmatrix}\mathbf{u}_2^T$$

and

$$R = \begin{bmatrix} -5 & -7 & -6 \\ 0 & -5 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Following the QR-Algorithm, set $A_1 = A$, $Q_1 = Q$, and $R_1 = R$. Then

$$A_2 = R_1Q_1 = \left(Q_1^TR_1^T\right)^T$$

Let $R_1^T = [\mathbf{c}_1 \mid \mathbf{c}_2 \mid \mathbf{c}_3]$ *and compute* $Q_1^T R_1^T$:

$$Q_1^T = \left(I - \frac{1}{10} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \mathbf{u}_1^T + \frac{1}{50} \begin{bmatrix} 4 \\ -10 \\ -3 \end{bmatrix} \mathbf{u}_2^T \right)^T$$

$$= I - \frac{1}{10} \begin{bmatrix} 8 \\ 0 \\ 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} + \frac{1}{50} \begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix} \begin{bmatrix} 4 & -10 & -3 \end{bmatrix}$$

$$= I - \frac{2}{5} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} + \frac{2}{25} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 4 & -10 & -3 \end{bmatrix}$$

and

$$Q^T \mathbf{c}_1 = \begin{bmatrix} -5 \\ -7 \\ -6 \end{bmatrix} + \frac{32}{5} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \frac{136}{25} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{39}{5} \\ \frac{97}{25} \\ \frac{146}{25} \end{bmatrix} = \begin{bmatrix} 7.8 \\ 3.88 \\ 5.84 \end{bmatrix}$$

$$Q^T \mathbf{c}_2 = \begin{bmatrix} 0 \\ -5 \\ -1 \end{bmatrix} + \frac{2}{5} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \frac{106}{25} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{87}{25} \\ \frac{91}{25} \end{bmatrix} = \begin{bmatrix} 0.8 \\ 3.48 \\ 3.64 \end{bmatrix}$$

$$Q^T \mathbf{c}_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} - \frac{4}{5} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{12}{25} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{8}{5} \\ -\frac{24}{25} \\ \frac{18}{25} \end{bmatrix} = \begin{bmatrix} -1.6 \\ -0.96 \\ 0.72 \end{bmatrix}.$$

Therefore

$$A_2 = \begin{bmatrix} 7.8 & 3.88 & 5.84 \\ 0.8 & 3.48 & 3.64 \\ -1.6 & -0.96 & 0.72 \end{bmatrix}.$$

Exercise 130 *Continue the calculations in Example 129.*

a. *Perform the second iteration of the QR-Algorithm and produce* A_3.

b. *Write a Mathematica routine to calculate* A_n *and use it to compute* A_{10}.

11-3-2012

Matrix Polynomials and the Cayley-Hamilton Theorem

The goal of this lecture is to prove the famous Cayley-Hamilton Theorem (Theorem 140), which asserts that if $p_A(t)$ is the characteristic polynomial of an $n \times n$ matrix A, then $p_A(A) = \mathbf{0}$. In the course of the argument, we prove that Krylov's Method produces the characteristic polynomial $p_H(t)$ of a Hessenberg matrix H (Corollary 137).

Definition 131 *Let $p(t) = c_0 + c_1 t + \cdots + c_n t^n$ and let A be an $n \times n$ matrix. The **matrix polynomial** $p(A)$ is defined*

$$p(A) := c_0 I + c_1 A + \cdots + c_n A^n.$$

Question: How are the eigenvalues of $p(A)$ related to the eigenvalues of A?

Exercise 132 *Let $q(t) = c_0 + c_1 t + \cdots + c_k t^k$ and let A be an $n \times n$ matrix such that $A\mathbf{x} = \lambda \mathbf{x}$, for some $\mathbf{x} \neq \mathbf{0}$. Prove that $q(A)\mathbf{x} = q(\lambda)\mathbf{x}$.*

In particular, if $p_A(t)$ is the characteristic polynomial of A and λ is an eigenvalue of A, then $p_A(\lambda) = 0$ is an eigenvalue of $p_A(A)$, by Exercise 132. But $p_A(A)\mathbf{x} = \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$ implies that $p_A(A)$ is a singular matrix (otherwise $\mathbf{x} = p_A(A)^{-1}\mathbf{0} = \mathbf{0}$, which is a contradiction). In fact, $p_A(A) = \mathbf{0}$ as we shall see.

As a first step towards a proof of the Cayley-Hamilton Theorem, let H be an unreduced $n \times n$ Hessenberg matrix, and apply Krylov's Method to obtain the polynomial

$$p(t) = a_0 + a_1 t + \cdots + a_{n-1} t^{n-1} + t^n$$

whose coefficients $a_0, \ldots, a_{n-1}$ are the unique solution of the linear system

$$a_0 \mathbf{w}_0 + a_1 \mathbf{w}_1 + \cdots + a_{n-1} \mathbf{w}_{n-1} = -\mathbf{w}_n. \tag{17}$$

Since $\mathbf{w}_i = H^i \mathbf{e}_1$ for $i = 0, 1, \ldots, n$, we can rewrite equation (17) in the form

$$a_0 \mathbf{e}_1 + a_1 H \mathbf{e}_1 + \cdots + a_{n-1} H^{n-1} \mathbf{e}_1 + H^n \mathbf{e}_1 = \mathbf{0}$$

and conclude that $p(H)\mathbf{e}_1 = 0$.

Definition 133 *A **monic** polynomial has the form $q(t) = a_0 + a_1 t + \cdots + a_{n-1} t^{n-1} + t^n$.*

Note that the polynomial $p(t)$ produced by Krylov's Method is monic.

Proposition 134 *The characteristic polynomial $p_A(t)$ of an $n \times n$ matrix A is monic.*

Proof. Let A be an $n \times n$ matrix and let $B = (b_{ij}) = tI - A$. The characteristic polynomial

$$p_A(t) = \det B = \det \begin{bmatrix} t - a_{11} & \cdots & -a_{1n} \\ \vdots & \ddots & \vdots \\ -a_{n1} & \cdots & t - a_{nn} \end{bmatrix}$$

is a sum of $n!$ products $\pm b_{1,j_1} b_{2,j_2} \cdots b_{n,j_n}$, where $(j_1, \ldots j_n)$ ranges over all permutations of $\{1, 2, \ldots, n\}$. Note that each of these products is a polynomial in t, exactly one of which has order n, namely, the monic polynomial $(t - a_{11}) \cdots (t - a_{nn})$. Therefore $p_A(t)$ is monic. ■

Theorem 135 *Let H be an $n \times n$ unreduced Hessenberg matrix, let $p(t) = a_0 + a_1 t + \cdots + a_{n-1} t^{n-1} + t^n$ be the polynomial produced by Krylov's Method, and let $q(t)$ be any monic polynomial. Then*

1. $p(H) = 0$.
2. *If* $q(H) = 0$, *then* $\deg q(t) \geq n$. Moreover, *if* $\deg q(t) = n$, *then* $q(t) = p(t)$.

Proof. (1) By Proposition 100, $\{\mathbf{e}_1, H\mathbf{e}_1, \ldots, H^{n-1}\mathbf{e}_1\}$ is a linearly independent subset of $\mathbb{R}^n$ and is therefore a basis. Given $\mathbf{y} \in \mathbb{R}^n$, write

$$\begin{aligned}\mathbf{y} &= c_0\mathbf{e}_1 + c_1 H\mathbf{e}_1 + \cdots + c_{n-1}H^{n-1}\mathbf{e}_1 \\ &= \left(c_0 I + c_1 H + \cdots + c_{n-1}H^{n-1}\right)\mathbf{e}_1.\end{aligned}$$

Then multiplying both sides by $p(H)$ gives

$$\begin{aligned}p(H)\mathbf{y} &= p(H)\left(c_0 I + c_1 H + \cdots + c_{n-1}H^{n-1}\right)\mathbf{e}_1 \\ &= \left(c_0 p(H) + c_1 p(H) H + \cdots + c_{n-1}p(H) H^{n-1}\right)\mathbf{e}_1 \\ &= c_0 p(H)\mathbf{e}_1 + c_1 p(H) H\mathbf{e}_1 + \cdots + c_{n-1}p(H) H^{n-1}\mathbf{e}_1.\end{aligned}$$

Note that $p(H) H^i = H^i p(H)$ for $1 \leq i \leq n:$

$$\begin{aligned}p(H) H^i &= \left(a_0 I + a_1 H + \cdots + a_{n-1}H^{n-1} + H^n\right) H^i \\ &= a_0 H^i + a_1 H^{i+1} + \cdots + a_{n-1}H^{i+n-1} + H^{i+n} \\ &= H^i\left(a_0 I + a_1 H + \cdots + a_{n-1}H^{n-1} + H^n\right) = H^i p(H).\end{aligned}$$

Hence for all $\mathbf{y} \in \mathbb{R}^n$ we have

$$\begin{aligned}p(H)\mathbf{y} &= c_0 p(H)\mathbf{e}_1 + c_1 H p(H)\mathbf{e}_1 + \cdots + c_{n-1}H^{n-1}p(H)\mathbf{e}_1 \\ &= \left(c_0 I + c_1 H + \cdots + c_{n-1}H^{n-1}\right) p(H)\mathbf{e}_1 = \mathbf{0}.\end{aligned}$$

In particular, $p(H)\mathbf{e}_i = \mathbf{0}$ for all i. Therefore $p(H) = 0$.
(2) Let $q(t) = b_0 + b_1 t + \cdots + b_{n-1}t^{k-1} + t^k$ be a monic polynomial such that $q(H) = 0$. Then $q(H)\mathbf{y} = \mathbf{0} \cdot \mathbf{y} = \mathbf{0}$ for all $\mathbf{y} \in \mathbb{R}^n$, and in particular we have

$$\mathbf{0} = q(H)\mathbf{e}_1 = b_0\mathbf{e}_1 + b_1 H\mathbf{e}_1 + \cdots + b_{k-1}H^{k-1}\mathbf{e}_1 + H^k\mathbf{e}_1$$

so that

$$-H^k\mathbf{e}_1 = b_0\mathbf{e}_1 + b_1 H\mathbf{e}_1 + \cdots + b_{k-1}H^{k-1}\mathbf{e}_1$$

and the set $\{\mathbf{e}_1, H\mathbf{e}_1, \ldots, H^k\mathbf{e}_1\}$ is linearly dependent. By Proposition 100, the set $\{\mathbf{e}_1, H\mathbf{e}_1, \ldots, H^i\mathbf{e}_1\}$ is linearly independent if $i \leq n-1$. Therefore $k \geq n$. Furthermore, if $k = n$, the set $\{\mathbf{e}_1, H\mathbf{e}_1, \ldots, H^{n-1}\mathbf{e}_1\}$ is a basis for $\mathbb{R}^n$ and the coefficients $b_0, b_1, \ldots, b_{n-1}$ uniquely determine the vector $-H^n\mathbf{e}_1$. But

$$p(H)\mathbf{e}_1 = a_0\mathbf{e}_1 + a_1 H\mathbf{e}_1 + \cdots + a_{n-1}H^{n-1}\mathbf{e}_1 + H^n\mathbf{e}_1 = \mathbf{0}$$

by part (1), and we have

$$\begin{aligned}-H^n\mathbf{e}_1 &= a_0\mathbf{e}_1 + a_1 H\mathbf{e}_1 + \cdots + a_{n-1}H^{n-1}\mathbf{e}_1 \\ &= b_0\mathbf{e}_1 + b_1 H\mathbf{e}_1 + \cdots + b_{n-1}H^{n-1}\mathbf{e}_1.\end{aligned}$$

Therefore $b_i = a_i$ for all i, and $q(t) = p(t)$. ■

Thus the polynomial $p(t)$ produced by Krylov's Method is the unique monic polynomial of degree n such that $p(H) = 0$.

Theorem 136 *Let H be an $n \times n$ unreduced Hessenberg matrix and let $p(t)$ be the monic polynomial produced by Krylov's Method. Then λ is an eigenvalue of H if and only if $p(\lambda) = 0$.*

Proof. Let λ is an eigenvalue of H and let $\mathbf{x}$ be a corresponding eigenvector; then $H\mathbf{x} = \lambda\mathbf{x}$. Exercise (132) and Theorem 135 imply

$$p(\lambda)\mathbf{x} = p(H)\mathbf{x} = \mathbf{0}.$$

Therefore $p(\lambda) = 0$. Conversely, if $p(\lambda) = 0$, dividing $p(t)$ by $\lambda - t$ produces a monic polynomial $q(t)$ of degree $n-1$ such that

$$p(t) = q(t)(\lambda - t).$$

Thus

$$p(H) = q(H)(\lambda I - H) = \mathbf{0}.$$

But $\lambda I - H$ is a singular matrix (otherwise $q(H) = (\lambda I - H)^{-1}\mathbf{0} = \mathbf{0}$ contradicts the assertion in Theorem 135, part (2), that $\deg q(t) \geq n$.) Therefore $0 = \det(\lambda I - H) = p_H(\lambda)$ and λ is an eigenvalue of H. ■

Corollary 137 *If H is an unreduced Hessenberg matrix, the monic polynomial produced by Krylov's Method is the characteristic polynomial of H.*

Proof. Since the characteristic polynomial $p_H(t)$ and the polynomial $p(t)$ produced by Krylov's Method are monic, and have exactly the same roots by Theorem 136, $p_H(t) = p(t)$. ■

Theorem 138 *If H is a Hessenberg matrix, then $p_H(H) = \mathbf{0}$.*

Proof. If H is unreduced, the conclusion follows from Corollary 137 and Theorem 135. If H is reduced, assume that H contains 2 unreduced Hessenberg blocks; the proof when H contains n unreduced blocks follows by induction and is left to the reader. Let

$$H = \begin{bmatrix} H_1 & V_1 \\ \mathbf{0} & H_2 \end{bmatrix},$$

where H_1 and H_2 are unreduced Hessenberg blocks. Then

$$p_H(t) = \det(tI - H) = \det(tI - H_1)\det(tI - H_2) = p_{H_1}(t)\,p_{H_2}(t)$$

by Theorem 105. Furthermore,

$$H^2 = \begin{bmatrix} H_1 & V_1 \\ \mathbf{0} & H_2 \end{bmatrix}^2 = \begin{bmatrix} H_1^2 & V_2 \\ \mathbf{0} & H_2^2 \end{bmatrix}$$

for some V_2 and inductively,

$$H^k = \begin{bmatrix} H_1^k & V_k \\ \mathbf{0} & H_2^k \end{bmatrix}$$

for some V_k. Therefore if $q(t) = c_0 + c_1 t + \cdots + c_k t^k$ is any polynomial, then

$$q(H) = c_0 I + c_1 H + \cdots + c_k H^k = c_0 \begin{bmatrix} I_1 & \mathbf{0} \\ \mathbf{0} & I_2 \end{bmatrix} + c_1 \begin{bmatrix} H_1 & V_1 \\ \mathbf{0} & H_2 \end{bmatrix} + \cdots + c_k \begin{bmatrix} H_1^k & V_k \\ \mathbf{0} & H_2^k \end{bmatrix}$$

$$= \begin{bmatrix} c_0 I_1 + c_1 H_1 + \cdots + c_k H_1^k & W \\ \mathbf{0} & c_0 I_2 + c_1 H_2 + \cdots + c_k H_2^k \end{bmatrix} = \begin{bmatrix} q(H_1) & W \\ \mathbf{0} & q(H_2) \end{bmatrix}.$$

In particular,

$$p_H(H) = p_{H_1}(H)\,p_{H_2}(H) = \begin{bmatrix} p_1(H_1) & R \\ \mathbf{0} & p_1(H_2) \end{bmatrix} \begin{bmatrix} p_2(H_1) & S \\ \mathbf{0} & p_2(H_2) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & R \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & S \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \mathbf{0}.$$

■

Corollary 139 *Let A be an $n \times n$ matrix. If H is a Hessenberg matrix similar to A, then $p_H(t) = p_A(t)$.*

Proof. Since $H \sim A$, the characteristic polynomials $p_A(t)$ and $p_H(t)$ have exactly the same roots by Proposition 96; hence $p_A(t) = c\,p_H(t)$ for some scalar c. Since $p_A(t)$ and $p_H(t)$ are monic, $c = 1$. ■

Theorem 140 (Cayley-Hamilton Theorem) *If A is an $n \times n$ matrix, then $p_A(A) = \mathbf{0}$.*

Proof. Let H be an $n \times n$ Hessenberg matrix such that $H = S^{-1}AS$. By Corollary 139, $p_A(t) = p_H(t) = a_0 + a_1 t + \cdots + a_{n-1}t^{n-1} + t^n$, and by Theorem 138 we have

$$\begin{aligned}
p_A(A) &= p_H(A) = p_H\left(SHS^{-1}\right) \\
&= a_0 I + a_1 SHS^{-1} + \cdots + a_{n-1}\left(SHS^{-1}\right)^{n-1} + \left(SHS^{-1}\right)^n \\
&= a_0 SIS^{-1} + a_1 SHS^{-1} + \cdots + a_{n-1}SH^{n-1}S^{-1} + SH^nS^{-1} \\
&= S\left(a_0 I + a_1 H + \cdots + a_{n-1}H^{n-1} + H^n\right)S^{-1} \\
&= S \cdot p_H(H) \cdot S^{-1} = S \cdot \mathbf{0} \cdot S^{-1} = \mathbf{0}.
\end{aligned}$$

■

11-7-2012

Given an $n \times n$ matrix A, consider the system of linear first order differential equations $\mathbf{x}' = A\mathbf{x}$ with initial value $\mathbf{x}(0) = \mathbf{x}_0$. Let $H = S^{-1}AS$ be a Hessenberg matrix and let

$$\mathbf{x} = S\mathbf{y} \text{ and } \mathbf{x}_0 = S\mathbf{y}_0.$$

Then $\mathbf{x}' = S\mathbf{y}'$ and

$$H\mathbf{y} = \left(S^{-1}AS\right)\mathbf{y} = S^{-1}A\mathbf{x} = S^{-1}\mathbf{x}' = \mathbf{y}'.$$

Thus the change of variables $\mathbf{x} = S\mathbf{y}$ transforms the given initial value problem (IVP) into the equivalent one

$$\mathbf{y}' = H\mathbf{y} \text{ with } \mathbf{y}(0) = \mathbf{y}_0. \tag{18}$$

When H is unreduced, we can solve this IVP by following the procedure in this lecture; but when H is reduced, we'll need Jordan Canonical Form (the final topic in this course).

