Quadratic Forms

Math 422

Definition 1 *A quadratic form is a function* $f : \mathbb{R}^n \to \mathbb{R}$ *of form*

$$f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x},$$

where A *is an* $n \times n$ *symmetric matrix.*

Example 2 $f(x, y) = 2x^2 + 3xy - 4y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$

Note that the Euclidean inner product (dot product) of two (column) vectors $\mathbf{a}$ and $\mathbf{b}$ can be expressed in terms of matrix multiplication as

$$\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{b}^T \mathbf{a}.$$

Thus, a quadratic form can be expressed in terms of the Euclidean inner product as

$$\mathbf{x}^T A\mathbf{x} = \langle A\mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{x}, A\mathbf{x} \rangle.$$

Let S^{n-1} denote the unit $(n-1)$-dimensional sphere in $\mathbb{R}^n$, i.e., relative to the Euclidean inner product

$$S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{x} \rangle = 1\}.$$

Since S^{n-1} is a closed and bounded subset of $\mathbb{R}^n$, continuous functions on S^{n-1} attain their maximum and minimum values.

Question #1: For $\mathbf{x} \in S^{n-1}$, what are the maximum and minimum values of a quadratic form $\mathbf{x}^T A\mathbf{x}$?

Theorem 3 *Let* A *be a symmetric* $n \times n$ *matrix with eigenvalues* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. *Then*

1. $\lambda_1 \geq \mathbf{x}^T A\mathbf{x} \geq \lambda_n$ *for all* $\mathbf{x} \in S^{n-1}$.
2. *If* $\mathbf{x}_1 \in S^{n-1}$ *is an eigenvalue associated with* λ_1, *then* $\lambda_1 = \mathbf{x}_1^T A\mathbf{x}_1$.
3. *If* $\mathbf{x}_n \in S^{n-1}$ *is an eigenvalue associated with* λ_n, *then* $\lambda_n = \mathbf{x}_n^T A\mathbf{x}_n$.

The maximum and minimum of a quadratic form $\mathbf{x}^T A\mathbf{x}$ can be found by computing the largest and smallest eigenvalue of A. The maximum (respectively, minimum) will always be attained at diametrically opposite points on the unit sphere $\pm\frac{\mathbf{x}}{\|\mathbf{x}\|}$, where $\mathbf{x}$ is any eigenvector associated with λ_1 (respectively, λ_n).

Example 4 *Consider* $f(x_1, x_2) = 2x_1x_2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. *Since the eigenvalues of* $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *are* $\lambda_1 = 1$ *and* $\lambda_2 = -1$, *the maximum and minimum values of* f *on the unit circle* S^1 *are* 1 *and* -1, *respectively. Furthermore, the maximum value is attained at the eigenvectors on* S^1 *associated with* $\lambda_1 = 1$, *namely* $\pm\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$; *the minimum value is attained at the eigenvectors on* S^1 *associated with* $\lambda_2 = -1$, *namely* $\pm\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

Question #2: Under what conditions is the quadratic form $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$?

Definition 5 $\mathbf{x}^T A\mathbf{x}$ *is positive definite iff* $\mathbf{x}^T A\mathbf{x} > 0$ *for all* $\mathbf{x} \neq \mathbf{0}$ *and* $\mathbf{x}^T A\mathbf{x} = 0$ *iff* $\mathbf{x} = \mathbf{0}$. *A symmetric matrix is positive definite iff* $\mathbf{x}^T A\mathbf{x}$ *is positive definite.*

Example 6 *The Euclidean inner product is a positive definite quadratic form since*

$$x_1^2 + \cdots + x_n^2 = \langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^T \mathbf{x} = \mathbf{x}^T I \mathbf{x}.$$

Theorem 7 *A symmetric matrix A is positive definite iff all eigenvalues of A are positive.*

Example 8 *The matrix* $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$ *is positive definite since the eigenvalues of A are* $\lambda_1 = 8$, $\lambda_2 = 2$ *and* $\lambda_3 = 2$. *Note that if* $\mathbf{x} \neq \mathbf{0}$, *then* $\mathbf{x}^T A\mathbf{x} = 2x_1^2 + 4x_2^2 + 4x_3^2 + 4x_1x_2 + 4x_1x_3 + 4x_2x_3 > 0$.

