**Solution to Exercises on General Jordan Form**

Math 422

**Exercise 11:** Find a Jordan Form $\mathcal{J}$ of

$$A = \begin{bmatrix} -1 & -18 & -7 \\ 1 & -13 & -4 \\ -1 & 25 & 8 \end{bmatrix}$$

and a similarity transformation $P^{-1}AP = \mathcal{J}$.

**Solution:** The matrix $A$ has a single eigenvalue $\lambda = -2$. Let

$$L = A + 2I = \begin{bmatrix} 1 & -18 & -7 \\ 1 & -11 & -4 \\ -1 & 25 & 10 \end{bmatrix} \overset{\text{row reduce}}{\longrightarrow} \begin{bmatrix} 1 & 0 & \frac{5}{7} \\ 0 & 1 & \frac{3}{7} \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $rank\,(L) = 2$ and $\dim N\,(L) = 1$. Furthermore,

$$L^2 = \begin{bmatrix} -10 & 5 & -5 \\ -6 & 3 & -3 \\ 14 & -7 & 7 \end{bmatrix} \overset{\text{row reduce}}{\longrightarrow} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so that $rank\left(L^2\right) = 1$. But $L^3 = \mathbf{0}$ so $L$ is nilpotent of index 3, and it follows that the JCF $J$ of $L$ consists of a single Jordan block:

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Following the algorithm for nilpotent reduction to JCF, set $i = 2$ and let $\mathcal{S}_2$ be the set containing $y = \begin{bmatrix} -10 & -6 & 14 \end{bmatrix}^T$, the single basic column of $L^2$. Note that $\mathcal{S}_2$ is a basis for $N\,(L)$ since $\mathcal{S}_2 \subset N\,(L)$ and $\dim N\,(L) = 1$. Thus $\mathcal{S}_1 = \mathcal{S}_0 = \varnothing$ and step (1) of the algorithm is complete. Proceeding with step (2), $y \in \mathcal{S}_2$ requires us to find a particular solution of $L^2x = y$:

$$\left[\begin{array}{ccc|c} -10 & 5 & -5 & -10 \\ -6 & 3 & -3 & -6 \\ 14 & -7 & 7 & 14 \end{array}\right] \overset{\text{row reduce}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution is

$$s\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};$$

setting $s = t = 0$ gives the particular solution $e_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. Now build a Jordan chain of length 3 on $L^2x$ and form the matrix

$$P = \left[L^2e_1 \mid Le_1 \mid e_1\right] = \begin{bmatrix} -10 & 1 & 1 \\ -6 & 1 & 0 \\ 14 & -1 & 0 \end{bmatrix}.$$

Then

$$J = P^{-1}LP = P^{-1}\,(A + 2I)\,P = P^{-1}AP + 2I.$$

And solving for $\mathcal{J} = P^{-1}AP = J - 2I$ we conclude that

$$\mathcal{J} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}.$$

**Exercise 12:** Find a Jordan Form $\mathcal{J}$ of

$$A = \begin{bmatrix} 4 & -4 & -11 & 11 \\ 3 & -12 & -42 & 42 \\ -2 & 12 & 37 & -34 \\ -1 & 7 & 20 & -17 \end{bmatrix}$$

and a similarity transformation $P^{-1}AP = \mathcal{J}$.

**Solution:** The matrix $A$ has a single eigenvalue $\lambda = 3$. Let

$$L = A - 3I = \begin{bmatrix} 1 & -4 & -11 & 11 \\ 3 & -15 & -42 & 42 \\ -2 & 12 & 34 & -34 \\ -1 & 7 & 20 & -20 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus $rank\,(L) = \dim N\,(L) = 2$. Furthermore,

$$L^2 = \begin{bmatrix} 0 & 1 & 3 & -3 \\ 0 & 3 & 9 & -9 \\ 0 & -2 & -6 & 6 \\ 0 & -1 & -3 & 3 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 0 & 1 & 3 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that $rank\left(L^2\right) = 1$. But $L^3 = \mathbf{0}$ so $L$ is nilpotent of index 3, and it follows that the JCF $J$ of $L$ consists of two Jordan blocks of sizes $3 \times 3$ and $1 \times 1$ :

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Following the algorithm for nilpotent reduction to JCF, set $i = 2$ and let $\mathcal{S}_2$ be the set containing $y = \begin{bmatrix} 1 & 3 & -2 & -1 \end{bmatrix}^T$, the single basic column of $L^2$.

