The Jordan Canonical Form of a Nilpotent Matrix

Math 422

Schur's Triangularization Theorem tells us that every matrix A is unitarily similar to an upper triangular matrix T. However, the only thing certain at this point is that the the diagonal entries of T are the eigenvalues of A. The off-diagonal entries of T seem unpredictable and out of control. Recall that the Core-Nilpotent Decomposition of a singular matrix A of index k produces a block diagonal matrix

$$\begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & L \end{bmatrix}$$

similar to A in which C is non-singular, $rank\,(C) = rank\left(A^k\right)$, and L is nilpotent of index k. Is it possible to simplify C and L via similarity transformations and obtain triangular matrices whose off-diagonal entries are predictable? The goal of this lecture is to do exactly this for nilpotent matrices.

Let L be an $n \times n$ nilpotent matrix of index k. Then $L^{k-1} \neq 0$ and $L^k = 0$. Let's compute the eigenvalues of L. Suppose $x \neq 0$ satisfies $Lx = \lambda x$; then $0 = L^k x = L^{k-1}(Lx) = L^{k-1}(\lambda x) = \lambda L^{k-1}x = \lambda L^{k-2}(Lx) = \lambda^2 L^{k-2}x = \cdots = \lambda^k x$; thus $\lambda = 0$ is the only eigenvalue of L.

Now if L is diagonalizable, there is an invertible matrix P and a diagonal matrix D such that $P^{-1}LP = D$. Since the diagonal entries of D are the eigenvalues of L, and $\lambda = 0$ is the only eigenvalue of L, we have $D = 0$. Solving $P^{-1}LP = 0$ for L gives $L = 0$. Thus a diagonalizable nilpotent matrix is the zero matrix, or equivalently, a non-zero nilpotent matrix L is not diagonalizable. And indeed, some off-diagonal entries in the "simplified" form of L will be non-zero.

Let L be a non-zero nilpotent matrix. Since L is triangularizable, there exists an invertible P such that

$$P^{-1}LP = \begin{bmatrix} 0 & * & \cdots & * \\ 0 & 0 & \ddots & * \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The simplification procedure given in this lecture produces a matrix similar to L whose non-zero entries lie exclusively on the superdiagonal of $P^{-1}LP$. An example of this procedure follows below.

Definition 1 *Let L be nilpotent of index k and define $L^0 = I$. For $p = 1, 2, \ldots, k$, let $x \in \mathbb{C}^n$ such that $y = L^{p-1}x \neq 0$. The* <u>*Jordan chain on y of length p*</u> *is the set $\left\{L^{p-1}x, \ldots, Lx, x\right\}$.*

Exercise 2 *Let L be nilpotent of index k. If $x \in \mathbb{C}^n$ satisfies $L^{k-1}x \neq 0$, prove that $\left\{L^{k-1}x, \ldots, \; Lx, x\right\}$ is linearly independent.*

Example 3 *Let* $L = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$; *then* $L^2 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *and* $L^3 = 0$. *Let* $x = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$; *then*

$$Lx = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b+2c \\ 3c \\ 0 \end{bmatrix} \quad \text{and} \quad L^2x = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3c \\ 0 \\ 0 \end{bmatrix}.$$

Note that $\left\{L^2x, Lx, x\right\}$ is linearly independent iff $c \neq 0$. Thus if $x = [1\ 2\ 3]^T$, then $y = L^2x = [9\ 0\ 0]^T$ and the Jordan chain on y is

$$\left\{ \begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

Form the matrix

$$P = \left[L^2x \mid Lx \mid x\right] = \begin{bmatrix} 9 & 8 & 1 \\ 0 & 9 & 2 \\ 0 & 0 & 3 \end{bmatrix};$$

then

$$J = P^{-1}LP = \frac{1}{243}\begin{bmatrix} 27 & -24 & 7 \\ 0 & 27 & -18 \\ 0 & 0 & 81 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 9 & 8 & 1 \\ 0 & 9 & 2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

is the "Jordan form" of L.