Example 141 *Consider the IVP* $\mathbf{x}' = A\mathbf{x}$ *with* $\mathbf{x}(0) = \mathbf{x}_0$, *where*

$$A = \begin{bmatrix} -1 & -8 & 1 \\ -1 & -3 & 2 \\ -4 & -16 & 7 \end{bmatrix} \text{ and } \mathbf{x}_0 = \begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix}.$$

The change of variables $\mathbf{x} = S\mathbf{y}$, *where*

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix},$$

transforms this IVP into $\mathbf{y}' = H\mathbf{y}$ *with* $\mathbf{y}(0) = \mathbf{y}_0$, *where*

$$H = S^{-1}AS = \begin{bmatrix} -1 & -4 & 1 \\ -1 & 5 & 2 \\ 0 & -8 & -1 \end{bmatrix}$$

is an unreduced Hessenberg matrix and

$$\mathbf{y}(0) = S^{-1}\mathbf{x}(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 9 \end{bmatrix}.$$

But before we can solve the IVP in Example 141, we need to develop some important theoretical ideas.

Definition 142 *Let A be an $n \times n$ matrix with characteristic polynomial $p_A(t) = (t - \lambda_1)^{r_1} \cdots (t - \lambda_k)^{r_k}$, where $r_1 + \cdots + r_k = n$ and $\lambda_i \in \mathbb{C}$. The* ***algebraic multiplicity*** *of λ_i is r_i, and the* ***geometric multiplicity*** *of λ_i is the dimension of the eigenspace corresponding to λ_i.*

The geometric multiplicity of an eigenvalue is at least 1 and at most its algebraic multiplicity.

Definition 143 *An $n \times n$ matrix A is* ***defective*** *if A has an eigenvalue whose geometric multiplicity is less than its algebraic multiplicity.*

Example 144 *The characteristic polynomial $p_A(t) = t^2$ of the triangular matrix*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has the single root $\lambda = 0$, which is an eigenvalue of algebraic multiplicity 2. The eigenspace of λ is one dimensional and is spanned by the single vector $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, so the geometric multiplicity of λ is 1. Therefore A is defective and is not diagonalizable.

Exercise 145 *Show that the following matrices are defective:*

a. $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$ b. $\begin{bmatrix} -1 & -1 & -2 \\ 8 & -11 & -8 \\ -10 & 11 & 7 \end{bmatrix}$ c. $\begin{bmatrix} -4 & 1 & 1 & 1 \\ -16 & 3 & 4 & 4 \\ -7 & 2 & 2 & 1 \\ -11 & 1 & 3 & 4 \end{bmatrix}$

For purposes of the application in this section, it is necessary to express the characteristic polynomial of an $n \times n$ matrix A in the form

$$p_A(t) = (-1)^n \det(A - tI).$$

Definition 146 *Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. A vector $\mathbf{v} \in \mathbb{R}^n$ is a* ***generalized eigenvector of order*** p ***corresponding to*** λ *if $(A - \lambda I)^{p-1} \mathbf{v} \neq \mathbf{0}$ and $(A - \lambda I)^p \mathbf{v} = \mathbf{0}$.*

A usual eigenvector corresponding to λ is a generalized eigenvector of order 1. For future reference, note that if $A - \lambda I$ is nilpotent with index of nilpotency k, and $\mathbf{y} = (A - \lambda I)^{p-1} \mathbf{v} \neq \mathbf{0}$, the set $\{\mathbf{y} = (A - \lambda I)^{p-1} \mathbf{v}, \ldots, (A - \lambda I)^2 \mathbf{v}, (A - \lambda I) \mathbf{v}, \mathbf{v}\}$ is called the *Jordan chain on* $\mathbf{y}$ *of length* p.

Theorem 147 *Let λ be an eigenvalue of an unreduced $n \times n$ Hessenberg matrix H. If the algebraic multiplicity of λ is m, then for each $p = 1, 2, \ldots, m$, there is a generalized eigenvector $\mathbf{v}_p$ of order p corresponding to λ.*

Proof. Let $p_H(t)$ be the characteristic polynomial of H. Since the algebraic multiplicity of λ is m, we have (up to sign)

$$p_H(t) = (-1)^n (t - \lambda)^m q(t) \text{ with } q(\lambda) \neq 0.$$

Consider the polynomial

$$(t - \lambda)^{m-1} q(t) = a_0 + a_1 t + \cdots + a_{n-2} t^{n-2} + t^{n-1}$$

and the corresponding matrix polynomial

$$(H - \lambda I)^{m-1} q(H) = a_0 I + a_1 H + \cdots + a_{n-2} H^{n-2} + H^{n-1}.$$

Then

$$(H - \lambda I)^{m-1} q(H) \mathbf{e}_1 = a_0 \mathbf{e}_1 + a_1 H \mathbf{e}_1 + \cdots + a_{n-2} H^{n-2} \mathbf{e}_1 + H^{n-1} \mathbf{e}_1 \neq \mathbf{0}$$

since $\{\mathbf{e}_1, H\mathbf{e}_1, \ldots, H^{n-1}\mathbf{e}_1\}$ is linearly independent by Theorem 100. On the other hand,

$$(H - \lambda I)^m q(H) \mathbf{e}_1 = p_H(H) \mathbf{e}_1 = \mathbf{0}$$

by the Cayley-Hamilton Theorem. Therefore the vector $\mathbf{v}_m = q(H) \mathbf{e}_1$ is a generalized eigenvector of order m corresponding to λ. Finally, for $1 \leq p \leq m$, let

$$\mathbf{v}_p = (H - \lambda I)^{m-p} \mathbf{v}_m;$$

then $(H - \lambda I)^{p-1} \mathbf{v}_p \neq \mathbf{0}$ and $(H - \lambda I)^p \mathbf{v}_p = \mathbf{0}$. Thus for each p, there is a generalized eigenvector $\mathbf{v}_p$ of order p corresponding to λ . ■

Theorem 148 *An unreduced $n \times n$ Hessenberg matrix has n linearly independent generalized eigenvectors.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the eigenvalues of H and let m_i be the algebraic multiplicity of λ_i. Then the characteristic polynomial of H is

$$p_H(t) = (-1)^n (t - \lambda_1)^{m_1} (t - \lambda_2)^{m_2} \cdots (t - \lambda_k)^{m_k} = (t - \lambda_1)^{m_1} q(t),$$

where $m_1 + \cdots + m_k = n$. By Theorem 147, λ_i has a corresponding generalized eigenvector $\mathbf{v}_{i,p}$ of order p for each $p = 1, 2, \ldots, m_i$. Denote these generalized eigenvector by

$$\{\underbrace{\mathbf{v}_{1,1}, \ldots, \mathbf{v}_{1,m_1}}_{\lambda_1}, \underbrace{\mathbf{v}_{2,1}, \ldots, \mathbf{v}_{2,m_2}}_{\lambda_2}, \ldots, \underbrace{\mathbf{v}_{k.1}, \ldots, \mathbf{v}_{k,m_k}}_{\lambda_k}\}$$

and suppose that

$$\left(a_{1,1}\mathbf{v}_{1,1}+\cdots+a_{1,m_1}\mathbf{v}_{1,m_1}\right)+\left(a_{2,1}\mathbf{v}_{2,1}+\cdots+a_{2,m_2}\mathbf{v}_{2,m_2}\right)+\cdots+\left(a_{k,1}\mathbf{v}_{k,1}+\cdots+a_{k,m_k}\mathbf{v}_{k,m_k}\right)=\mathbf{0}. \quad (19)$$

Since the factors of $q\left(t\right)=\left(t-\lambda_2\right)^{m_2}\cdots\left(t-\lambda_k\right)^{m_k}$ commute,

$$q\left(H\right)\mathbf{v}_{i,j}=\cdots\left(H-\lambda_i I\right)^j\mathbf{v}_{i,j}=\mathbf{0}$$

for all j and all $i\geq 2$. Now left-multiply both sides of (19) by $q\left(H\right)$; then

$$\begin{aligned} &q\left(H\right)\left(a_{1,1}\mathbf{v}_{1,1}+\cdots+a_{1,m_1}\mathbf{v}_{1,m_1}\right)\\ &\qquad+\underbrace{q\left(H\right)\left(a_{2,1}\mathbf{v}_{2,1}+\cdots+a_{2,m_2}\mathbf{v}_{2,m_2}+\cdots+a_{k,1}\mathbf{v}_{k,1}+\cdots+a_{k,m_k}\mathbf{v}_{k,m_k}\right)}_{\mathbf{0}}=\mathbf{0}\end{aligned}$$

reduces to

$$a_{1,1}q\left(H\right)\mathbf{v}_{1,1}+\cdots+a_{1,m_1}q\left(H\right)\mathbf{v}_{1,m_1}=\mathbf{0}. \quad (20)$$

Keeping in mind that $\left(t-\lambda_1\right)^p q\left(t\right)=q\left(t\right)\left(t-\lambda_1\right)^p$ and $\left(H-\lambda_1 I\right)^{m_1-1}\mathbf{v}_{1,m_1-j}=\mathbf{0}$ for $1\leq j\leq m_1-1$, left-multiply both sides of (20) by $\left(H-\lambda_1 I\right)^{m_1-1}$; then

$$a_{1,m_1}\underbrace{q\left(H\right)\left(H-\lambda_1 I\right)^{m_1-1}\mathbf{v}_{1,m_1}}_{\neq\mathbf{0}}=\mathbf{0}$$

implies $a_{1,m_1}=0$, and (20) reduces to

$$a_{1,1}q\left(H\right)\mathbf{v}_{1,1}+\cdots+a_{1,m_1-1}q\left(H\right)\mathbf{v}_{1,m_1-1}=\mathbf{0}. \quad (21)$$

Left-multiply both sides of (21) by $\left(H-\lambda_1 I\right)^{m_1-2}$; then

$$a_{1,m_1-1}\underbrace{q\left(H\right)\left(H-\lambda_1 I\right)^{m_1-2}\mathbf{v}_{1,m_1-1}}_{\neq\mathbf{0}}=\mathbf{0}$$

implies $a_{1,m_1-1}=0$, and so on inductively. Do this for each λ_i. Conclude that all coefficients vanish and the n generalized eigenvectors are linearly independent. ■

Example 149 *Since the characteristic polynomial of the unreduced Hessenberg matrix*

$$H=\begin{bmatrix}1 & -1\\ 1 & 3\end{bmatrix} \quad \textit{is} \quad p_H\left(t\right)=\det\begin{bmatrix}1-t & -1\\ 1 & 3-t\end{bmatrix}=\left(2-t\right)^2,$$

the eigenvalue $\lambda=2$ has algebraic multiplicity 2. A basis for the corresponding eigenspace is found by solving the system

$$\left(H-2I\right)\mathbf{x}=0:$$

$$\begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix}\xrightarrow{row\ reduce}\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}.$$

Then $\left\{\mathbf{v}_1=\left[\begin{smallmatrix}-1\\ 1\end{smallmatrix}\right]\right\}$ is a basis and the eigenvalue $\lambda=2$ has geometric multiplicity 1. To find a generalized eigenvector $\mathbf{v}_2$ corresponding to $\lambda=2$, solve the system

$$\left(H-2I\right)\mathbf{x}=\mathbf{v}_1:$$

$$\left[\begin{array}{cc:c}-1 & -1 & -1\\ 1 & 1 & 1\end{array}\right]\xrightarrow{row\ reduce}\left[\begin{array}{cc:c}1 & 1 & 1\\ 0 & 0 & 0\end{array}\right].$$

The general solution is

$$t\begin{bmatrix}-1\\ 1\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix};$$

set $t = 0$ and obtain the particular solution $\mathbf{v}_2 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. Since

$$(H - 2I)^2 \mathbf{v}_2 = (H - 2I)\,\mathbf{v}_1 = \mathbf{0},$$

$\mathbf{v}_2$ is a generalized eigenvector of order 2 corresponding to $\lambda = 2$. Therefore

$$\left\{ \mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

is a linearly independent set of generalized eigenvectors corresponding to $\lambda = 2$.

Problem: *Let H be a complex $n \times n$ unreduced Hessenberg matrix. Solve the IVP $y' = Hy$ with $\mathbf{y}(0) = \mathbf{y}_0$.*

Procedure:

1. Use Krylov's Method to compute the characteristic polynomial of H. Then
$$p_H(t) = (-1)^n (t - \lambda_1)^{m_1} (t - \lambda_2)^{m_2} \cdots (t - \lambda_k)^{m_k}$$
and the eigenvalue λ_i has algebraic multiplicity m_i.

2. Find a maximal linearly independent set of eigenvectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ of H. Then each $\mathbf{u}_j$ corresponds to some eigenvalue λ_i and $\mathbf{y} = e^{\lambda_i t}\mathbf{u}_j$ is a particular solution since
$$\mathbf{y}' = e^{\lambda_i t}(\lambda_i \mathbf{u}_j) = e^{\lambda_i t} H\mathbf{u}_j = H\left(e^{\lambda_i t}\mathbf{u}_j\right) = H\mathbf{y}.$$

3. Let d_i be the geometric multiplicity of λ_i. If $d_i < m_i$, compute $q_i = m_i - d_i$ generalized eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_{q_i}\}$ of H in the following way: Set $\mathbf{v}_1 = \mathbf{u}_1$. Successively solve each of the following systems of linear equations in which $\mathbf{v}_p$ is known and $\mathbf{v}_{p+1}$ is an unknown particular solution:
$$\begin{aligned}
(H - \lambda_i I)\,\mathbf{v}_2 = \mathbf{v}_1 \quad &\Rightarrow \quad H\mathbf{v}_2 = \mathbf{v}_1 + \lambda_i \mathbf{v}_2 \\
(H - \lambda_i I)\,\mathbf{v}_3 = \mathbf{v}_2 \quad &\Rightarrow \quad H\mathbf{v}_3 = \mathbf{v}_2 + \lambda_i \mathbf{v}_3 \\
(H - \lambda_i I)\,\mathbf{v}_4 = \mathbf{v}_3 \quad &\Rightarrow \quad H\mathbf{v}_4 = \mathbf{v}_3 + \lambda_i \mathbf{v}_4 \\
&\vdots \\
(H - \lambda_i I)\,\mathbf{v}_{q_i} = \mathbf{v}_{q_i - 1} \quad &\Rightarrow \quad H\mathbf{v}_{q_i} = \mathbf{v}_{q_i - 1} + \lambda_i \mathbf{v}_{q_i}.
\end{aligned}$$
Then
$$(H - \lambda_i I)^{q_i - 1}\mathbf{v}_{q_i} = (H - \lambda_i I)^{q_i - 2}\mathbf{v}_{q_{i-1}} = \cdots = (H - \lambda_i I)\,\mathbf{v}_2 = \mathbf{v}_1. \tag{22}$$
Since $(H - \lambda_i I)\,\mathbf{v}_1 = \mathbf{0}$, left-multiplying each expression in (22) by $H - \lambda_i I$ gives
$$(H - \lambda_i I)^{q_i}\mathbf{v}_{q_i} = (H - \lambda_i I)^{q_i - 1}\mathbf{v}_{q_{i-1}} = \cdots = (H - \lambda_i I)^2\mathbf{v}_2 = (H - \lambda_i I)\,\mathbf{v}_1 = \mathbf{0}.$$
But $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{q_i}\}$ is linearly independent by Theorem 148 and $\mathbf{v}_p$ is a generalized eigenvector of order p corresponding to λ_i for each $p = 1, 2, \ldots, q_i$. Let
$$\mathbf{y}_p = e^{\lambda_i t}\left(\mathbf{v}_p + t\mathbf{v}_{p-1} + \frac{t^2}{2!}\mathbf{v}_{p-2} + \cdots + \frac{t^{p-1}}{(p-1)!}\mathbf{v}_1\right);$$
then
$$\begin{aligned}
\mathbf{y}_p' &= \lambda_i e^{\lambda_i t}\left(\mathbf{v}_p + t\mathbf{v}_{p-1} + \frac{t^2}{2!}\mathbf{v}_{p-2} + \cdots + \frac{t^{p-1}}{(p-1)!}\mathbf{v}_1\right) + e^{\lambda_i t}\left(\mathbf{v}_{p-1} + t\mathbf{v}_{p-2} + \cdots + \frac{t^{p-2}}{(p-2)!}\mathbf{v}_1\right) \\
&= e^{\lambda_i t}\Bigg((\mathbf{v}_{p-1} + \lambda_i \mathbf{v}_p) + t(\mathbf{v}_{p-2} + \lambda_i \mathbf{v}_{p-1}) + \frac{t^2}{2!}(\mathbf{v}_{p-3} + \lambda_i \mathbf{v}_{p-2}) + \cdots \\
&\qquad + \frac{t^{p-2}}{(p-2)!}(\mathbf{v}_1 + \lambda_i \mathbf{v}_2) + \frac{t^{p-1}}{(p-1)!}\lambda_i \mathbf{v}_1\Bigg) \\
&= e^{\lambda_i t}\left(H\mathbf{v}_p + tH\mathbf{v}_{p-1} + \frac{t^2}{2!}H\mathbf{v}_{p-2} + \cdots + \frac{t^{p-1}}{(p-1)!}H\mathbf{v}_1\right) = H\mathbf{y}_p,
\end{aligned}$$

and $\mathbf{y}_p$ is a particular solution.