1. Extend $\mathcal{S}_2$ to a basis for $R\,(L) \cap N\,(L)$.

   (a) Let $\{b_1, b_2\}$ denote the basic columns of $L$; then

   $$B = [b_1 \mid b_2] = \begin{bmatrix} 1 & -4 \\ 3 & -15 \\ -2 & 12 \\ -1 & 7 \end{bmatrix}.$$

   (b) Solve $LBx = \mathbf{0}$ :

   $$LB = \begin{bmatrix} 0 & 1 \\ 0 & 3 \\ 0 & -2 \\ 0 & -1 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

   Then $\left\{v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ is a basis for $N\,(LB)$ and $\left\{Bv = \begin{bmatrix} 1 & 3 & -2 & -1 \end{bmatrix}^T\right\}$ is a basis for $R\,(L) \cap N\,(L)$.

   (c) Since $y = Bv$, the matrix $[y \mid Bv]$ has one basic column. Thus $\mathcal{S}_2$ is in fact a basis for $R\,(L) \cap N\,(L)$ and $\mathcal{S}_1 = \varnothing$.

   (d) Decrement $i$. Now $i = 1$ and we repeat step (1).

1′. Extend $\mathcal{S}_2 \cup \mathcal{S}_1$ to a basis for $N(L)$ :

(a) The basic columns of $L^0 = I$ are $\{e_1, e_2, e_3\}$ and $B = I$.

(b) Then $LB = L$ and we solve $Lx = \mathbf{0}$ : A basis for $N(LB) = N(L)$ is

$$\left\{ v_1 = \begin{bmatrix} 1 & 3 & -1 & 0 \end{bmatrix}^T, \; v_2 = \begin{bmatrix} 1 & 3 & 0 & 1 \end{bmatrix}^T \right\}.$$

(c) The basic columns $y$ and $v_1$ of

$$[y \mid v_1 \mid v_2] = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \\ -2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

form the desired basis for $N(L)$. We set $\mathcal{S}_0 = \{v_1\}$.

(d) Decrement $i$. Now $i = 0$ and we proceed to step (2).

2. Since $y \in \mathcal{S}_2$, we solve $L^2 x = y$ :

$$\left[\begin{array}{cccc|c} 0 & 1 & 3 & -3 & 1 \\ 0 & 3 & 9 & -9 & 3 \\ 0 & -2 & -6 & 6 & -2 \\ 0 & -1 & -3 & 3 & -1 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{cccc|c} 0 & 1 & 3 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

A particular solution is $e_2$. Build a Jordan chain of length 3 on $L^2 e_2$ and form the matrix

$$p_1 = \left[L^2 e_2 \mid L e_2 \mid e_2\right] = \begin{bmatrix} 1 & -4 & 0 \\ 3 & -15 & 1 \\ -2 & 12 & 0 \\ -1 & 7 & 0 \end{bmatrix}.$$

Since $v_1 \in S_0$, we solve $L^0 x = v_1$. But $L^0 = I$ so that $x = v_1$ and $p_2 = v_1$. Thus

$$P = [p_1 \mid p_2] = \begin{bmatrix} 1 & -4 & 0 & 1 \\ 3 & -15 & 1 & 3 \\ -2 & 12 & 0 & -1 \\ -1 & 7 & 0 & 0 \end{bmatrix}$$

and

$$J = P^{-1}(A - 3I)P = P^{-1}AP - 3I.$$

Solving for $\mathcal{J} = P^{-1}AP = J + 3I$ we conclude that

$$\mathcal{J} = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$