Since $\lambda = 0$ is the only eigenvalue of an $n \times n$ non-zero nilpotent L, the eigenvectors of L are exactly the non-zero vectors in $N(L)$. But $\dim N(L) < n$ since L is not diagonalizable and possesses an *incomplete* set of linearly independent eigenvectors. So the process by which one constructs the desired similarity transformation $P^{-1}LP$ involves appropriately extending a deficient basis for $N(L)$ to a basis for $\mathbb{C}^n$. This process has essentially two steps:

- Construct a somewhat special basis $\mathcal{B}$ for $N(L)$.
- Extend $\mathcal{B}$ to a basis for $\mathbb{C}^n$ by building Jordan chains on the elements of $\mathcal{B}$.

Definition 4 *A nilpotent Jordan block is a matrix of the form*

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

A nilpotent Jordan matrix is a block diagonal matrix of the form

$$\begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & J_m \end{bmatrix}, \tag{1}$$

where each J_i is a nilpotent Jordan block.

When the context is clear we refer to a nilpotent Jordan matrix as a Jordan matrix.

Theorem 5 *Every $n \times n$ nilpotent matrix L of index k is similar to an $n \times n$ Jordan matrix J in which*

- *the number of Jordan blocks is* $\dim N(L)$;
- *the size of the largest Jordan block is $k \times k$;*
- *for $1 \leq j \leq k$, the number of $j \times j$ Jordan blocks is*

$$rank\left(L^{j-1}\right) - 2rank\left(L^{j}\right) + rank\left(L^{j+1}\right);$$

- *the ordering of the J_i's is arbitrary.*

Whereas the essentials of the proof appear in the algorithm below, we omit the details.

Definition 6 *If L is a nilpotent matrix, a Jordan form of L is a Jordan matrix $J = P^{-1}LP$. The Jordan structure of L is the number and size of the Jordan blocks in every Jordan form J of L.*

Theorem 5 tells us that Jordan form is unique up to ordering of the blocks J_i. Indeed, given any prescribed ordering, there is a Jordan form whose Jordan blocks appear in that prescribed order.

Definition 7 *The* <u>*Jordan Canonical Form*</u> *(JCF) of a nilpotent matrix L is the Jordan form of L in which the Jordan blocks are distributed along the diagonal in order of decreasing size.*

Example 8 *Let us determine the Jordan structure and JCF of the nilpotent matrix*

$$L = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -1 \\ 3 & 1 & 5 & 1 & -1 & 3 \\ -2 & -1 & 0 & 0 & -1 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -5 & -3 & -1 & -1 & -1 & -1 \\ -3 & -2 & -1 & -1 & 0 & -1 \end{bmatrix} \overset{row\text{-}reduce}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 & -\frac{2}{3} & \frac{4}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{4}{3} & -\frac{5}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

the number of Jordan blocks is $\dim N(L) = 3$. *Then*

$$L^2 = \begin{bmatrix} 6 & 3 & 3 & 1 & 1 & 2 \\ -6 & -3 & -3 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & -3 & -3 & -1 & -1 & -2 \\ -6 & -3 & -3 & -1 & -1 & -2 \end{bmatrix} \overset{row\text{-}reduce}{\longrightarrow} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

has rank 1 *and* $L^3 = 0$; *therefore the index* $(L) = 3$ *and the size of the largest Jordan block is* 3×3. *Let*

$$\begin{aligned} r_0 &= rank\left(L^0\right) = 6 \\ r_1 &= rank\left(L^1\right) = 3 \\ r_2 &= rank\left(L^2\right) = 1 \\ r_3 &= rank\left(L^3\right) = 0 \\ r_4 &= rank\left(L^4\right) = 0 \end{aligned};$$

then the number n_i of $i \times i$ Jordan blocks is

$$\begin{aligned} n_1 &= r_0 - 2r_1 + r_2 = 1 \\ n_2 &= r_1 - 2r_2 + r_3 = 1 \\ n_3 &= r_2 - 2r_3 + r_4 = 1 \end{aligned}.$$

Obtain the JCF by distributing these blocks along the diagonal in order of decreasing size. Then the JCF of L is

$$\left[\begin{array}{ccc|cc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

This leaves us with the task of determining a non-singular matrix P such that $P^{-1}LP$ is the JCF of L. As mentioned above, this process has essentially two steps. A detailed algorithm follows our next definition and a key exercise.