4. Finally, the general solution of the given system of linear differential equations consists of all linear combinations of the solutions found in steps 2 and 3. Given this, one can find the initial vector $\mathbf{y}_0$ that solves the IVP by solving the system of equations obtained by setting $t = 0$.

Example 150 *Solve the IVP*

$$\begin{aligned} y_1' &= y_1 - y_2 \\ y_2' &= y_1 + 3y_2 \end{aligned}$$

with initial conditions $y_1(0) = 5$, $y_2(0) = -7$. *In matrix form* $\mathbf{y}' = H\mathbf{y}$, *this is*

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \text{ with } \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \end{bmatrix}. \tag{23}$$

In Example 149 we found linearly independent generalized eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

corresponding to the eigenvalue $\lambda = 2$. *Let*

$$\mathbf{y} = e^{2t}\mathbf{v}_1;$$

differentiating gives

$$\mathbf{y}' = e^{2t}(2\mathbf{v}_1) = e^{2t}H\mathbf{v}_1 = H\left(e^{2t}\mathbf{v}_1\right) = H\mathbf{y};$$

thus $\mathbf{y} = e^{2t}\mathbf{v}_1$ *is a particular solution. Now*

$$(H - 2I)\mathbf{v}_2 = \mathbf{v}_1 \quad \Rightarrow \quad H\mathbf{v}_2 = \mathbf{v}_1 + 2\mathbf{v}_2.$$

Let

$$\mathbf{y} = e^{2t}(\mathbf{v}_2 + t\mathbf{v}_1);$$

differentiating gives

$$\begin{aligned} \mathbf{y}' &= 2e^{2t}(\mathbf{v}_2 + t\mathbf{v}_1) + e^{2t}\mathbf{v}_1 = e^{2t}\left[(\mathbf{v}_1 + 2\mathbf{v}_2) + t(2\mathbf{v}_1)\right] \\ &= e^{2t}(H\mathbf{v}_2 + tH\mathbf{v}_1) = He^{2t}(\mathbf{v}_2 + t\mathbf{v}_1) = H\mathbf{y}. \end{aligned}$$

Then $\mathbf{y} = e^{2t}(\mathbf{v}_2 + t\mathbf{v}_1)$ *is a particular solution, and the general solution of (23) is*

$$\mathbf{y} = c_1e^{2t}\mathbf{v}_1 + c_2e^{2t}(\mathbf{v}_2 + t\mathbf{v}_1), \; c_i \in \mathbb{R}. \tag{24}$$

To find the desired initial values, set $t = 0$ *in (24) and solve* $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{y}(0)$:

$$\left[\begin{array}{cc:c} -1 & 1 & 5 \\ 1 & 0 & -7 \end{array}\right] \xrightarrow{row\ reduce} \left[\begin{array}{cc:c} 1 & 0 & -7 \\ 0 & 1 & -2 \end{array}\right].$$

Therefore

$$\mathbf{y} = -7e^{2t}\mathbf{v}_1 - 2e^{2t}(\mathbf{v}_2 + t\mathbf{v}_1)$$

solves the IVP.

11-11-2012

Schur's Triangularization Theorem

The characteristic polynomial $p_A(t)$ of an $n \times n$ complex matrix A splits as a product of linear factors of the form $(t-\lambda)^m$. Of course, finding these factors is a difficult problem, but *having factored* $p_A(t)$, Schur's Triangularization Theorem tells us that A can always be triangularized whether or not A is diagonalizable. In fact, we proved a special case of Schur's Triangularization Theorem when A is a real matrix with strictly real eigenvalues (see Theorem 57). As a consequence of Schur's Triangularization Theorem, we'll obtain another proof of the Cayley-Hamilton Theorem.

Let $\mathbf{v} = (v_1, \ldots, v_n)$ be a non-zero vector in $\mathbb{C}^n$ and set $\mathbf{x} = \dfrac{\overline{v_1}\mathbf{v}}{\|\overline{v_1}\mathbf{v}\|}$. Then $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{C}^n$ is a unit vector with $x_1 \in \mathbb{R}$. Set $\mathbf{u} = \mathbf{x} - \mathbf{e}_1$ and note that

$$\|\mathbf{u}\|^2 = \langle \mathbf{x}-\mathbf{e}_1, \mathbf{x}-\mathbf{e}_1\rangle = \langle \mathbf{x},\mathbf{x}\rangle - \langle \mathbf{x},\mathbf{e}_1\rangle - \langle \mathbf{e}_1,\mathbf{x}\rangle + \langle \mathbf{e}_1,\mathbf{e}_1\rangle = 2 - 2x_1 = 2(1-x_1)$$

and

$$\mathbf{u}^*\mathbf{x} = \langle \mathbf{x},\mathbf{u}\rangle = \langle \mathbf{x},\mathbf{x}-\mathbf{e}_1\rangle = \langle \mathbf{x},\mathbf{x}\rangle - \langle \mathbf{x},\mathbf{e}_1\rangle = 1 - x_1.$$

If $\mathbf{x} \neq \mathbf{e}_1$, then $\mathbf{u} \neq 0$ and we may apply the Householder transformation Q associated with $\mathbf{u}$ to $\mathbf{x}$ and $\mathbf{e}_1$:

$$Q\mathbf{x} = \mathbf{x} - \frac{2\mathbf{u}^*\mathbf{x}}{\|\mathbf{u}\|^2}\mathbf{u} = \mathbf{x} - \frac{2(1-x_1)}{2(1-x_1)}\mathbf{u} = \mathbf{x} - \mathbf{u} = \mathbf{x} - (\mathbf{x}-\mathbf{e}_1) = \mathbf{e}_1; \tag{25}$$

applying Q to both sides of (25) and using the fact that $Q^2 = I$ we have

$$\mathbf{x} = Q\mathbf{e}_1.$$

If $\mathbf{x} = \mathbf{e}_1$, set $Q = I$; then in either case

$$\mathbf{x} = Q\mathbf{e}_1 \text{ and } \mathbf{e}_1 = Q\mathbf{x}.$$

Thus when $\mathbf{x} \neq \mathbf{e}_1$, the unit vectors $\mathbf{x}$ and $\mathbf{e}_1$ are reflections of each other in the subspace $(\mathbf{x}-\mathbf{e}_1)^\perp$. In particular, if $\mathbf{x}$ is a unit vector in $\mathbb{R}^2$, the line $(\mathbf{x}-\mathbf{e}_1)^\perp$ bisects the angle between $\mathbf{x}$ and $\mathbf{e}_1$.

We are ready to prove the main theorem in this lecture:

Theorem 151 (Schur's Triangulation Theorem) *Every $n \times n$ complex matrix A is unitarily triangularizable, i.e., there exists a triangular matrix T and unitary matrix U such that $U^*AU = T$.*

Proof. We proceed by induction on the size of A. For $n = 1$ there is nothing to prove. So assume $n > 1$ and that the result holds for all matrices of size less than n. Since every complex matrix has an eigenvalue, choose an eigenvalue λ of A and an associated eigenvector $\mathbf{v} = (v_1, \ldots, v_n)$. Let $\mathbf{x} = \dfrac{\overline{v_1}\mathbf{v}}{\|\overline{v_1}\mathbf{v}\|}$ and set $\mathbf{u} = \mathbf{x} - \mathbf{e}_1$; if $\mathbf{x} \neq \mathbf{e}_1$, let Q be the Householder matrix associated with $\mathbf{u}$; if $\mathbf{x} = \mathbf{e}_1$ let $Q = I$. Then $\mathbf{x} = Q\mathbf{e}_1$ by the discussion above, and $\mathbf{x}$ is the first column of Q. By Exercise 114, Q is unitary and Hermitian, so $\mathbf{x}^*$ is the first row of Q. Since $Q = Q^*$ we have

$$Q = [\mathbf{x} \mid V] = \begin{bmatrix} \mathbf{x}^* \\ V^* \end{bmatrix}$$

and

$$\begin{aligned} QAQ &= QA[\mathbf{x} \mid V] = Q[A\mathbf{x} \mid AV] = Q[\lambda\mathbf{x} \mid AV] = [\lambda Q\mathbf{x} \mid QAV] \\ &= \left[\lambda\mathbf{e}_1 \mid \begin{bmatrix} \mathbf{x}^* \\ V^* \end{bmatrix} AV\right] = \left[\begin{array}{c|c} \lambda & \mathbf{x}^*AV \\ \hline \mathbf{0} & V^*AV \end{array}\right]. \end{aligned}$$

Since $B = V^*AV$ is an $(n-1) \times (n-1)$ matrix, we may apply the induction hypothesis and obtain an $(n-1) \times (n-1)$ unitary matrix R such that $T_{n-1} = R^*BR$ is upper-triangular. Let

$$U = Q\left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & R \end{array}\right];$$

then

$$U^*U = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & R^* \end{array}\right] QQ \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & R \end{array}\right] = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & R^*R \end{array}\right] = I$$

so that U is unitary, and

$$\begin{aligned} U^*AU &= \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & R^* \end{array}\right] QAQ \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & R \end{array}\right] \\ &= \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & R^* \end{array}\right] \left[\begin{array}{c|c} \lambda & \mathbf{x}^*AV \\ \hline \mathbf{0} & B \end{array}\right] \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & R \end{array}\right] \\ &= \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & R^* \end{array}\right] \left[\begin{array}{c|c} \lambda & \mathbf{x}^*AVR \\ \hline \mathbf{0} & BR \end{array}\right] \\ &= \left[\begin{array}{c|c} \lambda & \mathbf{x}^*AVR \\ \hline \mathbf{0} & R^*BR \end{array}\right] = \left[\begin{array}{c|c} \lambda & \mathbf{x}^*AVR \\ \hline \mathbf{0} & T_{n-1} \end{array}\right], \end{aligned}$$

is upper triangular. ■

Since similar matrices have the same eigenvalues, the eigenvalues of A are the diagonal entries of every Schur triangularization of A. In summary, every matrix is triangularizable but only non-defective matrices are diagonalizable.

Example 152 *In Exercise 145 you were asked to prove that the matrix*

$$A = \begin{bmatrix} -1 & -1 & -2 \\ 8 & -11 & -8 \\ -10 & 11 & 7 \end{bmatrix}$$

is defective, and hence not diagonalizable. Let's numerically approximate the Schur triangularization of the matrix. Since A is real with real eigenvalues $\lambda_1 = 1$, $\lambda_2 = -3$, $\lambda_3 = -3$, Schur's Triangularization Theorem tells us that A is orthogonally triangularizable. Arbitrarily choose an eigenvalue, say $\lambda_1 = 1$, then

$$A - I = \begin{bmatrix} -2 & -1 & -2 \\ 8 & -12 & -8 \\ -10 & 11 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and $\mathbf{x} = \begin{bmatrix} -1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$ *is a corresponding unit eigenvector. Let* $\mathbf{u} = \mathbf{x} - \mathbf{e}_1 = \begin{bmatrix} -4/3 \\ -2/3 \\ 2/3 \end{bmatrix}$; *then* $\|\mathbf{u}\|^2 = \dfrac{8}{3}$ *and the associated Householder matrix is*

$$Q_1 = I - \frac{3}{4}\mathbf{u}\mathbf{u}^T = \frac{1}{3}\begin{bmatrix} -1 & -2 & 2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} = [\mathbf{x} \mid V].$$

Then

$$Q_1AQ_1 = \frac{1}{3}\begin{bmatrix} 3 & 64 & 13 \\ 0 & -13 & -1 \\ 0 & 16 & -5 \end{bmatrix} \text{ and } V^TAV = \frac{1}{3}\begin{bmatrix} -13 & -1 \\ 16 & -5 \end{bmatrix}.$$

Now the 2×2 matrix $B = V^TAV$ has a single eigenvalue $\lambda = -3$. To triangularize B, consider the unit eigenvector $\mathbf{x} = \frac{1}{\sqrt{17}}\left[\begin{smallmatrix} 1 \\ -4 \end{smallmatrix}\right]$ corresponding to $\lambda = -3$. Then $\mathbf{u} = \mathbf{x} - \mathbf{e}_1 = \frac{1}{\sqrt{17}}\left[\begin{smallmatrix} 1-\sqrt{17} \\ -4 \end{smallmatrix}\right]$, $\|\mathbf{u}\|^2 = 2\left(\dfrac{\sqrt{17}-1}{\sqrt{17}}\right)$, and the associated Householder matrix (rounded to five decimal places) is

$$Q_2 = I - \frac{\sqrt{17}}{\sqrt{17}-1}\mathbf{u}\mathbf{u}^T \approx \begin{bmatrix} 0.242\,54 & -0.970\,14 \\ -0.970\,14 & -0.242\,54 \end{bmatrix}.$$

Thus

$$Q_2 B Q_2 = \begin{bmatrix} -3 & 17/3 \\ 0 & -3 \end{bmatrix}$$

is a Schur triangularization of B. Finally, let

$$U = Q_1 \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & Q_2 \end{array}\right] \approx \begin{bmatrix} -1/3 & -0.80845 & 0.48507 \\ -2/3 & -0.16169 & -0.72761 \\ 2/3 & -0.56591 & -0.48507 \end{bmatrix};$$

then $U^T U = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & Q_2 \end{array}\right] Q_1 Q_1 \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & Q_2 \end{array}\right] = I$, *so U is orthogonal, and*

$$U^T A U \approx \begin{bmatrix} 1 & 0.970\,25 & -21.747 \\ 0 & -3.000 & 5.6667 \\ 0 & 0 & -3.000 \end{bmatrix}$$

is an numerically approximate Schur triangularization of A.

Exercise 153 *Following the proof of Schur's Triangularization Theorem, find an orthogonal matrix P such that $P^T AP$ is upper triangular:*

a. $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

b. $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

c. $A = \begin{bmatrix} 13 & -9 \\ 16 & -11 \end{bmatrix}$

11-15-12

Another Proof of the Cayley-Hamilton Theorem

The Cayley-Hamilton Theorem is a direct consequence of Schur's Triangularization Theorem giving a proof quite different from our previous one. First we need a definition and a fact about products of upper triangular block matrices.

Definition 154 *An $n \times n$ matrix N is* ***nilpotent*** *if and only if $N^k = \mathbf{0}$ for some positive integer k. If N is nilpotent, its* ***index of nilpotency*** *is the smallest positive integer k such that $N^k = \mathbf{0}$.*

Example 155 *Polynomial differentiation $\mathcal{D}$ is a nilpotent operator on $\mathcal{P}_3$ whose matrix in the standard basis $\mathcal{B} = \{1, x, x^2, x^3\}$ is*

$$[\mathcal{D}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad [\mathcal{D}]^2 = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad [\mathcal{D}]^3 = \begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad \textit{and}\, [\mathcal{D}]^4 = 0.$$

So the index of nilpotency of is 4.

Exercise 156 *Nilpotent matrices are singular.*

Exercise 157 *Let T be an upper triangular block matrix of the form*

$$T = \begin{bmatrix} T_1 & * & \cdots & * \\ \mathbf{0} & T_2 & \cdots & * \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & T_k \end{bmatrix}, \quad \textit{where } T_i \textit{ is an } m_i \times m_i \textit{ block.}$$

Let Z_i be the matrix obtained from T by replacing T_i with the zero matrix. Then $Z_1 Z_2 \cdots Z_k = \mathbf{0}$.