Definition 9 *For* $1 \leq k \leq n$, *the* k^{th} *principal submatrix of an* $n \times n$ *matrix* $A = [a_{ij}]$ *is*

$$\begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix}.$$

Theorem 10 *A symmetric matrix A is positive definite iff every principal subdeterminant of A is positive.*

Example 11 *The principal subdeterminants of the matrix* $A = \begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & 4 \\ -3 & 4 & 9 \end{bmatrix}$ *are* $\det[2] = 2$, $\det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 3$ *and* $\det A = 1$. *Since all are positive, the quadratic form* $\mathbf{x}^T A\mathbf{x}$ *is positive definite.*

Question #3: If $\mathbf{x}^T A\mathbf{x}$ is a quadratic form with non-diagonal A, under what conditions does there exist an orthogonal change variables $\mathbf{x} = P\mathbf{y}$ so that $(P\mathbf{y})^T A (P\mathbf{y}) = \mathbf{y}^T \left(P^T A P\right) \mathbf{y}$ has no cross-terms?

Definition 12 *An* $n \times n$ *matrix A is orthogonally diagonalizable iff there exists an orthogonal matrix P such that* $P^T A P$ *is a diagonal matrix.*

Theorem 13 *If A is an* $n \times n$ *matrix, then the following are equivalent:*

1. *A is orthogonally diagonalizable.*
2. *A has an orthonormal set of n eigenvectors.*
3. *A is symmetric.*

Theorem 14 *If A is symmetric, then*

1. *The eigenvalues of A are real numbers.*
2. *Eigenvectors from different eigenspaces are orthogonal with respect to the Euclidean inner product.*

Use the following procedure to orthogonally diagonalize A:

Example 15

1. *Find a basis for each eigenspace of A.*
2. *Apply Gram-Schmidt and obtain an orthonormal basis for each eigenspace.*
3. *Form the matrix P whose columns are the basis vectors constructed in step 2.*

Example 16 *Consider the matrix* $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$ *whose eigenvalues are* $\lambda_1 = 8$, $\lambda_2 = 2$ *and* $\lambda_3 = 2$. *Canonical bases for the eigenspaces are* $\left\{ \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ $\leftrightarrow 8$ *and* $\left\{ \mathbf{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\} \leftrightarrow 2$. *Note that* $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle = \langle \mathbf{x}_1, \mathbf{x}_3 \rangle = 0$. *Applying Gram-Schmidt gives* $\left\{ \mathbf{v}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\}$ *and* $\left\{ \mathbf{v}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \ \mathbf{v}_3 = \frac{\mathbf{x}_3 - \langle \mathbf{x}_3, \mathbf{v}_2 \rangle \mathbf{v}_2}{\|\mathbf{x}_3 - \langle \mathbf{x}_3, \mathbf{v}_2 \rangle \mathbf{v}_2\|} = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \right\}$. *The matrix* $P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$ *and* $P^T A P = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ *is a diagonal matrix.*