Definition 9 *Let E be any row-echelon form of a matrix A. Let $c_1, \ldots, c_q$ index the columns of E containing leading 1's and let y_i denote the c_i^{th} column of A. The columns $y_1, \ldots, y_q$ are called the* <u>*basic columns of A*</u>.

Exercise 10 *Let $B = [b_1 \mid \cdots \mid b_p]$ be an $n \times p$ matrix with linearly independent columns. Prove that multiplication by B preserves linear independence, i.e., if $\{v_1, \ldots, v_s\}$ is linearly independent in $\mathbb{C}^p$, then $\{Bv_1, \ldots, Bv_s\}$ is linearly independent in $\mathbb{C}^n$.*

Nilpotent Reduction to JCF

Given a nilpotent matrix L of index k, set $i = k-1$ and let $\mathcal{S}_{k-1} = \{y_1, \ldots, y_q\}$ be the basic columns of L^{k-1}.

1. Extend $\mathcal{S}_{k-1} \cup \cdots \cup \mathcal{S}_i$ to a basis for $R\left(L^{i-1}\right) \cap N(L)$ in the following way:

   (a) Let $\{b_1, \ldots, b_p\}$ be the basic columns of L^{i-1} and let $B = [b_1 \mid \cdots \mid b_p]$.

   (b) Solve $LBx = 0$ for x and obtain a basis $\{v_1, \ldots, v_s\}$ for $N(LB)$; then $\{Bv_1, \ldots, Bv_s\}$ is a basis for $R\left(L^{i-1}\right) \cap N(L)$ by Lemma 11 below.

   (c) Form the matrix $[y_1 \mid \cdots \mid y_q \mid Bv_1 \mid \cdots \mid Bv_s]$; its basic columns $\left\{y_1, \ldots, y_q, Bv_{\beta_1}, \ldots, Bv_{\beta_j}\right\}$ form a basis for $R\left(L^{i-1}\right) \cap N(L)$ containing $\mathcal{S}_{k-1} \cup \cdots \cup \mathcal{S}_i$. Let
   $$\mathcal{S}_{i-1} = \left\{Bv_{\beta_1}, \ldots, Bv_{\beta_j}\right\}.$$

   (d) Decrement i, let $\{y_1, \ldots, y_q\}$ be the basic columns of L^i, and repeat step 1 until $i = 0$. Then $\mathcal{S}_{k-1} \cup \cdots \cup \mathcal{S}_0 = \{b_1, \ldots, b_t\}$ is a basis for $N(L)$.

2. For each j, if $b_j \in \mathcal{S}_i$, find a particular solution x_j of $L^i x = b_j$ and build a Jordan chain $\left\{L^i x_j, \ldots, Lx_j, x_j\right\}$. Set
   $$p_j = \left[L^i x_j \mid \cdots \mid Lx_j \mid x_j\right];$$
   the desired similarity transformation is defined by the matrix
   $$P = [p_1 \mid \cdots \mid p_t].$$

Lemma 11 *In the notation of the algorithm above, $\{Bv_1, \ldots, Bv_s\}$ is a basis for $R\left(L^{i-1}\right) \cap N(L)$.*

Proof. Note that $Bv_j \in R(B)$ and $LBv_j = 0$ implies $Bv_j \in N(L)$. But $R(B) = R\left(L^{i-1}\right)$ implies that $Bv_j \in R(B) \cap N(L) = \mathcal{M}_{i-1}$ for all j. The set $\{Bv_1, \ldots, Bv_s\}$ is linearly independent by Exercise 10. To show that $\{Bv_1, \ldots, Bv_s\}$ spans $\mathcal{M}_{i-1}$, let $y \in R(B) \cap N(L)$. Since $y \in R(B)$, there is some $u \in \mathbb{C}^n$ such that $y = Bu$; since $y \in N(L)$, $0 = Ly = LBu$. Thus $u \in N(LB)$ and we may express u in the basis $\{v_1, \ldots, v_s\}$ as $u = c_1 v_1 + \cdots + c_s v_s$. Therefore $y = Bu = B(c_1 v_1 + \cdots + c_s v_s) = c_1 Bv_1 + \cdots + c_s Bv_s$ and $\{Bv_1, \ldots, Bv_s\}$ spans. ■