Theorem 158 *(Cayley-Hamilton Theorem) If $p(t)$ is the characteristic polynomial of a square complex matrix A, then $p(A) = \mathbf{0}$.*

Proof. By Schur's triangularization Theorem there is a unitary matrix U and an upper triangular matrix T such that $U^*AU = T$, and we may construct U in such a way that the eigenvalues of A are distributed along the diagonal in any order. So if $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of A and m_i is the multiplicity of λ_i, construct U so that

$$T = U^*AU = \begin{bmatrix} T_1 & * & \cdots & * \\ \mathbf{0} & T_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & T_k \end{bmatrix},$$

where T_i is the $m_i \times m_i$ matrix

$$T_i = \begin{bmatrix} \lambda_i & * & \cdots & * \\ 0 & \lambda_i & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_i \end{bmatrix}.$$

The characteristic polynomial of T_i is $p_i(t) = (t - \lambda_i)^{m_i}$, and

$$T_i - \lambda_i I = \begin{bmatrix} 0 & * & \cdots & * & * \\ 0 & 0 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & * \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

is nilpotent with index of nilpotency $\leq m_i$; therefore $p_i(T_i) = (T_i - \lambda_i I)^{m_i} = \mathbf{0}$. Furthermore, the characteristic polynomial $p_T(t) = (t-\lambda_1)^{m_1} \cdots (t-\lambda_k)^{m_k}$, and the matrix polynomial $p_T(T) = (T - \lambda_1 I)^{m_1} \cdots (T - \lambda_k I)^{m_k}$ is a product $Z_1 Z_2 \cdots Z_k$, where the i^{th} block $(T_i - \lambda_1 I)^{m_i}$ of $Z_i = (T - \lambda_i I)^{m_i}$ is $\mathbf{0}$. Then $p(T) = \mathbf{0}$ by Exercise 157. Now consider the matrix

$$U^* p(A) U = U^* (A - \lambda_1 I)^{m_1} \cdots (A - \lambda_k I)^{m_k} U$$

and insert U^*U between each adjacent pair of linear factors. Then

$$\begin{aligned} U^* p(A) U &= \underbrace{U^*(A-\lambda_1 I) UU^* \cdots UU^* (A - \lambda_1 I) U}_{m_1 \text{ factors } U^*(A-\lambda_1 I)U} \cdots \underbrace{U^*(A-\lambda_k I) UU^* \cdots UU^* (A - \lambda_k I) U}_{m_k \text{ factors } U^*(A-\lambda_k I)U} \\ &= (U^*AU - \lambda_1 U^*U)^{m_1} \cdots (U^*AU - \lambda_k U^*U)^{m_k} \\ &= (T - \lambda_1 I)^{m_1} \cdots (T - \lambda_k I)^{m_k} = p(T) = \mathbf{0}. \end{aligned}$$

Solving for $p(A)$ completes the proof. ■

Here are some more interesting fact about upper triangular block matrices:

Exercise 159 *Let T be as above and let $p(t) = a_n t^n + \cdots + a_1 t + a_0$ be any polynomial. Then*

(a) $T^n = \begin{bmatrix} T_1^n & * & \cdots & * \\ \mathbf{0} & T_2^n & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & T_k^n \end{bmatrix}$ *for all n.*

(b) $p(T) = \begin{bmatrix} p(T_1) & * & \cdots & * \\ \mathbf{0} & p(T_2) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & p(T_k) \end{bmatrix}$.

11-11-12

Let A be a square matrix over a field F (with entries from F). Recall that the nullspace $N(A)$ is the solution space of the linear system $A\mathbf{x} = \mathbf{0}$ and the column space $R(A)$ is the subspace spanned by the columns of A. Since $N(A)$ and $R(A)$ are the kernel and range of the matrix transformation $T_A(\mathbf{x}) = A\mathbf{x}$, we use the terms "nullspace" and "kernel", and "column space" and "range", interchangeably.

Recall that if U and V are subspaces of a vector space W such that $U \cap V = \mathbf{0}$, the direct sum $U \oplus V$ is the subspace $\{\mathbf{u} + \mathbf{v} \mid \mathbf{u} \in U \text{ and } \mathbf{v} \in V\} \subseteq W$. When $U \oplus V = W$ we refer to $U \oplus V$ as a *direct sum decomposition of* W and to U and V as *complementary subspaces.*

Exercise 160 *Prove that if* $\mathbf{w} \in U \oplus V$, *there exist unique vectors* $\mathbf{u} \in U$ *and* $\mathbf{v} \in V$ *such that* $\mathbf{w} = \mathbf{u} + \mathbf{v}$.

Example 161 *The subspaces* $U = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$ *and* $V = \{(0, 0, z) \mid z \in \mathbb{R}\}$ *are complementary subspaces in* $\mathbb{R}^3$ *and* $U \oplus V$ *is a direct sum decomposition of* $\mathbb{R}^3$. *An element* $\mathbf{w} = (x, y, z) \in \mathbb{R}^3$ *is the unique sum* $\mathbf{w} = \mathbf{u} + \mathbf{v}$, *where* $\mathbf{u} = (x, y, 0) \in U$ *and* $\mathbf{v} = (0, 0, z) \in V$. *Note that* $V = U^\perp$ *with respect to the Euclidean inner product. More generally, if* U *is a subspace of an inner product space* V, *then* $V = U \oplus U^\perp$.

Given an $n \times n$ matrix A over F, we wish to define a direct sum decomposition of the n-fold Cartesian product F^n in terms of $N(A)$ and $R(A)$. Of course, if A is nonsingular, $N(A) = \mathbf{0}$, $R(A) = F^n$, and there is the trivial decomposition $F^n = \{\mathbf{0}\} \oplus R(A)$. So for the remainder of this lecture, A is assumed to be singular unless otherwise indicated.

To begin, observe that the nullspace and column space of a singular $n \times n$ matrix may intersect non-trivially:

Example 162 *Consider the (singular) matrix* $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$. *A basis for* $N(A)$ *is* $\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ *and a basis for* $R(A)$ *is* $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$; *hence* $N(A) \cap R(A) = N(A) \neq \mathbf{0}$.

Theorem 163 (Range-Nullspace Decomposition) *If* A *is an* $n \times n$ *matrix over* F, *there is a smallest integer* $k \geq 1$ *such that*

$$F^n = N\left(A^k\right) \oplus R\left(A^k\right).$$

By the remark above, Theorem 163 holds trivially for nonsingular matrices with $k = 1$; hence we must prove Theorem 163 when A is singular. The proof follows from several propositions, which are independently interesting and useful.

Proposition 164 *Let* A *be an* $n \times n$ *matrix. There are the following sequences of inclusions:*

$$\{\mathbf{0}\} \subseteq N(A) \subseteq N\left(A^2\right) \subseteq \cdots \subseteq F^n$$

$$F^n \supseteq R(A) \supseteq R\left(A^2\right) \supseteq \cdots \supseteq \{\mathbf{0}\}.$$

Proof. To show that $N\left(A^k\right) \subseteq N\left(A^{k+1}\right)$, let $\mathbf{x} \in N\left(A^k\right)$ and show that $\mathbf{x} \in N\left(A^{k+1}\right)$. But $\mathbf{x} \in N\left(A^k\right)$ if and only if $A^k\mathbf{x} = \mathbf{0}$, in which case $A^{k+1}\mathbf{x} = A\left(A^k\mathbf{x}\right) = A(\mathbf{0}) = \mathbf{0}$ and $\mathbf{x} \in N\left(A^{k+1}\right)$. To show that $R\left(A^k\right) \supseteq R\left(A^{k+1}\right)$, let $\mathbf{y} \in R\left(A^{k+1}\right)$ and exhibit a vector $\mathbf{u}$ such that $A^k\mathbf{u} = \mathbf{y}$. But $\mathbf{y} \in R\left(A^{k+1}\right)$ if and only if $\mathbf{y} = A^{k+1}\mathbf{x} = A^k(A\mathbf{x})$ for some $\mathbf{x} \in F^n$. Set $\mathbf{u} = A\mathbf{x}$; then $A^k\mathbf{u} = \mathbf{y}$. ■

Proposition 165 *Let* A *be an* $n \times n$ *matrix. There exist integers* $p, q \geq 1$ *such that* $N(A^p) = N\left(A^{p+1}\right)$ *and* $R(A^q) = R\left(A^{q+1}\right)$.

Proof. By Proposition 164, $\dim N(A^p)$ increases with i while $\dim R(A^q)$ decreases with j, i.e.,

$$0 \leq \dim N(A) \leq \dim N\left(A^2\right) \leq \cdots \leq \dim N(A^p) \leq \cdots \leq n$$

$$n \geq \dim R(A) \geq \dim R\left(A^2\right) \geq \cdots \geq \dim R(A^q) \geq \cdots \geq 0.$$

These inequalities hold for all $p, q > 0$ and cannot always be strict (otherwise $\dim N(A^p)$ would eventually exceed n and $\dim R(A^q)$ would eventually become negative). Hence there exist $p > 0$ and $q > 0$ such that $\dim N(A^p) = \dim N\left(A^{p+1}\right)$ and $\dim R(A^q) = \dim R\left(A^{q+1}\right)$. But $N(A^p) \subseteq N\left(A^{p+1}\right)$ and $R(A^q) \supseteq \dim R\left(A^{q+1}\right)$ by Proposition 164, and the subspaces on either side of these containments have the same dimension. Therefore $N(A^p) = N\left(A^{p+1}\right)$ and $R(A^q) = R\left(A^{q+1}\right)$. ■

In fact, the sequences of inclusions in Proposition 164 eventually *stabilize* at some k^{th} power of A, which means there exist smallest positive integer k such that $N\left(A^k\right) = N\left(A^{k+i}\right)$ and $R\left(A^k\right) = R\left(A^{k+i}\right)$ for all $i \geq 1$. But proving this requires some notation and a lemma.

Notation 166 *Let V be an n-dimensional vector space over F, let U be a subspace of V, and let A be an $n \times n$ matrix over F. Then AU is the subspace*

$$AU = \{A\mathbf{u} \mid \mathbf{u} \in U\} \subseteq V.$$

For example, consider the x-axis $U = \{(x, 0) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$ and the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; then $AU = \{(0, x) \mid x \in \mathbb{R}\}$ is the y-axis.

Lemma 167 *Let A be a square matrix. Then $R\left(A^{k+i}\right) = A^i R\left(A^k\right)$ for all $i, k \geq 1$.*

Proof. First note that $R\left(A^{k+1}\right) \subseteq AR\left(A^k\right)$ for all $k \geq 1$: If $\mathbf{y} \in R\left(A^{k+1}\right)$, there exists $\mathbf{x} \in F^n$ such that $\mathbf{y} = A^{k+1}\mathbf{x} = AA^k\mathbf{x} \in AR\left(A^k\right)$. Also, $AR\left(A^k\right) \subseteq R\left(A^{k+1}\right)$ for all $k \geq 1$: If $\mathbf{y} \in AR\left(A^k\right)$, there exists $\mathbf{x} \in R\left(A^k\right)$ such that $\mathbf{y} = A\mathbf{x}$. Since $\mathbf{x} \in R\left(A^k\right)$, there exists $\mathbf{u} \in F^n$ such that $\mathbf{x} = A^k\mathbf{u}$. Hence $\mathbf{y} = A\mathbf{x} = AA^k\mathbf{u} = A^{k+1}\mathbf{u}$ and $\mathbf{y} \in R\left(A^{k+1}\right)$. Inductively, for each $i \geq 1$ we have $R\left(A^{i+k}\right) = AR\left(A^{k+i-1}\right) = A^2R\left(A^{k+i-2}\right) = \cdots = A^iR\left(A^k\right)$. ■

Proposition 168 *Let A be an $n \times n$ matrix and let k be the smallest positive integer such that $R\left(A^k\right) = R\left(A^{k+1}\right)$. Then*

1. *the sequences of inclusions in Proposition 164 stabilize at A^k, i.e.,*

$$\{\mathbf{0}\} \subset N(A) \subset \cdots \subset N\left(A^k\right) = N\left(A^{k+1}\right) = \cdots \subseteq F^n$$

$$F^n \supset R(A) \supset \cdots \supset R\left(A^k\right) = R\left(A^{k+1}\right) = \cdots \supseteq \{\mathbf{0}\},$$

2. $N\left(A^k\right) \cap R\left(A^k\right) = \mathbf{0}$.

Proof. (*1*) By Lemma 167 and Proposition 165 we have $R\left(A^{k+i}\right) = A^iR\left(A^k\right) = A^iR\left(A^{k+1}\right) = R\left(A^{k+i+1}\right)$, for each $i \geq 1$. Therefore $R\left(A^k\right) = R\left(A^{k+i}\right)$ for all $i > 0$. Furthermore, by the Dimension Theorem, $\dim N\left(A^{k+i}\right) = n - \dim R\left(A^{k+i}\right) = n - \dim R\left(A^k\right) = \dim N\left(A^k\right)$. Therefore $N\left(A^k\right) = N\left(A^{k+i}\right)$. Thus if $i < k$, then $R\left(A^i\right) \supset R\left(A^k\right)$ and consequently, $N\left(A^i\right) \subset N\left(A^k\right)$ by the Dimension Theorem.

(*2*) Let $\mathbf{y} \in R\left(A^k\right) \cap N\left(A^k\right)$. Since $\mathbf{y} \in R\left(A^k\right)$, there exists $\mathbf{x} \in F^n$ such that $\mathbf{y} = A^k\mathbf{x}$. On the other hand, since $\mathbf{y} \in N\left(A^k\right)$ we have $\mathbf{0} = A^k\mathbf{y} = A^kA^k\mathbf{x} = A^{2k}\mathbf{x}$. Then $\mathbf{x} \in N\left(A^{2k}\right) = N\left(A^k\right)$ and consequently, $\mathbf{y} = A^k\mathbf{x} = \mathbf{0}$. Therefore $R\left(A^k\right) \cap N\left(A^k\right) = \mathbf{0}$. ■

Definition 169 *Let A be an $n \times n$ matrix. If $\det A \neq 0$, define the **index of** A to be 0. Otherwise, define the **index of** A to be the smallest positive integer such that $N\left(A^k\right) \cap R\left(A^k\right) = \mathbf{0}$.*

Example 170 *Continuing Example 162, we have*

$$A^2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

a basis for $N\left(A^2\right)$ *is* $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ *and a basis for* $R\left(A^2\right)$ *is* $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$. *Since* $N\left(A^2\right) \cap R\left(A^2\right) = \mathbf{0}$, *the index of* A *is* 2.

If A is an $n \times n$ matrix of index k, Proposition 168 tells us that $N\left(A^k\right) \oplus R\left(A^k\right)$ is a subspace of F^n. All that remains to be proved is that $N\left(A^k\right) \oplus R\left(A^k\right) = F^n$.

Proposition 171 *If* A *be an* $n \times n$ *matrix of index* k, *then* $N\left(A^k\right) \oplus R\left(A^k\right) = F^n$.

Proof. Choose bases $\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ for $N\left(A^k\right)$ and $\mathbf{Y} = \{\mathbf{y}_1, \ldots, \mathbf{y}_q\}$ for $R\left(A^k\right)$; then $p + q = \dim N\left(A^k\right) + \dim R\left(A^k\right) = n$ by the Dimension Theorem. Since $\mathbf{X} \cup \mathbf{Y}$ has n elements, it is a basis for F^n if linearly independent. Suppose $\mathbf{X} \cup \mathbf{Y}$ is linearly dependent. Then there exist coefficients $a_1, \ldots, a_p, b_1, \ldots, b_q$ not all zero such that

$$a_1\mathbf{x}_1 + \cdots + a_p\mathbf{x}_p + b_1\mathbf{y}_1 + \cdots + b_q\mathbf{y}_q = \mathbf{0}.$$

Assume $a_i \neq 0$ for some i. Since $\mathbf{X}$ and $\mathbf{Y}$ are bases, $\mathbf{z} = a_1\mathbf{x}_1 + \cdots + a_p\mathbf{x}_p \neq \mathbf{0}$; consequently, $\mathbf{z} = -b_1\mathbf{y}_1 - \cdots - b_q\mathbf{y}_q \neq \mathbf{0}$. Therefore $\mathbf{z}$ is a non-zero vector in $N\left(A^k\right) \cap R\left(A^k\right)$, contradicting Proposition 168, part (2). Therefore $\mathbf{X} \cup \mathbf{Y}$ is linearly independent. ∎

The proof of the Range-Nullspace Decomposition Theorem is now complete.

Proposition 172 *If* L *is a nilpotent matrix with index of nilpotency* p, *then the index of* L *is* p.

Exercise 173 *Prove Proposition 172.*

Exercise 174 *Find the index of* A *and the corresponding Range-Nullspace decomposition of* $\mathbb{R}^5$*:*

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

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11-24-12

In this lecture we observe that a singular matrix A is similar to a block diagonal matrix $\begin{bmatrix} L & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix}$, where L is nilpotent and C is nonsingular.