Theorem 17 *Let* $\mathbf{x}^T A \mathbf{x}$ *be a quadratic form in variables* $x_1, \ldots, x_n$. *Let* P *be an orthogonal matrix that orthogonally diagonalizes* A. *If* $\boldsymbol{\lambda}_1, \ldots, \lambda_n$ *are the eigenvalues of* A *and* $y_1, \ldots, y_n$ *are new variables such that* $\mathbf{x} = P\mathbf{y}$, *then*

$$\mathbf{x}^T A \mathbf{x} = \mathbf{y}^T \left(P^T A P \right) \mathbf{y} = \boldsymbol{\lambda}_1 y_1^2 + \cdots + \lambda_n y_n^2$$

and

$$P^T A P = \begin{bmatrix} \boldsymbol{\lambda}_1 & 0 & \cdots & 0 \\ 0 & \boldsymbol{\lambda}_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Example 18 *Let* $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$. *By the calculations in Example 16,* $2x_1^2 + 4x_2^2 + 4x_3^2 + 4x_1x_2 + 4x_1x_3 + 4x_2x_3 = \mathbf{x}^T A \mathbf{x} = \mathbf{y}^T \left(P^T A P \right) \mathbf{y} = 8y_1^2 + 2y_2^2 + 2y_3^2$.

<u>**Question #4:**</u> If $\mathbf{x}^T A \mathbf{x}$ is a quadratic form in two or three variables and c is a constant, what does the graph of the level set $\mathbf{x}^T A \mathbf{x} = c$ look like?

Theorem 19 *If* $\mathbf{x}^T A \mathbf{x}$ *is a quadratic form in two variables and* c *is a constant, the level curve given by* $\mathbf{x}^T A \mathbf{x} = c$ *is a conic. If* $\mathbf{x}^T A \mathbf{x}$ *is a quadratic form in three variables and* c *is a constant, the level surface given by* $\mathbf{x}^T A \mathbf{x} = c$ *is a quadric.*

Example 20 *In Example 4, let* $P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$; *then* $P^T A P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. *The level curve given by* $2x_1x_2 = 1$ *is the hyperbola* $y_1^2 - y_2^2 = 1$ *since*

$$2x_1x_2 = \mathbf{x}^T A \mathbf{x} = \mathbf{y}^T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y} = y_1^2 - y_2^2.$$

From Example 18 we observe that the level surface $2x_1^2 + 4x_2^2 + 4x_3^2 + 4x_1x_2 + 4x_1x_3 + 4x_2x_3 = 1$ *is the ellipsoid* $8y_1^2 + 2y_2^2 + 2y_3^2 = 1$.

Exercise 21 *Since the quadratic form in Example 11 is positive definite, the quadric given by* $\mathbf{x}^T A\mathbf{x} = 1$ *is an ellipsoid. Eliminate the cross-terms by performing an orthogonal change of variables. Express this ellipsoid in the standard form* $\frac{y_1^2}{a^2} + \frac{y_2^2}{b^2} + \frac{y_3^2}{c^2} = 1$.

Definition 22 *A quadratic form* $\mathbf{x}^T A\mathbf{x}$ *is* non-degenerate *if all eigenvalues of* A *are non-zero.*

Definition 23 *The* signature *of a non-degenerate quadratic form* $\mathbf{x}^T A\mathbf{x}$*, denoted by* $sig(A)$*, is the number of negative eigenvalues of* A.

Theorem 24 *Let* $\mathbf{x}^T A\mathbf{x}$ *be a non-degenerate quadratic form in two variables.*

1. *If* $sig(A) = 0$*, then* $\mathbf{x}^T A\mathbf{x} = 1$ *is an ellipse.*
2. *If* $sig(A) = 1$*, then* $\mathbf{x}^T A\mathbf{x} = 1$ *is an hyperbola.*

Theorem 25 *Let* $\mathbf{x}^T A\mathbf{x}$ *be a non-degenerate quadratic form in three variables.*

1. *If* $sig(A) = 0$*, then* $\mathbf{x}^T A\mathbf{x} = 1$ *is an ellipsoid.*
2. *If* $sig(A) = 1$*, then* $\mathbf{x}^T A\mathbf{x} = 1$ *is an hyperboloid of one sheet.*
3. *If* $sig(A) = 2$*, then* $\mathbf{x}^T A\mathbf{x} = 1$ *is an hyperboloid of two sheets.*

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