In particular, suppose L is an $n \times n$ nilpotent matrix of index n. Choose a non-zero vector x_1 such that $L^{n-1}x_1 \neq 0$. Then

$$\begin{array}{rcl} x_1 & \in & N\left(L^n\right) \setminus N\left(L^{n-1}\right) \\ x_2 & = & L^{n-1}x_1 \Rightarrow x_2 \in N(L) \\ x_3 & = & L^{n-2}x_1 \Rightarrow x_3 \in N\left(L^2\right) \setminus N(L) \\ & & \vdots \\ x_n & = & Lx_1 \Rightarrow x_n \in N\left(L^{n-1}\right) \setminus N\left(L^{n-2}\right). \end{array}$$

Thus $\{x_2, \ldots, x_n, x_1\} = \left\{L^{n-1}x_1, \ldots, Lx_1, x_1\right\}$ is linearly independent in C^n and $\{x_2\}$ is a basis for the 1-dimensional subspace $N(L)$. Thus $\left\{L^{n-1}x_1, \ldots, Lx_1, x_1\right\}$ is a Jordan chain on x_2 of length n and the Jordan form of L consists of one $n \times n$ Jordan block J. Let $P = \left[L^{n-1}x_1 \mid \cdots \mid Lx_1 \mid x_1\right]$; then

$$J = P^{-1}LP = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Example 12 *Let*

$$L = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$L^2 = \begin{bmatrix} 0 & 0 & 4 & 17 \\ 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad L^3 = \begin{bmatrix} 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad L^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and L is nilpotent of index 4. Let $x_1 = \mathbf{e}_4$; then

$$\begin{aligned} L^3 x_1 &= [24, 0, 0, 0]^T \\ L^2 x_1 &= [17, 24, 0, 0]^T \\ L x_1 &= [3, 5, 6, 0]^T \end{aligned}$$

and $\{L^3x_1, L^2x_1, Lx_1, x_1\}$ is a Jordan chain on $y = L^3x_1$. Form the matrix

$$P = \left[L^3x_1 | L^2x_1 | Lx_1 | x_1\right] = \begin{bmatrix} 24 & 17 & 3 & 0 \\ 0 & 24 & 5 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

Then

$$J = P^{-1}LP = \begin{bmatrix} 24 & 17 & 3 & 0 \\ 0 & 24 & 5 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 24 & 17 & 3 & 0 \\ 0 & 24 & 5 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example 13 *Let's construct the matrix P that produces the Jordan form of L in Example 8. Since L has index 3, we row-reduce L^2 and find one basic column*

$$y_1 = [6, -6, 0, 0, -6, -6]^T.$$

Then $\mathcal{S}_2 = \{y_1\}$ contains the basic column of $R\left(L^2\right)$; set $i = 2$.

1. *Extend $\mathcal{S}_2$ to a basis for $R(L) \cap N(L)$:*

   (a) *Row-reducing, we find that the basic columns of L are its first three columns. Thus we let*

   $$B = \begin{bmatrix} 1 & 1 & -2 \\ 3 & 1 & 5 \\ -2 & -1 & 0 \\ 2 & 1 & 0 \\ -5 & -3 & -1 \\ -3 & -2 & -1 \end{bmatrix}.$$

   (b) *Compute LB and solve $LBx = 0$ to obtain a basis for $N(LB)$:*

$$LB = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -1 \\ 3 & 1 & 5 & 1 & -1 & 3 \\ -2 & -1 & 0 & 0 & -1 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -5 & -3 & -1 & -1 & -1 & -1 \\ -3 & -2 & -1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 3 & 1 & 5 \\ -2 & -1 & 0 \\ 2 & 1 & 0 \\ -5 & -3 & -1 \\ -3 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 3 \\ -6 & -3 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -6 & -3 & -3 \\ -6 & -3 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 3 & 3 \\ -6 & -3 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -6 & -3 & -3 \\ -6 & -3 & -3 \end{bmatrix} \xrightarrow{row\text{-}reduce} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left\{ v_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix},\ v_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

Then $\left\{ Bv_1 = [1\ \ -1\ 0\ 0\ \ -1\ \ -1]^T, Bv_2 = [-5\ 7\ 2\ \ -2\ 3\ 1]^T \right\}$ *is a basis for* $R(L) \cap N(L)$.