Definition 175 *Let A be an $n \times n$ matrix. A subspace $S \subseteq F^n$ is **invariant under multiplication by** A if $AS \subseteq S$.*

Proposition 176 *If A is an $n \times n$ matrix of index k, then $R\left(A^k\right)$ and $N\left(A^k\right)$ are invariant under multiplication by A.*

Proof. Of course, $R\left(A^k\right)$ is invariant under multiplication by A since $AR\left(A^k\right) = R\left(A^{k+1}\right) = R\left(A^k\right)$ by Lemma 167. To see that $N\left(A^k\right)$ is invariant under multiplication by A, let $\mathbf{x} \in AN\left(A^k\right)$. Then there exists $\mathbf{w} \in N\left(A^k\right) = N\left(A^{k+1}\right)$ such that $\mathbf{x} = A\mathbf{w}$. Then $A^k\mathbf{x} = A^{k+1}\mathbf{w} = \mathbf{0}$ so that $\mathbf{x} \in N\left(A^k\right)$. Therefore $AN\left(A^k\right) \subseteq N\left(A^k\right)$ as claimed. ■

Let A be a singular $n \times n$ matrix of index k and recall that $N\left(A^k\right) \oplus R\left(A^k\right) = F^n$. Let

$$r = \dim R\left(A^k\right) = n - \dim N\left(A^k\right)$$

and choose bases $\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_{n-r}\}$ for $N\left(A^k\right)$ and $\mathbf{Y} = \{\mathbf{y}_1, \ldots, \mathbf{y}_r\}$ for $R\left(A^k\right)$; then $\mathcal{B} = \mathbf{X} \cup \mathbf{Y}$ is a basis for F^n. Let $T_A : F^n \to F^n$ be the matrix transformation $T_A(\mathbf{x}) = A\mathbf{x}$. Since $N\left(A^k\right)$ and $R\left(A^k\right)$ are invariant under multiplication by A, for all $\mathbf{x} \in N\left(A^k\right)$ and $\mathbf{y} \in R\left(A^k\right)$ we have

$$T_A(\mathbf{x}) = l_1\mathbf{x}_1 + \cdots + l_{n-r}\mathbf{x}_{n-r} \text{ and } T_A(\mathbf{y}) = c_1\mathbf{y}_1 + \cdots + c_r\mathbf{y}_r,$$

and in particular,

$$T_A(\mathbf{x}_i) = l_{1,i}\mathbf{x}_1 + \cdots + l_{n-r,i}\mathbf{x}_{n-r} \text{ and } T_A(\mathbf{y}_j) = c_{1,j}\mathbf{y}_1 + \cdots + c_{r,j}\mathbf{y}_r,$$

for $1 \leq i \leq n-r$ and $1 \leq j \leq r$. Given $\mathbf{u} \in F^n$, consider the $\mathcal{B}$-coordinate matrix $[\mathbf{u}]_{\mathcal{B}}$ and the change-of-coordinates matrix

$$Q = [X|Y] = [\mathbf{x}_1 \mid \cdots \mid \mathbf{x}_{n-r} \mid \mathbf{y}_1 \mid \cdots \mid \mathbf{y}_r].$$

Then $Q[\mathbf{u}]_{\mathcal{B}} = \mathbf{u}$ and we have

$$Q^{-1}AQ[\mathbf{u}]_{\mathcal{B}} = Q^{-1}A\mathbf{u} = Q^{-1}T_A(\mathbf{u}) = [T_A(\mathbf{u})]_{\mathcal{B}} = [T_A]_{\mathcal{B}}[\mathbf{u}]_{\mathcal{B}}.$$

Therefore

$$\begin{aligned} Q^{-1}AQ &= [T_A]_{\mathcal{B}} = [[T_A(\mathbf{x}_1)]_{\mathcal{B}} \mid \cdots \mid [T_A(\mathbf{x}_{n-r})]_{\mathcal{B}} \mid [T_A(\mathbf{y}_1)]_{\mathcal{B}} \mid \cdots \mid [T_A(\mathbf{y}_r)]_{\mathcal{B}}] \\ &= \begin{bmatrix} l_{1,1} & \cdots & l_{1,n-r} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ l_{n-r,1} & \cdots & l_{n-r,n-r} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & c_{1,1} & \cdots & c_{1,r} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & c_{r,1} & \cdots & c_{r,r} \end{bmatrix} = \begin{bmatrix} L & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix}. \end{aligned}$$

Theorem 177 (Nilpotent-Nonsingular Form) *Let A be a $n \times n$ singular matrix of index k and let $r = \operatorname{rank}\left(A^k\right)$. Then there exists a nonsingular matrix Q such that*

$$Q^{-1}AQ = \begin{bmatrix} L & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix},$$

where L is nilpotent of index k and C is nonsingular of rank r.

Proof. Let L, C and $Q = [X|Y]$ as above, and let $Q^{-1} = \begin{bmatrix} U^{(n-r)\times n} \\ V^{r\times n} \end{bmatrix}$. Since $A^k \mathbf{x}_j = \mathbf{0}$ for all j we have

$$\begin{bmatrix} L^k & \mathbf{0} \\ \mathbf{0} & C^k \end{bmatrix} = \begin{bmatrix} L & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix}^k = \left(Q^{-1}AQ\right)^k = Q^{-1}A^kQ$$

$$= \begin{bmatrix} U \\ V \end{bmatrix} A^k \,[X \mid Y] = \begin{bmatrix} U \\ V \end{bmatrix} \left[A^kX \mid A^kY\right] = \begin{bmatrix} \mathbf{0} & UA^kY \\ \mathbf{0} & VA^kY \end{bmatrix}.$$

Therefore $L^k = \mathbf{0}$ and L is nilpotent with index of nilpotency is at most k. Before proving that the index of nilpotency is exactly k, we compute the rank of C. Since rank is a similarity invariant we have

$$\operatorname{rank}\left(C^k\right) = \operatorname{rank}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C^k \end{bmatrix} = \operatorname{rank}\begin{bmatrix} L^k & \mathbf{0} \\ \mathbf{0} & C^k \end{bmatrix} = \operatorname{rank}\left(Q^{-1}A^kQ\right) = \operatorname{rank}\left(A^k\right) = r.$$

Hence C^k is nonsingular since it is an $r \times r$ matrix of rank r, and furthermore, $[\det(C)]^k = \det\left(C^k\right) \neq 0$ implies $\det C \neq 0$ so that C is nonsingular and $\operatorname{rank}(C^p) = r$ for $1 \leq p \leq k$. Now suppose L has index of nilpotency less than k. Then $L^{k-1} = \mathbf{0}$ and

$$\begin{aligned} \operatorname{rank}\left(A^{k-1}\right) &= \operatorname{rank}\left(Q^{-1}A^{k-1}Q\right) = \operatorname{rank}\begin{bmatrix} L^{k-1} & \mathbf{0} \\ \mathbf{0} & C^{k-1} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C^{k-1} \end{bmatrix} \\ &= \operatorname{rank}\left(C^{k-1}\right) = r = \operatorname{rank}\left(A^k\right) \end{aligned}$$

so that $\dim R\left(A^{k-1}\right) = \dim R\left(A^k\right)$. But $R\left(A^{k-1}\right) \supset R\left(A^k\right)$ by Proposition 168; hence $\dim R\left(A^{k-1}\right) > \dim R\left(A^k\right)$, which is a contradiction. Therefore L has index of nilpotency k. ■

Example 178 *Let us compute the Nilpotent-Nonsingular form of the singular matrix*

$$A = \begin{bmatrix} -2 & 0 & -4 \\ 4 & 2 & 4 \\ 3 & 2 & 2 \end{bmatrix}.$$

A basis for $N(A)$ *is* $\left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\}$ *and a basis for* $R(A)$ *is* $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\}$; *hence* $N(A) \cap R(A) = N(A) \neq \mathbf{0}$. *Squaring gives* $A^2 = \begin{bmatrix} -8 & -8 & 0 \\ 12 & 12 & 0 \\ 8 & 8 & 0 \end{bmatrix}$. *A basis for* $N\left(A^2\right)$ *is* $\mathbf{X} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ *and a basis for* $R\left(A^2\right)$ *is* $\mathbf{Y} = \left\{ \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} \right\}$. *Thus* $N\left(A^2\right) \cap R\left(A^2\right) = \mathbf{0}$ *and* A *has index* 2. *Let* $Q = [X \mid Y] = \begin{bmatrix} -1 & 0 & -2 \\ 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$; *then the Nilpotent-Nonsingular form of* A *is*

$$Q^{-1}AQ = \begin{bmatrix} -2 & 4 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

where $L = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$ *is nilpotent of index* 2 *and* $C = [2]$ *is nonsingular of rank* 1.

Exercise 179 *In Exercise 174 you computed the index of A and the corresponding Range-Nullspace decomposition of $\mathbb{R}^5$. Find the Nilpotent-Nonsingular form of A.*

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

11-24-12

Jordan Form of a Nilpotent Matrix

Schur's Triangularization Theorem tells us that every matrix complex A is unitarily similar to an upper triangular matrix T. However, the only thing certain at this point is that the the diagonal entries of T are the eigenvalues of A. The off-diagonal entries of T seem unpredictable and out of control. In the previous lecture we observed that singular matrix A of index k is similar to a block diagonal matrix

$$\begin{bmatrix} L & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix},$$

where L is nilpotent of index k and C is nonsingular with the same rank as A^k. Is it possible to simplify L and C via similarity transformations and obtain triangular matrices whose off-diagonal entries are predictable? The goal of this lecture is to do exactly this for nilpotent matrices.

Of course, the only nilpotent matrix of index 1 is the zero matrix, and the index of a non-zero nilpotent matrix is greater than 1.

Proposition 180 *An $n \times n$ nilpotent matrix L of index $k > 1$ has the following properties:*

1. *L has exactly one eigenvalue $\lambda = 0$.*
2. *The eigenvectors of L are the non-zero vectors in $N(L)$.*
3. *L is defective.*
4. *If P is an invertible $n \times n$ matrix, $P^{-1}LP$ has non-zero off-diagonal entries.*

Proof. (1) First note that $L^{k-1} \neq \mathbf{0}$ and $L^k = \mathbf{0}$ since L has index $k > 1$. Let λ be an eigenvalue of L and let $\mathbf{x}$ be a corresponding eigenvector. Then

$$\mathbf{0} = L^k\mathbf{x} = L^{k-1}(L\mathbf{x}) = L^{k-1}(\lambda\mathbf{x}) = \lambda L^{k-1}\mathbf{x} = \lambda L^{k-2}(L\mathbf{x}) = \lambda^2 L^{k-2}\mathbf{x} = \cdots = \lambda^k\mathbf{x}.$$

But $\mathbf{x} \neq \mathbf{0}$ implies $\lambda = 0$.

(2) Since $\lambda = 0$ is the only eigenvalue of L, we have $L - \lambda I = L$, and it follows that the eigenvectors of L are the non-zero vectors in $N(L)$.

(3) Suppose L is diagonalizable. Then there exists an invertible matrix P and a diagonal matrix $D = P^{-1}LPD$ whose diagonal entries are the eigenvalues of L. Since $\lambda = 0$ is its only eigenvalue, $D = \mathbf{0} = P^{-1}LP$, and solving for L gives $L = \mathbf{0}$. But $L \neq \mathbf{0}$; therefore L is defective.

(4) Since L is not-diagonalizable, $P^{-1}LP$ is a non-zero non-diagonal matrix, which has non-zero off-diagonal entries. ■

Definition 181 *Let L be an $n \times n$ nilpotent matrix of index k and define $L^0 = I$. Let $\mathbf{x} \in \mathbb{C}^n$ such that $\mathbf{y} = L^{k-1}\mathbf{x} \neq \mathbf{0}$ and let $1 \leq p \leq k$. The* ***Jordan chain on $\mathbf{y}$ of length*** *p is the set $\left\{L^{p-1}\mathbf{x}, \ldots, L^1\mathbf{x}, L^0\mathbf{x}\right\}$.*

Exercise 182 *Let L be an $n \times n$ nilpotent matrix of index k. If $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{y} = L^{k-1}\mathbf{x} \neq \mathbf{0}$, prove that the Jordan chain on $\mathbf{y}$ of length k is linearly independent.*

Example 183 *Consider the matrices*

$$L = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \quad L^2 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \textit{and} \quad L^3 = \mathbf{0}.$$

Thus L is nilpotent of index 3. The Jordan chain of length 3 on

$$\mathbf{y} = L^2\mathbf{e}_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

is the linearly independent set

$$\left\{ L^2\mathbf{e}_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \; L^1\mathbf{e}_3 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \; L^0\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Form the matrix

$$P = \left[L^2\mathbf{x} \mid L\mathbf{x} \mid \mathbf{x}\right] = \begin{bmatrix} 3 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \text{ then } P^{-1} = \frac{1}{9}\begin{bmatrix} 3 & -2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

and

$$J = P^{-1}LP = \frac{1}{9}\begin{bmatrix} 3 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & -2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

is the "Jordan form" of L.

When $L \neq \mathbf{0}$, the reduction algorithm presented in this lecture produces a triangular matrix $P^{-1}LP$ whose non-zero entries lie exclusively on the first superdiagonal. Since L is defective, a basis for $N(L)$ contains fewer than n eigenvectors of L. So to construct the matrix P, we must appropriately extend a basis for $N(L)$ to a basis for $\mathbb{C}^n$. This process has essentially two steps:

- Construct a somewhat special basis $\mathcal{B}$ for $N(L)$.
- Extend $\mathcal{B}$ to a basis for $\mathbb{C}^n$ by constructing Jordan chains on the elements of $\mathcal{B}$.

Definition 184 *A **nilpotent Jordan block** is a matrix of the form*

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

*A **nilpotent Jordan matrix** is a block diagonal matrix of the form*

$$\begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & J_m \end{bmatrix}, \tag{26}$$

where each J_i *is a nilpotent Jordan block.*

When the context is clear we refer to a nilpotent Jordan block or matrix as a Jordan block or matrix. Note that $[0]$ is a 1×1 Jordan block and the $n \times n$ zero matrix is a Jordan matrix with exactly n Jordan blocks.

Theorem 185 (Jordan Structure of a Nilpotent Matrix) *Every* $n \times n$ *nilpotent matrix* L *of index* $k > 1$ *is similar to an* $n \times n$ *Jordan matrix* J *with the following properties:*

1. *The number of Jordan blocks is the* $\operatorname{nullity}(L)$.
2. *The size of the largest Jordan block is* $k \times k$.
3. *For* $1 \leq j \leq k$, *the number of* $j \times j$ *Jordan blocks is* $\operatorname{rank}\left(L^{j-1}\right) - 2\operatorname{rank}\left(L^{j}\right) + \operatorname{rank}\left(L^{j+1}\right)$.
4. *The ordering of the Jordan blocks is arbitrary.*

Proof. The essential ingredients in the proof appear in the Reduction Algorithm below. However, note that (3) implies (1): The total number of Jordan blocks is

$$\sum_{j=1}^{k} \operatorname{rank}\left(L^{j-1}\right) - 2\operatorname{rank}\left(L^{j}\right) + \operatorname{rank}\left(L^{j+1}\right)$$

$$\begin{array}{rlll}
= & \operatorname{rank}(I) - & 2\operatorname{rank}(L) + & \operatorname{rank}\left(L^2\right) \\
+ & \operatorname{rank}(L) - & 2\operatorname{rank}\left(L^2\right) + & \vdots \\
+ & \operatorname{rank}\left(L^2\right) - & \vdots & \vdots \\
 & \vdots & \vdots & \vdots \\
+ & \vdots & \vdots & +\operatorname{rank}\left(L^{k-1}\right) \\
+ & \vdots & -2\operatorname{rank}\left(L^{k-1}\right) & \\
+ & \operatorname{rank}\left(L^{k-1}\right) & & \\
= & n - \operatorname{rank}(L) = \operatorname{nullity}(L). & &
\end{array}$$

■

Definition 186 *A* ***Jordan form of*** *a nilpotent matrix L is a Jordan matrix J similar to L. The* ***Jordan structure of*** *L is the number and size of the Jordan blocks in every Jordan form J of L.*

Example 187 *Continuing Example 183, note that*

$$L = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

has $\operatorname{nullity}(L) = 1$. *Therefore the Jordan form of L has exactly one Jordan block by Theorem 185, part (1), and the (unique) Jordan form of L is*

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since L has index 3, Theorem 185, part (2), confirms the fact that the largest (only) Jordan block has size 3×3, and Theorem 185, part (3), confirms the fact that the number of 3×3 Jordan blocks is

$$\operatorname{rank}\left(L^2\right) - 2\operatorname{rank}\left(L^3\right) + \operatorname{rank}\left(L^4\right) = 1 - 2(0) + 0 = 1.$$

Theorem 185 tells us that Jordan form is unique up to ordering of the blocks J_i. Indeed, given any prescribed ordering, there is a Jordan form whose Jordan blocks appear in that prescribed order.

Definition 188 *The* ***Jordan Canonical Form*** *(JCF)* ***of a nilpotent matrix*** *L is the Jordan form of L in which the Jordan blocks are distributed along the diagonal in order of decreasing size.*

Exercise 189 *Compute the Jordan structure of the following 6×6 nilpotent matrix and find its JCF:*

$$L = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -1 \\ 3 & 1 & 5 & 1 & -1 & 3 \\ -2 & -1 & 0 & 0 & -1 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -5 & -3 & -1 & -1 & -1 & -1 \\ -3 & -2 & -1 & -1 & 0 & -1 \end{bmatrix}.$$

This leaves us with the task of determining a nonsingular matrix P such that $P^{-1}LP$ is the JCF of L. As mentioned above, this process has essentially two steps. A detailed algorithm follows our next definition and two key exercises.

Definition 190 *Let E be any row-echelon form of a matrix A. Let $c_1 < c_2 < \cdots < c_q$ index the columns of E containing leading ones and let $\mathbf{y}_i$ denote the c_i^{th} column of A. The columns $\mathbf{y}_1, \ldots, \mathbf{y}_q$ are called the **basic columns** of A.*

Exercise 191 *Let $B = [\mathbf{b}_1 \mid \cdots \mid b_p]$ be an $n \times p$ matrix with linearly independent columns. Prove that multiplication by B preserves linear independence, i.e., if $\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$ is linearly independent in $\mathbb{C}^p$, then $\{B\mathbf{v}_1, \ldots, B\mathbf{v}_s\}$ is linearly independent in $\mathbb{C}^n$.*

Exercise 192 *Let L be a nilpotent matrix of index $k > 1$, let $1 \leq i \leq k-1$, and let $\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ be the basic columns of L^{i-1}. Form the matrix $B = [\mathbf{b}_1 \mid \cdots \mid \mathbf{b}_p]$ and prove that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$ be a basis for $N(LB)$, then $\{B\mathbf{v}_1, \ldots, B\mathbf{v}_s\}$ is a basis for $R\left(L^{i-1}\right) \cap N(L)$.*

Algorithm (Nilpotent Matrix Reduction to JCF)

Given a nilpotent matrix L of index k, let $\mathcal{S}_{k-1} = \{\mathbf{y}_1, \ldots, \mathbf{y}_q\}$ be the set of basic columns of L^{k-1}.

1 For $i = k-1$ to 1 step -1 :
If $\mathcal{S}_{k-1} \cup \cdots \cup \mathcal{S}_i$ is a basis for $N(L) \cap R\left(L^{i-1}\right)$, let $\mathcal{S}_{i-1} = \varnothing$ and go to step 1d.
Otherwise, extend $\mathcal{S}_{k-1} \cup \cdots \cup \mathcal{S}_i$ to a basis for $N(L) \cap R\left(L^{i-1}\right)$ as follows:

- (a) Form the matrix $B = [\mathbf{b}_1 \mid \cdots \mid \mathbf{b}_p]$ of basic columns of L^{i-1}.
- (b) Find a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$ for $N(LB)$; then $\{B\mathbf{v}_1, \ldots, B\mathbf{v}_s\}$ is a basis for $N(L) \cap R\left(L^{i-1}\right)$ (see Exercise 192).
- (c) The basic columns $\left\{\mathbf{y}_1, \ldots, \mathbf{y}_q, B\mathbf{v}_{\beta_1}, \ldots, B\mathbf{v}_{\beta_j}\right\}$ of the matrix $[\mathbf{y}_1 \mid \cdots \mid \mathbf{y}_q \mid B\mathbf{v}_1 \mid \cdots \mid B\mathbf{v}_s]$ form a basis for $R\left(L^{i-1}\right) \cap N(L)$ containing $\mathcal{S}_{k-1} \cup \cdots \cup \mathcal{S}_i$. Let

$$\mathcal{S}_{i-1} = \left\{B\mathbf{v}_{\beta_1}, \ldots, B\mathbf{v}_{\beta_j}\right\},$$

set $q := \#\left(\mathcal{S}_{k-1} \cup \cdots \cup \mathcal{S}_{i-1}\right)$, and let $\mathcal{S}_{k-1} \cup \cdots \cup \mathcal{S}_{i-1} = \{\mathbf{y}_1, \ldots, \mathbf{y}_q\}$.
- (d) Next i.

  Construct a matrix P such that $P^{-1}LP$ is the JCF of L as follows:

2 For $j = 1$ to q :

- (a) Let i be the positive integer such that $\mathbf{y}_j \in \mathcal{S}_i$. Find a particular solution $\mathbf{x}_j$ of $L^i\mathbf{x} = \mathbf{y}_j$.
- (b) Build the Jordan chain $\left\{L^i\mathbf{x}_j, \ldots, L\mathbf{x}_j, \mathbf{x}_j\right\}$ and form the matrix

$$P_j = \left[L^i\mathbf{x}_j \mid \cdots \mid L\mathbf{x}_j \mid \mathbf{x}_j\right].$$

- (c) Next j.

  Form the block matrix
$$P = [P_1 \mid \cdots \mid P_q].$$

Consider the Jordan chain $\left\{L^i\mathbf{x}_j, \ldots, L\mathbf{x}_j, \mathbf{x}_j\right\}$ in the Reduction Algorithm above. Since $L^i\mathbf{x}_j = \mathbf{y}_j \in N(L)$, the vector $\mathbf{y}_j$ is an eigenvector of L corresponding to $\lambda = 0$, and $L\left(L^{i-1}\mathbf{x}_j\right) = L^i\mathbf{x}_j = \mathbf{y}_j$ implies that $L^{i-1}\mathbf{x}_j$ is a generalized eigenvector of L. Similarly, $L\left(L^{i-2}\mathbf{x}_j\right) = L^{i-1}\mathbf{x}_j$ implies that $L^{i-2}\mathbf{x}_j$ is a generalized eigenvector of L, and inductively, $L^{i-r}\mathbf{x}_j$ is a generalized eigenvector of L for each $r = 1, 2, \ldots, i$. Thus we have proved:

Corollary 193 *The columns of P form a basis for $\mathbb{C}^n$ consisting of generalized eigenvectors of L.*

Now consider the special case of an $n \times n$ nilpotent matrix L of index $n > 1$. Since the size of the largest Jordan block is $n \times n$, by the Jordan Structure Theorem, there is exactly one Jordan block and JCF of L is

$$J = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

To construct the similarity transformation $P^{-1}LP = J$, note that $L^{n-1} \neq \mathbf{0}$ implies that there is a smallest positive integer i such that $\mathbf{y} = L^{n-1}\mathbf{e}_i \neq \mathbf{0}$, and furthermore,

$$\begin{aligned} L^{n-1}\mathbf{e}_i &\in N(L) \cap R\left(L^{n-1}\right) \\ L^{n-2}\mathbf{e}_i &\in N\left(L^2\right) \smallsetminus N(L) \\ &\vdots \\ L\mathbf{e}_i &\in N\left(L^{n-1}\right) \smallsetminus N\left(L^{n-2}\right) \\ \mathbf{e}_i &\in \mathbb{C}^n \smallsetminus N\left(L^{n-1}\right). \end{aligned}$$

Consequently, there are the following properly nested sequences of n subspaces, each with positive dimension:

$$\begin{array}{ccccccccc} N(L) & \subset & N\left(L^2\right) & \subset & \cdots & \subset & N\left(L^{n-1}\right) & \subset & \mathbb{C}^n \\ R\left(L^{n-1}\right) & \subset & R\left(L^{n-2}\right) & \subset & \cdots & \subset & R(L) & \subset & \mathbb{C}^n \end{array}$$

It follows that nullity $\left(L^i\right) = i$ and rank $\left(L^i\right) = n - i$ for $1 \leq i \leq n-1$. In particular, rank $\left(L^{n-1}\right) = 1$ implies L^{n-1} has exactly one basic column, namely $L^{n-1}\mathbf{e}_i = \mathbf{y}$; rank $\left(L^{n-1}\right) =$ nullity (L) together with $\mathbf{y} \in N(L) \cap R\left(L^{n-1}\right)$ implies $N(L) = R\left(L^{n-1}\right)$ and $N(L) \cap R\left(L^{i-1}\right) = N(L)$ for each i; and nullity $(L) = 1$ implies that $\mathcal{S}_{n-1} = \{\mathbf{y}\}$ is a basis for $N(L)$. Thus we proceed immediately to step 2 of the Reduction Algorithm. Since $\mathbf{e}_i$ is a particular solution of $L^{n-1}\mathbf{x} = \mathbf{y}$, we build the Jordan chain on $\mathbf{y}$ of length n and form the matrix $P = \left[L^{n-1}\mathbf{e}_i \mid \cdots \mid L\mathbf{e}_i \mid \mathbf{e}_i\right]$. Then $P^{-1}LP = J$. We summarize this discussion as a corollary:

Corollary 194 *Let L is an $n \times n$ nilpotent matrix of index $n > 1$, let i be the smallest positive integer such that $L^{n-1}\mathbf{e}_i \neq \mathbf{0}$, and let $P = \left[L^{n-1}\mathbf{e}_i \mid \cdots \mid L\mathbf{e}_i \mid \mathbf{e}_i\right]$. Then the JCF of L consists of one $n \times n$ Jordan block $J = P^{-1}LP$.*

Example 195 *Let*

$$L = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$L^2 = \begin{bmatrix} 0 & 0 & 4 & 17 \\ 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad L^3 = \begin{bmatrix} 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad L^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and L is nilpotent of index 4. Build the Jordan chain of length 4 on $L^3\mathbf{e}_4$:

$$\begin{aligned} L^3\mathbf{e}_4 &= \begin{bmatrix} 24 & 0 & 0 & 0 \end{bmatrix}^T \\ L^2\mathbf{e}_4 &= \begin{bmatrix} 17 & 24 & 0 & 0 \end{bmatrix}^T \\ L\mathbf{e}_4 &= \begin{bmatrix} 3 & 5 & 6 & 0 \end{bmatrix}^T \\ \mathbf{e}_4 &= \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T \end{aligned}$$

and form the matrix

$$P = \left[L^3\mathbf{e}_4|L^2\mathbf{e}_4|L\mathbf{e}_4|\mathbf{e}_4\right] = \begin{bmatrix} 24 & 17 & 3 & 0 \\ 0 & 24 & 5 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$J = P^{-1}LP = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example 196 *Let's construct the matrix P such that $P^{-1}LP$ is the JCF of the 6×6 matrix L in Exercise 189. Row-reducing we find that the* nullity $(L) = 3$:

$$L = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -1 \\ 3 & 1 & 5 & 1 & -1 & 3 \\ -2 & -1 & 0 & 0 & -1 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -5 & -3 & -1 & -1 & -1 & -1 \\ -3 & -2 & -1 & -1 & 0 & -1 \end{bmatrix} \xrightarrow{row\text{-}reduce} \begin{bmatrix} 1 & 0 & 0 & -\frac{2}{3} & \frac{4}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{4}{3} & -\frac{5}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Computing powers we find that the index of L is 3 :

$$L^2 = \begin{bmatrix} 6 & 3 & 3 & 1 & 1 & 2 \\ -6 & -3 & -3 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & -3 & -3 & -1 & -1 & -2 \\ -6 & -3 & -3 & -1 & -1 & -2 \end{bmatrix} \quad \textit{and} \quad L^3 = \mathbf{0}.$$

Since L has index 3, *we row-reduce L^2 and find that L^2 has one basic column:*

$$\mathbf{y}_1 = \begin{bmatrix} 6 & -6 & 0 & 0 & -6 & -6 \end{bmatrix}^T.$$

Let $\mathcal{S}_2 = \{\mathbf{y}_1\}$; since nullity $(L) = 3$, *$\mathcal{S}_2$ is not a basis for $N(L)$. Hence we proceed to step 1 of the Reduction Algorithm.*

1. *Extend $\mathcal{S}_2$ to a basis for $N(L) \cap R(L)$ as follows: Set $i := 2$.*

   (a) *By row-reducing L, we find that its basic columns are its first three columns; thus we form the matrix*

   $$B = \begin{bmatrix} 1 & 1 & -2 \\ 3 & 1 & 5 \\ -2 & -1 & 0 \\ 2 & 1 & 0 \\ -5 & -3 & -1 \\ -3 & -2 & -1 \end{bmatrix}.$$

   (b) *Next we obtain a basis for $N(LB)$ by solving $LB\mathbf{x} = 0$:*

   $$LB = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -1 \\ 3 & 1 & 5 & 1 & -1 & 3 \\ -2 & -1 & 0 & 0 & -1 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -5 & -3 & -1 & -1 & -1 & -1 \\ -3 & -2 & -1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 3 & 1 & 5 \\ -2 & -1 & 0 \\ 2 & 1 & 0 \\ -5 & -3 & -1 \\ -3 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 3 \\ -6 & -3 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -6 & -3 & -3 \\ -6 & -3 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 3 & 3 \\ -6 & -3 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -6 & -3 & -3 \\ -6 & -3 & -3 \end{bmatrix} \xrightarrow{row\text{-}reduce} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left\{ \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

Then $\left\{ B\mathbf{v}_1 = \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & -1 \end{bmatrix}^T, B\mathbf{v}_2 = \begin{bmatrix} -5 & 7 & 2 & -2 & 3 & 1 \end{bmatrix}^T \right\}$ *is a basis for* $N(L) \cap R(L)$.

(c) *Form the matrix*

$$[\mathbf{y}_1 \mid B\mathbf{v}_1 \mid B\mathbf{v}_2] = \begin{bmatrix} 6 & 1 & -5 \\ -6 & -1 & 7 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \\ -6 & -1 & 3 \\ -6 & -1 & 1 \end{bmatrix} \xrightarrow{row\text{-}reduce} \begin{bmatrix} 1 & \frac{1}{6} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since columns 1 and 3 are its basic columns, $\{\mathbf{y}_1, \mathbf{y}_2 = B\mathbf{v}_2\}$ *is a basis for* $N(L) \cap R(L)$ *containing* $\mathcal{S}_2$. *Let* $\mathcal{S}_1 = \{\mathbf{y}_2\}$.

(d) *Set* $i := i - 1$; *since* $i = 1$, *we return to step 1a.*

(a) *The basic columns of* L *are its first three columns. Extend* $\mathcal{S}_2 \cup \mathcal{S}_1 = \{\mathbf{y}_1, \mathbf{y}_2\}$ *to a basis for* $N(L) \cap R\left(L^0\right) = \mathbb{C}^6 \cap N(L) = N(L)$:

(b) *Since* $L^0 = I$, *we let* $B = I$.

(c) *Since* $LB = L$, *we find a basis for* $N(LB) = N(L)$:

$$\begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -1 \\ 3 & 1 & 5 & 1 & -1 & 3 \\ -2 & -1 & 0 & 0 & -1 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -5 & -3 & -1 & -1 & -1 & -1 \\ -3 & -2 & -1 & -1 & 0 & -1 \end{bmatrix} \xrightarrow{row\text{-}reduce} \begin{bmatrix} 1 & 0 & 0 & -\frac{2}{3} & \frac{4}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{4}{3} & -\frac{5}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left\{ \mathbf{v}_1 = \begin{bmatrix} 2 \\ -4 \\ -1 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -4 \\ 5 \\ 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \ v_3 = \begin{bmatrix} 1 \\ -2 \\ -2 \\ 0 \\ 0 \\ 3 \end{bmatrix} \right\}.$$

Then a basis for $N(L)$ *is*

$$\{B\mathbf{v}_1, B\mathbf{v}_2, B\mathbf{v}_3\} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}.$$

(d) *Form the matrix*

$$[\mathbf{y}_1 \mid \mathbf{y}_2 \mid \mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3] = \begin{bmatrix} 6 & -5 & 2 & -4 & 1 \\ -6 & 7 & -4 & 5 & -2 \\ 0 & 2 & -1 & 2 & -2 \\ 0 & -2 & 3 & 0 & 0 \\ -6 & 3 & 0 & 3 & 0 \\ -6 & 1 & 0 & 0 & 3 \end{bmatrix} \xrightarrow{row\text{-}reduce} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{4} & -\frac{3}{4} \\ 0 & 1 & 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

its basic columns are columns 1,2, and 3, hence $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{v}_1\}$ *is the desired basis for* $N(L)$ *containing* $\mathcal{S}_2 \cup \mathcal{S}_1$. *Let* $\mathcal{S}_0 = \{\mathbf{v}_1\}$ *and set* $i := i - 1$; *then* $i = 0$ *and the process terminates having produced the basis* $\mathcal{S}_2 \cup \mathcal{S}_1 \cup \mathcal{S}_0 = \{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 = \mathbf{v}_1\}$ *with* $\mathbf{y}_1 \in \mathcal{S}_2$, $\mathbf{y}_2 \in \mathcal{S}_1$, *and* $\mathbf{y}_3 \in \mathcal{S}_0$.

2. *For each $j = 1, 2, 3$, build a Jordan chain on $\mathbf{y}_j \in \mathcal{S}_i$ of length $i+1$ by finding a particular solution $\mathbf{x}_j$ of $L^i\mathbf{x} = \mathbf{y}_j$.*

   (a) *Since $\mathbf{y}_1 \in \mathcal{S}_2$, we see by inspection that $L^2\mathbf{e}_1 = \mathbf{y}_1$. Thus*

$$P_1 = \left[L^2\mathbf{e}_1 \mid L\mathbf{e}_1 \mid \mathbf{e}_1\right] = \begin{bmatrix} 6 & 1 & 1 \\ -6 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 2 & 0 \\ -6 & -5 & 0 \\ -6 & -3 & 0 \end{bmatrix}.$$

Since $\mathbf{y}_2 \in \mathcal{S}_1$, we solve $L\mathbf{x} = \mathbf{y}_2$:

$$\left[\begin{array}{rrrrrr|r} 1 & 1 & -2 & 0 & 1 & -1 & -5 \\ 3 & 1 & 5 & 1 & -1 & 3 & 7 \\ -2 & -1 & 0 & 0 & -1 & 0 & 2 \\ 2 & 1 & 0 & 0 & 1 & 0 & -2 \\ -5 & -3 & -1 & -1 & -1 & -1 & 3 \\ -3 & -2 & -1 & -1 & 0 & -1 & 1 \end{array}\right] \xrightarrow{\text{row-reduce}} \left[\begin{array}{rrrrrr|r} 1 & 0 & 0 & -\frac{2}{3} & \frac{4}{3} & -\frac{1}{3} & -1 \\ 0 & 1 & 0 & \frac{4}{3} & -\frac{5}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

A particular solution is $\mathbf{x}_2 = \begin{bmatrix} -1 & 0 & 2 & 0 & 0 & 0 \end{bmatrix}^T$; hence

$$P_2 = [L\mathbf{x}_2 \mid \mathbf{x}_2] = \begin{bmatrix} -5 & -1 \\ 7 & 0 \\ 2 & 2 \\ -2 & 0 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}.$$

Since $\mathbf{y}_3 \in \mathcal{S}_0$ and $L^0 = I$, the unique solution of $L^0\mathbf{x} = \mathbf{y}_3$ is $\mathbf{x} = \mathbf{y}_3$. Hence the Jordan chain on $\mathbf{y}_3$ consists only of $\mathbf{y}_3$ and

$$P_3 = \begin{bmatrix} 2 \\ -4 \\ -1 \\ 3 \\ 0 \\ 0 \end{bmatrix}.$$

   (b) *Form the matrix*

$$P = [P_1 \mid P_2 \mid P_3] = \begin{bmatrix} 6 & 1 & 1 & -5 & -1 & 2 \\ -6 & 3 & 0 & 7 & 0 & -4 \\ 0 & -2 & 0 & 2 & 2 & -1 \\ 0 & 2 & 0 & -2 & 0 & 3 \\ -6 & -5 & 0 & 3 & 0 & 0 \\ -6 & -3 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Then

$$J = P^{-1}LP = \left[\begin{array}{ccc|cc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Exercise 197 *An unreduced Hessenberg matrix H and a Jordan matrix J appear below. Find an invertible matrix P such that $J = P^{-1}HP$. (Note: Some texts define the Jordan form with ones below the main diagonal as in H.)*

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \;;\; J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Exercise 198 *Prove that the Jordan matrices*

$$J_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are not similar. (Hint: Show that if $P = (p_{ij})$ is a 4×4 matrix such that $PJ_1 = J_2P$, then P is not invertible.)