(c) *Form the matrix*

$$[y_1 \mid Bv_1 \mid Bv_2] = \begin{bmatrix} 6 & 1 & -5 \\ -6 & -1 & 7 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \\ -6 & -1 & 3 \\ -6 & -1 & 1 \end{bmatrix} \xrightarrow{row\text{-}reduce} \begin{bmatrix} 1 & \frac{1}{6} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

columns 1 and 3 are its basic columns, hence $\{y_1, y_2 = Bv_2\}$ *is a basis for* $R(L) \cap N(L)$ *containing* $\mathcal{S}_2$. *Let*

$$\mathcal{S}_1 = \left\{ [-5\ 7\ 2\ \ -2\ 3\ 1]^T \right\}$$

and decrement i; *then* $i = 1$.

2. *Extend* $\mathcal{S}_2 \cup \mathcal{S}_1 = \{y_1, y_2\}$ *to a basis for* $R\left(L^0\right) \cap N(L) = N(L)$:

(a) *Since* $L^0 = I$, *we let* $B = I$.

(b) *Since* $LB = L$, *we find a basis for* $N(LB) = N(L)$:

$$\begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -1 \\ 3 & 1 & 5 & 1 & -1 & 3 \\ -2 & -1 & 0 & 0 & -1 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -5 & -3 & -1 & -1 & -1 & -1 \\ -3 & -2 & -1 & -1 & 0 & -1 \end{bmatrix} \xrightarrow{row\text{-}reduce} \begin{bmatrix} 1 & 0 & 0 & -\frac{2}{3} & \frac{4}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{4}{3} & -\frac{5}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left\{ v_1 = \begin{bmatrix} 2 \\ -4 \\ -1 \\ 3 \\ 0 \\ 0 \end{bmatrix},\ v_2 = \begin{bmatrix} -4 \\ 5 \\ 2 \\ 0 \\ 3 \\ 0 \end{bmatrix},\ v_3 = \begin{bmatrix} 1 \\ -2 \\ -2 \\ 0 \\ 0 \\ 3 \end{bmatrix} \right\}.$$

Then a basis for $\mathcal{M}_0$ *is*

$$\{Bv_1, Bv_2, Bv_3\} = \{v_1, v_2, v_3\}.$$

(c) *Form the matrix*

$$[y_1 \mid y_2 \mid v_1 \mid v_2 \mid v_3] = \begin{bmatrix} 6 & -5 & 2 & -4 & 1 \\ -6 & 7 & -4 & 5 & -2 \\ 0 & 2 & -1 & 2 & -2 \\ 0 & -2 & 3 & 0 & 0 \\ -6 & 3 & 0 & 3 & 0 \\ -6 & 1 & 0 & 0 & 3 \end{bmatrix} \xrightarrow{row\text{-}reduce} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{4} & -\frac{3}{4} \\ 0 & 1 & 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

columns 1,2, and 3 are its basic columns, hence $\{y_1, y_2, v_1\}$ *is a a basis for* $N(L)$ *containing* $\mathcal{S}_2 \cup \mathcal{S}_1$*. Let*

$$\mathcal{S}_0 = \left\{ [2 \ -4 \ -1 \ 3 \ 0 \ 0]^T \right\}$$

and decrement i*; then* $i = 0$ *and the process terminates having produced the basis* $\mathcal{S}_2 \cup \mathcal{S}_1 \cup \mathcal{S}_0$ *for* $N(L)$*.*

3. *Let* $\mathcal{S}_2 = \{b_1\}$, $\mathcal{S}_1 = \{b_2\}$, *and* $\mathcal{S}_0 = \{b_3\}$*. For* $j = 1, 2, 3$, *build a Jordan chain on* $b_j \in \mathcal{S}_i$ *of length* $i + 1$ *by finding a particular solution* x_j *of* $L^i x = b_j$*. When* $j = 1$ *we see by inspection that* $L^2 e_1 = b_1$*. Thus*

$$p_1 = \left[L^2 e_1 \mid L e_1 \mid e_1 \right] = \begin{bmatrix} 6 & 1 & 1 \\ -6 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 2 & 0 \\ -6 & -5 & 0 \\ -6 & -3 & 0 \end{bmatrix}.$$