Exercise 199 *For each of the matrices below:*

- *Show that L is nilpotent and determine its index.*
- *Find the JCF J of L.*
- *Find an invertible matrix P such that $J = P^{-1}LP$.*

(a)

$$L = \begin{bmatrix} 3 & 3 & 2 & 1 \\ -2 & -1 & -1 & -1 \\ 1 & -1 & 0 & 1 \\ -5 & -4 & -3 & -2 \end{bmatrix}$$

(b)

$$L = \begin{bmatrix} 2 & 1 & 2 & 0 & -1 \\ 3 & 1 & 3 & -1 & 1 \\ -3 & -1 & -2 & 0 & 2 \\ 3 & 2 & 4 & 0 & -1 \\ 2 & 1 & 2 & 0 & -1 \end{bmatrix}$$

(c)

$$L = \begin{bmatrix} 41 & 30 & 15 & 7 & 4 & 6 & 1 & 3 \\ -54 & -39 & -19 & -9 & -6 & -8 & -2 & -4 \\ 9 & 6 & 2 & 1 & 2 & 1 & 0 & 1 \\ -6 & -5 & -3 & -2 & 1 & -1 & 0 & 0 \\ -32 & -24 & -13 & -6 & -2 & -5 & -1 & -2 \\ -10 & -7 & -2 & 0 & -3 & 0 & 3 & -2 \\ -4 & -3 & -2 & -1 & 0 & -1 & -1 & 0 \\ 17 & 12 & 6 & 3 & 2 & 3 & 2 & 1 \end{bmatrix}$$

12-4-12

Recall that if A is a singular matrix of index k and $r = \operatorname{rank}\left(A^k\right)$, then the Nilpotent-Nonsingular form of A is a block diagonal matrix

$$Q^{-1}AQ = \begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & L \end{bmatrix},$$

where C is nonsingular of rank r and L is nilpotent of index k. In the previous lecture we determined the Jordan structure of L and discussed an algorithm for constructing a similar Jordan matrix, which is a block diagonal matrix whose diagonal blocks are Jordan blocks with ones along the first super-diagonal and zeros elsewhere. In this lecture we'll determine the Jordan structure of a general square matrix A and observe that A is similar to a Jordan matrix whose diagonal entries are the eigenvalues of A, whose first super-diagonal entries are ones or zeros, and whose other entries are zero. Since similar matrices have the same Jordan structure, square matrices are classified up to similarity by their Jordan structure. We begin with some notation and terminology.

Definition 200 *Let A be an $n \times n$ matrix A. the function $\operatorname{ind}_A : \mathbb{C} \to \mathbb{N} \cup \{0\}$, called the **index function of** A, is defined by*

$$\operatorname{ind}_A(z) = \operatorname{index}(A - zI).$$

Since the index of a nonsingular matrix is defined to be 0 and the index of a singular matrix is positive, $\operatorname{ind}_A(z) > 0$ if and only if z is an eigenvalue of A. Thus from our discussion of the Range-Nullspace Decomposition of $\mathbb{C}^n$ we have:

Proposition 201 *Let λ be an eigenvalue of an $n \times n$ matrix A and let $k = \operatorname{ind}_A(\lambda)$. Then k is the smallest positive integer such that*

1. $R\left[(A - \lambda I)^k\right] = R\left[(A - \lambda I)^{k+1}\right]$
2. $N\left[(A - \lambda I)^k\right] = N\left[(A - \lambda I)^{k+1}\right]$
3. $R\left[(A - \lambda I)^k\right] \cap N\left[(A - \lambda I)^k\right] = \{\mathbf{0}\}$
4. $\mathbb{C}^n = N\left[(A - \lambda I)^k\right] \oplus R\left[(A - \lambda I)^k\right]$.

Let $A = A_0$ be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_s$, let $k_i = \operatorname{ind}_A(\lambda_i)$, and let $r_p(\lambda_i) = \operatorname{rank}(A - \lambda_i I)^p$. The Nilpotent-Nonsingular form of $A_0 - \lambda_1 I$ produces a nonsingular matrix Q_1 such that

$$Q_1^{-1}(A_0 - \lambda_1 I)\, Q_1 = \begin{bmatrix} L_1 & \mathbf{0} \\ \mathbf{0} & C_1 \end{bmatrix},$$

where L_1 is nilpotent of index k_1 and C_1 is nonsingular of rank $r_{k_1}(\lambda_1)$. Let $J(\lambda_1)$ denote the JCF of L_1; there is a nonsingular matrix X_1 such that

$$X_1^{-1} L_1 X_1 = J(\lambda_1) = \begin{bmatrix} J_1(\lambda_1) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & J_2(\lambda_1) & & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & J_{t_1}(\lambda_1) \end{bmatrix}.$$

Since C_1 is nonsingular, so are its powers, and for each $p \geq 1$ we have

$$r_p(\lambda_1) = \operatorname{rank}\left(\begin{bmatrix} L_1 & \mathbf{0} \\ \mathbf{0} & C_1 \end{bmatrix}^p\right) = \operatorname{rank}\left(\begin{bmatrix} L_1^p & \mathbf{0} \\ \mathbf{0} & C_1^p \end{bmatrix}\right)$$

$$= \operatorname{rank}(L_1^p) + \operatorname{rank}(C_1),$$

or equivalently,

$$\operatorname{rank}\left(L_1^p\right) = r_p\left(\lambda_1\right) - \operatorname{rank}\left(C_1\right).$$

Therefore the number of $j \times j$ blocks in $J\left(\lambda_1\right)$ is

$$\begin{aligned}
v_j\left(\lambda_1\right) &= \operatorname{rank}\left(L_1^{j-1}\right) - 2\operatorname{rank}\left(L_1^{j}\right) + \operatorname{rank}\left(L_1^{j+1}\right) \\
&= \left[r_{j-1}\left(\lambda_1\right) - \operatorname{rank}\left(C_1\right)\right] - 2\left[r_j\left(\lambda_1\right) + \operatorname{rank}\left(C_1\right)\right] + \left[r_{j+1}\left(\lambda_1\right) - \operatorname{rank}\left(C_1\right)\right] \\
&= r_{j-1}\left(\lambda_1\right) - 2r_j\left(\lambda_1\right) + r_{j+1}\left(\lambda_1\right).
\end{aligned}$$

Let

$$Y_1 = Q_1 \begin{bmatrix} X_1 & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}.$$

Then Y_1 is nonsingular and

$$Y_1^{-1} = \begin{bmatrix} X_1^{-1} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} Q_1^{-1}.$$

Therefore

$$\begin{aligned}
Y_1^{-1} A_0 Y_1 - \lambda_1 I &= Y_1^{-1}\left(A_0 - \lambda_1 I\right) Y_1 \\
&= \begin{bmatrix} X_1^{-1} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} Q_1^{-1}\left(A - \lambda_1 I\right) Q_1 \begin{bmatrix} X_1 & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \\
&= \begin{bmatrix} X_1^{-1} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} C_1 & \mathbf{0} \\ \mathbf{0} & L_1 \end{bmatrix} \begin{bmatrix} X_1 & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \\
&= \begin{bmatrix} X_1^{-1} L_1 X_1 & \mathbf{0} \\ \mathbf{0} & C_1 \end{bmatrix} = \begin{bmatrix} J\left(\lambda_1\right) & \mathbf{0} \\ \mathbf{0} & C_1 \end{bmatrix}
\end{aligned}$$

so that

$$Y_1^{-1} A_0 Y_1 = \begin{bmatrix} J\left(\lambda_1\right) & \mathbf{0} \\ \mathbf{0} & C_1 \end{bmatrix} + \lambda_1 I = \begin{bmatrix} J\left(\lambda_1\right) + \lambda_1 I & \mathbf{0} \\ \mathbf{0} & C_1 + \lambda_1 I \end{bmatrix} = \begin{bmatrix} \mathcal{J}\left(\lambda_1\right) & \mathbf{0} \\ \mathbf{0} & A_1 \end{bmatrix},$$

where

$$\mathcal{J}\left(\lambda_1\right) = \begin{bmatrix} \mathcal{J}_1\left(\lambda_1\right) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathcal{J}_2\left(\lambda_1\right) & & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathcal{J}_{t_1}\left(\lambda_1\right) \end{bmatrix}$$

and

$$\mathcal{J}_i\left(\lambda_1\right) = J_i\left(\lambda_1\right) + \lambda_1 I = \begin{bmatrix} \lambda_1 & 1 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 1 & & 0 \\ 0 & 0 & \lambda_1 & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_1 \end{bmatrix}.$$

Thus the Jordan structure of $\mathcal{J}\left(\lambda_1\right)$ is inherited directly from the Jordan structure of L_1.

Lemma 202 *Let A be a square matrix, let λ_1 be an eigenvalue of A, and consider the Nilpotent-Nonsingular form*

$$Q^{-1}\left(A - \lambda_1 I\right) Q = \begin{bmatrix} L & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix}. \tag{27}$$

If the distinct eigenvalues of A are $\lambda_1, \ldots, \lambda_s$, then the distinct eigenvalues of $C + \lambda_1 I$ are $\lambda_2, \ldots, \lambda_s$.

Proof. Given an eigenvalue λ_1 of A, note that

$$\left(A - \lambda_1 I\right)\mathbf{x} = \lambda \mathbf{x} \text{ if and only if } A\mathbf{x} = \left(\lambda_1 + \lambda\right)\mathbf{x}.$$

Thus λ is an eigenvalue of $A-\lambda_1 I$ if and only if $\lambda_1+\lambda$ is an eigenvalue of A. Consequently, if $\lambda_1,\ldots,\lambda_s$ are the distinct eigenvalues of A, then $\lambda_1+\lambda\in\{\lambda_1,\ldots,\lambda_s\}$ if and only if $\lambda\in\{0,\lambda_2-\lambda_1,\lambda_3-\lambda_1,\ldots,\lambda_s-\lambda_1\}$ and it follows that the distinct eigenvalues of $A-\lambda_1 I$ and its Nilpotent-Nonsingular form (27) are $0,\lambda_2-\lambda_1,\lambda_3-\lambda_1,\ldots,\lambda_s-\lambda_1$. But L has exactly one eigenvalue $\lambda=0$ and every eigenvalue of C is non-zero (C is singular otherwise), therefore the distinct eigenvalues of C are exactly $\lambda_2-\lambda_1,\lambda_3-\lambda_1,\ldots,\lambda_s-\lambda_1$. Furthermore,

$$C\mathbf{x}=(\lambda-\lambda_1)\,\mathbf{x} \text{ if and only if } (C+\lambda_1 I)\,\mathbf{x}=\lambda\mathbf{x}$$

so that $\lambda-\lambda_1$ is an eigenvalue of C if and only if λ is an eigenvalue of $C+\lambda_1 I$. Consequently, the distinct eigenvalues of $C+\lambda_1 I$ are exactly $\lambda_2,\ldots,\lambda_s$. ■

Now continuing the discussion above, apply Lemma 202 to the matrix A_0 with distinct eigenvalues $\lambda_1,\ldots,\lambda_s$ and obtain the matrix $A_1=C+\lambda_1 I$ with distinct eigenvalues $\lambda_2,\ldots,\lambda_s$. Continue inductively: Find the Nilpotent-Nonsingular form of $A_1-\lambda_2 I$. Then

$$Q_2^{-1}(A_1-\lambda_2 I)\,Q_2=\begin{bmatrix} L_2 & \mathbf{0}\\ \mathbf{0} & C_2\end{bmatrix},$$

$$X_2^{-1}L_2X_2=J(\lambda_2)=\begin{bmatrix} J_1(\lambda_2) & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & J_2(\lambda_2) & & \mathbf{0}\\ \vdots & & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & J_{t_2}(\lambda_2)\end{bmatrix}.$$

Let

$$Y_2=Q_2\begin{bmatrix} X_2 & \mathbf{0}\\ \mathbf{0} & I\end{bmatrix},\ Y_2^{-1}=\begin{bmatrix} X_2^{-1} & \mathbf{0}\\ \mathbf{0} & I\end{bmatrix}Q_2^{-1};$$

then

$$Y_2^{-1}A_1Y_2-\lambda_2 I=\begin{bmatrix} X_2^{-1}L_2X_2 & \mathbf{0}\\ \mathbf{0} & C_2\end{bmatrix}=\begin{bmatrix} J(\lambda_2) & \mathbf{0}\\ \mathbf{0} & C_2\end{bmatrix}$$

and

$$Y_2^{-1}A_1Y_2=\begin{bmatrix} J(\lambda_2)+\lambda_2 I & \mathbf{0}\\ \mathbf{0} & C_2+\lambda_2 I\end{bmatrix}=\begin{bmatrix} \mathcal{J}(\lambda_2) & \mathbf{0}\\ \mathbf{0} & A_2\end{bmatrix},$$

where the distinct eigenvalues of A_2 are $\{\lambda_3,\ldots,\lambda_s\}$,

$$\mathcal{J}(\lambda_2)=\begin{bmatrix} \mathcal{J}_1(\lambda_2) & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \mathcal{J}_2(\lambda_2) & & \mathbf{0}\\ \vdots & & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & \mathcal{J}_{t_2}(\lambda_2)\end{bmatrix} \text{ and } \mathcal{J}_i(\lambda_2)=\begin{bmatrix} \lambda_2 & 1 & 0 & \cdots & 0\\ 0 & \lambda_2 & 1 & & 0\\ 0 & 0 & \lambda_2 & \ddots & \vdots\\ \vdots & \vdots & & \ddots & 1\\ 0 & 0 & 0 & \cdots & \lambda_2\end{bmatrix}.$$

Continue in this manner until all eigenvalues have been digested and obtain the sequence of matrices $\{Y_1,Y_2,\ldots,Y_s\}$, where

$$Y_i^{-1}A_{i-1}Y_i=\begin{cases}\begin{bmatrix} \mathcal{J}(\lambda_i) & 0\\ 0 & A_i\end{bmatrix}, & 1\le i\le s-1\\[2ex] \begin{bmatrix} \mathcal{J}(\lambda_i)\end{bmatrix}, & i=s\end{cases}$$

Let $P_1=Y_1$; then

$$P_1^{-1}AP_1=\begin{bmatrix} \mathcal{J}(\lambda_1) & \mathbf{0}\\ \mathbf{0} & A_1\end{bmatrix}.$$

For $i=2,3,\ldots,s$, let

$$P_i=P_{i-1}\begin{bmatrix} I & 0\\ 0 & Y_i\end{bmatrix};$$

then P_i is nonsingular and

$$P_i^{-1}AP_i = \begin{cases} \begin{bmatrix} \mathcal{J}(\lambda_1) & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathcal{J}(\lambda_2) & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathcal{J}(\lambda_i) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & A_i \end{bmatrix}, & 1 \leq i \leq s-1 \\ \begin{bmatrix} \mathcal{J}(\lambda_1) & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathcal{J}(\lambda_s) \end{bmatrix}, & i = s. \end{cases}.$$

Finally,

$$P_s = Y_1 \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & Y_2 \end{bmatrix} \cdots \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & Y_s \end{bmatrix} = Q_1 \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & Q_2 \end{bmatrix} \cdots \begin{bmatrix} I & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & I & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & Q_s \end{bmatrix} \begin{bmatrix} X_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & X_2 & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & X_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & I \end{bmatrix}$$

and

$$\mathcal{J} = P_s^{-1}AP_s = \begin{bmatrix} \mathcal{J}(\lambda_1) & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathcal{J}(\lambda_s) \end{bmatrix}.$$

Definition 203 *The matrix $\mathcal{J}$ is called a* ***Jordan Form*** *of A. The matrix $\mathcal{J}(\lambda_j)$ is called the* ***Jordan segment associated with*** *λ_j; its individual blocks $\mathcal{J}_i(\lambda_j)$ are called the* ***Jordan blocks associated with*** *λ_j. The* ***Jordan structure of*** *A is the number of Jordan segments in $\mathcal{J}$ and the number and sizes of the Jordan blocks in each segment.*

To summarize:

Theorem 204 *Every $n \times n$ matrix A with distinct eigenvalues $\{\lambda_1, \dots, \lambda_s\}$ is similar to a Jordan matrix*

$$\mathcal{J} = \begin{bmatrix} \mathcal{J}(\lambda_1) & \mathbf{0} & \cdots & \mathbf{0} \\ 0 & \mathcal{J}(\lambda_2) & \ddots & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathcal{J}(\lambda_s) \end{bmatrix},$$

where

$$\mathcal{J}(\lambda_j) = \begin{bmatrix} \mathcal{J}_1(\lambda_j) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathcal{J}_2(\lambda_j) & & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathcal{J}_{t_j}(\lambda_j) \end{bmatrix}, \quad \mathcal{J}_i(\lambda_j) = \begin{bmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ 0 & \lambda_j & 1 & & 0 \\ 0 & 0 & \lambda_j & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_j \end{bmatrix}$$

and

- *the number t_j of Jordan blocks in $\mathcal{J}(\lambda_j)$ is the* $\text{nullity}(A - \lambda_j I)$;
- *the size of the largest Jordan block in $\mathcal{J}(\lambda_j)$ is $k_j \times k_j$, where $k_j = \text{ind}_A(\lambda_j)$;*
- *the number of $i \times i$ Jordan blocks in $\mathcal{J}(\lambda_j)$ is*

$$v_i(\lambda_j) = r_{i-1}(\lambda_j) - 2r_i(\lambda_j) + r_{i+1}(\lambda_j).$$

- *The Jordan structure of A is uniquely determined by the entries of A.*
- *Jordan Form is unique up to order of Jordan segments.*
- *Two matrices are similar if and only if they have the same Jordan structure.*

Let A be an $n \times n$ matrix, and let λ be an eigenvalue of A. If $A - \lambda I$ is nilpotent of index n, there is a Jordan matrix J and an invertible matrix P such that

$$J = P^{-1}(A - \lambda I)P = P^{-1}AP - \lambda I,$$

and the Jordan form of A is

$$\mathcal{J}(\lambda) = P^{-1}AP = J + \lambda I = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}.$$

Example 205 *Consider the 4×4 matrix*

$$A = \begin{bmatrix} 7 & 1 & 2 & 3 \\ 0 & 7 & 4 & 5 \\ 0 & 0 & 7 & 6 \\ 0 & 0 & 0 & 7 \end{bmatrix},$$

which has a single eigenvalue $\lambda = 7$ with algebraic multiplicity 4. Let

$$L = A - 7I = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In Example 195, we computed the JCF

$$J = P^{-1}LP = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ where } P = \begin{bmatrix} 24 & 17 & 3 & 0 \\ 0 & 24 & 5 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The Jordan form of A, namely $\mathcal{J}(7) = P^{-1}AP$, is related to J in the following way:

$$J = P^{-1}LP = P^{-1}(A - 7I)P = P^{-1}AP - 7I = \mathcal{J}(7) - 7I$$

so that

$$\mathcal{J}(7) = J + 7I = \begin{bmatrix} 7 & 1 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix}.$$

Example 206 *The matrix*

$$A = \begin{bmatrix} 5 & 4 & 0 & 0 & 4 & 3 \\ 2 & 3 & 1 & 0 & 5 & 1 \\ 0 & -1 & 2 & 0 & 2 & 0 \\ -8 & -8 & -1 & 2 & -12 & -7 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -8 & -8 & -1 & 0 & -9 & -5 \end{bmatrix},$$

has two distinct eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$. Hence a Jordan Form $\mathcal{J}$ of A has two Jordan segments $\mathcal{J}(2)$ and $\mathcal{J}(-1)$:

$$\mathcal{J} = \begin{bmatrix} \mathcal{J}(2) & \mathbf{0} \\ \mathbf{0} & \mathcal{J}(-1) \end{bmatrix}.$$

To determine $\mathcal{J}(2)$ and $\mathcal{J}(-1)$, compute $r_i(2)$ until $r_k(2) = r_{k+1}(2)$ and $r_i(-1)$ until $r_l(-1) = r_{l+1}(-1)$:

$$\begin{array}{llll} r_0(2) = \operatorname{rank}(A-2I)^0 = & 6 & r_0(-1) = \operatorname{rank}(A+I)^0 = & 6 \\ r_1(2) = \operatorname{rank}(A-2I)^1 = & 4 & r_1(-1) = \operatorname{rank}(A+I)^1 = & 4 \\ r_2(2) = \operatorname{rank}(A-2I)^2 = & 3 & r_2(-1) = \operatorname{rank}(A+I)^2 = & 4 \\ r_3(2) = \operatorname{rank}(A-2I)^3 = & 2 & & \\ r_4(2) = \operatorname{rank}(A-2I)^4 = & 2 & & \end{array}$$

Thus $\operatorname{ind}_A(2) = 3$ and $\operatorname{ind}_A(-1) = 1$ implies that the largest Jordan block in $\mathcal{J}(2)$ is a 3×3 and the largest in $\mathcal{J}(-1)$ is a 1×1 (a diagonal matrix). Furthermore, the numbers of $i \times i$ Jordan blocks are

$$\begin{array}{ll} v_3(2) & = r_2(2) - 2r_3(2) + r_4(4) = 1 \\ v_2(2) & = r_1(2) - 2r_2(2) + r_3(4) = 0 \\ v_1(2) & = r_0(2) - 2r_1(2) + r_2(4) = 1 \\ & \\ v_1(-1) & = r_0(-1) - 2r_1(-1) + r_2(-1) \quad = 2 \end{array}.$$

Thus

$$\mathcal{J} = \left[\begin{array}{cccc||cc} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ \hline\hline 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

is a Jordan Form of A.

Example 207 *Since the matrix*

$$A = \begin{bmatrix} 3 & 0 & 1 \\ -4 & 1 & -2 \\ -4 & 0 & -1 \end{bmatrix}$$

has a single eigenvalue $\lambda = 1$, its Jordan Form has one Jordan segment $\mathcal{J}(1)$. Let's find the Jordan Form $\mathcal{J}$ of A and a similarity transformation $P^{-1}AP = \mathcal{J}$. Let

$$L = A - I = \begin{bmatrix} 2 & 0 & 1 \\ -4 & 0 & -2 \\ -4 & 0 & -2 \end{bmatrix};$$

then $L^2 = 0$ so L is nilpotent of index 2. Note that L is its own Nilpotent-Nonsingular form (there no nonsingular part). Compute the JCF of L. Row reducing L gives

$$\begin{bmatrix} 2 & 0 & 1 \\ -4 & 0 & -2 \\ -4 & 0 & -2 \end{bmatrix} \xrightarrow{row\text{-}reduce} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so $\operatorname{nullity}(L) = 2$ and there are two Jordan blocks in the JCF of L the largest of which is 2×2. Since L is 3×3, there is exactly one 2×2 and one 1×1 block. Thus the JCF of L is

$$J(1) = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

Next let's a similarity transformation $P^{-1}LP = J$. Following the Reduction Algorithm, set $i := 1$ and let $\mathcal{S}_1 = \{\mathbf{y}_1\} = \left\{[2, -4, -4]^T\right\}$ be the basic column of L.

1. *Extend* $\mathcal{S}_1$ *to a basis for* $R\left(L^0\right) \cap N(L) = \mathbb{C}^3 \cap N(L) = N(L)$.

   (a) $L^0 = I$ *has three basic columns* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$; *hence* $B = [\mathbf{e}_1|\mathbf{e}_2|\mathbf{e}_3] = I$.

   (b) *Solve* $LB\mathbf{x} = L\mathbf{x} = 0$ *and obtain the basis* $\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{[0,1,0]^T, \left[-\frac{1}{2}, 0, 1\right]^T\right\}$ *for* $N(L)$.

   (c) *Form the matrix*

$$[\mathbf{y}_1|\mathbf{v}_1|\mathbf{v}_2] = \begin{bmatrix} 2 & 0 & -\frac{1}{2} \\ -4 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \xrightarrow{row\text{-}reduce} \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

   The basic columns are $\left\{[2,-4,-4]^T, [0,1,0]^T\right\}$ *so* $\mathcal{S}_0 = \left\{[0,1,0]^T\right\}$.

   (d) $\mathcal{S}_1 \cup \mathcal{S}_0 = \{\mathbf{b}_1, \mathbf{b}_2\} = \left\{[2,-4,-4]^T, [0,1,0]^T\right\}$ *is the desired basis for* $N(L)$.

2. *Now* $\mathbf{e}_1$ *is a particular solution of* $L\mathbf{x} = [2,-4,-4]^T$. *Since* $\mathbf{e}_1 \in \mathcal{S}_1$, *we build the Jordan chain on* $\mathbf{e}_1$ *of length* 2 *and form the matrix*

$$P_1 = [L\mathbf{e}_1|\mathbf{e}_1] = \begin{bmatrix} 2 & 1 \\ -4 & 0 \\ -4 & 0 \end{bmatrix}.$$

   Furthermore, $\mathbf{e}_2$ *is a particular solution of* $L^0\mathbf{x} = I\mathbf{x} = [0,1,0]^T$. *Since* $\mathbf{e}_2 \in \mathcal{S}_0$, *there we set* $P_2 = \mathbf{e}_2$. *Then*

$$P = [P_1|P_2] = \begin{bmatrix} 2 & 1 & 0 \\ -4 & 0 & 1 \\ -4 & 0 & 0 \end{bmatrix} \quad \text{and} \quad J(1) = P^{-1}LP = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right].$$

   But $J(1) = P^{-1}LP = P^{-1}(A-I)P = P^{-1}AP - I$. *Therefore*

$$\mathcal{J} = P^{-1}AP = J(1) + I = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right].$$

Example 208 *The matrix*

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 1 & -1 & 3 & 0 & 0 \\ 3 & -3 & 3 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 \end{bmatrix}$$

has eigenvalues of A *are* $\lambda_1 = 2$, $\lambda_2 = -1+i$ *and* $\lambda_2 = -1-i$. *Thus a Jordan Form* $\mathcal{J}$ *of* A *has three Jordan segments* $\mathcal{J}(2)$, $\mathcal{J}(-1+i)$, *and* $\mathcal{J}(-1-i)$. *To find a nonsingular matrix* P *such that* $P^{-1}AP = \mathcal{J}$, *digest the eigenvalues of* A *one at a time, beginning with* $\lambda_1 = 2$. *The matrix* $A - 2I$ *has index 3, and*

$$(A-2I)^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 18 & -18 & 18 & -18 & 26 \\ 26 & -26 & 26 & -26 & -18 \end{bmatrix} \xrightarrow{row\ reduce} \begin{bmatrix} 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first and last columns of $(A-2I)^3$ *form a basis for its column space and one can read off a basis for its nullspace from the row reduction. Using these basis vectors as columns, form the matrix*

$$Q = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 9 & 13 \\ 0 & 0 & 0 & 13 & -9 \end{bmatrix};$$

then the Nilpotent-Nonsingular form of $A - 2I$ *is*

$$Q^{-1}(A-2I)Q = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -1 & -3 \end{array}\right] = \begin{bmatrix} J(2) & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix}.$$

Quite fortunately, the 3×3 *nilpotent block* $J(2)$ *is in JCF. Since*

$$Q^{-1}(A-2I)Q = Q^{-1}AQ - 2I$$

we have

$$Q^{-1}AQ = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -1 & -3 \end{array}\right] + \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} = \left[\begin{array}{ccc|cc} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right].$$

Now the eigenvalues of the nonsingular block

$$A_1 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

are $\lambda_2 = -1 + i$ *and* $\lambda_3 = -1 - i$*. Let's find bases for the associated eigenspaces. For* $\lambda_2 = -1 + i$ *we have*

$$A_1 - (-1+i)I = A_1 + (1-i)I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \xrightarrow{row\ reduce} \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix},$$

so $\left\{\left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right]\right\}$ *is a basis for the associated eigenspace. Similarly,* $\left\{\left[\begin{smallmatrix} -i \\ 1 \end{smallmatrix}\right]\right\}$ *is a basis for the eigenspace associated with* $\lambda_3 = -1 - i$*. Consequently, neither eigenvalue is deficient and* A_1 *is diagonalizable. Let* $X = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$*. Then*

$$X^{-1}A_1X = \begin{bmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1+i & 0 \\ 0 & -1-i \end{bmatrix}.$$

Finally, let

$$P = Q\begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 9 & 13 \\ 0 & 0 & 0 & 13 & -9 \end{bmatrix} \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & i & -i \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$= \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 13+9i & 13-9i \\ 0 & 0 & 0 & -9+13i & -9-13i \end{bmatrix}.$$

Then

$$\mathcal{J} = P^{-1}AP = \left[\begin{array}{ccc|cc} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & -1+i & 0 \\ 0 & 0 & 0 & 0 & -1-i \end{array}\right].$$

Exercise 209 *Consider the following Jordan matrix $\mathcal{J}$:*

$$\mathcal{J} = \begin{bmatrix} \boxed{\begin{matrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{matrix}} & & & & & \mathbf{0} \\ & \boxed{\begin{matrix} 4 & 1 \\ 0 & 4 \end{matrix}} & & & \\ & & \boxed{\begin{matrix} 3 & 1 \\ 0 & 3 \end{matrix}} & & \\ \mathbf{0} & & & \boxed{2} & \\ & & & & \boxed{2} \end{bmatrix}$$

1. *List the eigenvalues of $\mathcal{J}$.*
2. *Determine the algebraic multiplicity of each eigenvalue.*
3. *Determine the geometric multiplicity of each eigenvalue.*
4. *Determine* $\mathrm{ind}_{\mathcal{J}}(\lambda)$ *for each eigenvalue* λ *of* $\mathcal{J}$.

Exercise 210 *Find a Jordan Form of the following matrix A, whose eigenvalues are $\{-1, 0, 1\}$. Don't let the size of A frighten you!*

$$A = \begin{bmatrix} -4 & -5 & -3 & 1 & -2 & 0 & 1 & -2 \\ 4 & 7 & 3 & -1 & 3 & 0 & -1 & 2 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 2 & -4 & 2 & 0 & -3 & 1 \\ -8 & -14 & -5 & 1 & -6 & 0 & 1 & -4 \\ 4 & 7 & 4 & -3 & 3 & -1 & -3 & 4 \\ 2 & -2 & -2 & 5 & -3 & 0 & 4 & -1 \\ 6 & 7 & 3 & 0 & 2 & 0 & 0 & 3 \end{bmatrix}.$$

Exercise 211 *Find a Jordan Form $\mathcal{J}$ of*

$$A = \begin{bmatrix} -1 & -18 & -7 \\ 1 & -13 & -4 \\ 1 & 25 & 8 \end{bmatrix}.$$

and a similarity transformation $P^{-1}AP = \mathcal{J}$.

Exercise 212 *Find a Jordan Form $\mathcal{J}$ of*

$$A = \begin{bmatrix} 4 & -4 & -11 & 11 \\ 3 & -12 & -42 & 42 \\ -2 & 12 & 37 & -34 \\ -1 & 7 & 20 & -17 \end{bmatrix}$$

and a similarity transformation $P^{-1}AP = \mathcal{J}$.

Exercise 213 *Find a Jordan Form $\mathcal{J}$ of*

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

and a similarity transformation $P^{-1}AP = \mathcal{J}$.

Theorem 214 (Powers of a Jordan Block) *Consider a Jordan block*

$$\mathcal{J}(a) = \begin{bmatrix} a & 1 & & & & \\ & a & 1 & & \mathbf{0} & \\ & & a & \ddots & & \\ & & & \ddots & 1 & \\ & \mathbf{0} & & & a & 1 \\ & & & & & a \end{bmatrix}.$$

The $(i,j)^{th}$ *entry of* $\mathcal{J}(a)^p$ *is*

$$\tbinom{p}{j-i} a^{p-(j-i)},$$

where $\binom{p}{j-i} = 0$ *when* $j-i<0$ *or* $j-i>p$.

Proof. The result holds trivially for $p=1$. Assume $\left[\mathcal{J}(a)^{p-1}\right]_{ij} = \binom{p-1}{j-i} a^{(p-1)-(j-i)}$. Then

$$\begin{aligned}
\left[\mathcal{J}(a)^p\right]_{ij} &= \left[\mathcal{J}(a)^{p-1}\right]_{i,*} \cdot \left[\mathcal{J}(a)\right]_{*,j} \\
&= \begin{bmatrix} \cdots & \binom{p-1}{j-i-1} a^{(p-1)-(j-i-1)} & \binom{p-1}{j-i} a^{(p-1)-(j-i)} & \cdots \end{bmatrix} \cdot \begin{bmatrix} \vdots \\ 0 \\ 1 \\ a \\ 0 \\ \vdots \end{bmatrix} \begin{array}{l} \\ \\ \\ \leftarrow j^{th} \\ \\ \\ \end{array} \\
&= \tbinom{p-1}{j-i-1} a^{p-(j-i)} + \tbinom{p-1}{j-i} a^{(p-1)-(j-i)} a = \tbinom{p}{j-i} a^{p-(j-i)}.
\end{aligned}$$

■

Example 215 *For the* 5×5 *Jordan block* $\mathcal{J}(a)$ *we have*

$$\mathcal{J}(a)^3 = \begin{bmatrix} a^3 & 3a^2 & 3a & 1 & 0 \\ 0 & a^3 & 3a^2 & 3a & 1 \\ 0 & 0 & a^3 & 3a^2 & 3a \\ 0 & 0 & 0 & a^3 & 3a^2 \\ 0 & 0 & 0 & 0 & a^3 \end{bmatrix}.$$

Exercise 216 *In Example 207 we observed that the Jordan Form of*

$$A = \begin{bmatrix} 3 & 0 & 1 \\ -4 & 1 & -2 \\ -4 & 0 & -1 \end{bmatrix}$$

is

$$\mathcal{J} = P^{-1}AP = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ where } P = \begin{bmatrix} 2 & 1 & 0 \\ -4 & 0 & 1 \\ -4 & 0 & 0 \end{bmatrix}.$$

Establish a formula for A^p *for each* $p \geq 1$ *and use your formula to evaluate* $A^{10^{10}}$. *Write the entries of* $A^{10^{10}}$ *in terms of powers of* 10.

12-8-12