When $j = 2$ *we solve* $Lx = b_2$*:*

$$\left[\begin{array}{cccccc|c} 1 & 1 & -2 & 0 & 1 & -1 & -5 \\ 3 & 1 & 5 & 1 & -1 & 3 & 7 \\ -2 & -1 & 0 & 0 & -1 & 0 & 2 \\ 2 & 1 & 0 & 0 & 1 & 0 & -2 \\ -5 & -3 & -1 & -1 & -1 & -1 & 3 \\ -3 & -2 & -1 & -1 & 0 & -1 & 1 \end{array}\right] \xrightarrow{\textit{row-reduce}} \left[\begin{array}{cccccc|c} 1 & 0 & 0 & -\frac{2}{3} & \frac{4}{3} & -\frac{1}{3} & -1 \\ 0 & 1 & 0 & \frac{4}{3} & -\frac{5}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

A particular solution is $x_2 = [-1 \ 0 \ 2 \ 0 \ 0 \ 0]^T$ *; hence*

$$p_2 = [L x_2 \mid x_2] = \begin{bmatrix} -5 & -1 \\ 7 & 0 \\ 2 & 2 \\ -2 & 0 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}.$$

When $j = 3$ *we have* $L^0 = I$*, and the unique solution of* $L^0 x = b_3$ *is* $x = b_3 \in \mathcal{S}_0$*. Thus the Jordan chain on* b_3 *consists only of* b_3 *and we have* $p_3 = [2 \ -4 \ -1 \ 3 \ 0 \ 0]^T$ *. Finally, we form the matrix*

$$P = [p_1 \mid p_2 \mid p_3] = \begin{bmatrix} 6 & 1 & 1 & -5 & -1 & 2 \\ -6 & 3 & 0 & 7 & 0 & -4 \\ 0 & -2 & 0 & 2 & 2 & -1 \\ 0 & 2 & 0 & -2 & 0 & 3 \\ -6 & -5 & 0 & 3 & 0 & 0 \\ -6 & -3 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Then as expected, the JCF of L is

$$J = P^{-1} L P = \left[\begin{array}{ccc|cc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Exercise 14 *A Hessenberg matrix H and a Jordan matrix J appear below. Find an invertible matrix P such that $J = P^{-1}HP$. (Note: Some texts define the JCF with 1's below the main diagonal as in H.)*

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \;;\; J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Exercise 15 *Prove that the Jordan matrices*

$$J_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are not similar. (Hint: Show that if $P = (p_{ij})$ is a 4×4 matrix such that $PJ_1 = J_2P$, then P is not invertible.)

Exercise 16 *A 4×4 nilpotent matrix L is given below. Find matrices P and J such that $P^{-1}LP = J$ has Jordan form:*

$$L = \begin{bmatrix} 3 & 3 & 2 & 1 \\ -2 & -1 & -1 & -1 \\ 1 & -1 & 0 & 1 \\ -5 & -4 & -3 & -2 \end{bmatrix}.$$

Exercise 17 *Consider the 5×5 matrix:*

$$L = \begin{bmatrix} 2 & 1 & 2 & 0 & -1 \\ 3 & 1 & 3 & -1 & 1 \\ -3 & -1 & -2 & 0 & 2 \\ 3 & 2 & 4 & 0 & -1 \\ 2 & 1 & 2 & 0 & -1 \end{bmatrix}.$$

a. Show that L is nilpotent and determine its index of nilpotency.

b. Find the Jordan Form J of L.

c. Find an invertible matrix P such that $J = P^{-1}LP$.

Exercise 18 *Determine the Jordan structure of the following 8×8 nilpotent matrix:*

$$L = \begin{bmatrix} 41 & 30 & 15 & 7 & 4 & 6 & 1 & 3 \\ -54 & -39 & -19 & -9 & -6 & -8 & -2 & -4 \\ 9 & 6 & 2 & 1 & 2 & 1 & 0 & 1 \\ -6 & -5 & -3 & -2 & 1 & -1 & 0 & 0 \\ -32 & -24 & -13 & -6 & -2 & -5 & -1 & -2 \\ -10 & -7 & -2 & 0 & -3 & 0 & 3 & -2 \\ -4 & -3 & -2 & -1 & 0 & -1 & -1 & 0 \\ 17 & 12 & 6 & 3 & 2 & 3 & 2 & 1 \end{bmatrix}$$

12-7-10