The Linear Algebra of Space-Time:
Computing Length Contraction and Time Dilation Near the Speed of Light

Math 422

Minkowski Space

For simplicity, we consider 2-dimensional space-time, or *Minkowski space,* which is the *pseudo inner product space*

$$\mathbb{R}_1^2 = \{(t, x) : t, x \in \mathbb{R}\}$$

with *pseudo inner product* defined by

$$\langle (t_1, x_1), (t_2, x_2) \rangle = t_1 t_2 - x_1 x_2.$$

The *Minkowski norm*

$$\|(t, x)\| = \sqrt{t^2 - x^2}$$

ranges over all non-negative real and positive imaginary values.

Curves of constant Minkowski norm a satisfy the equation

$$t^2 - x^2 = a^2. \tag{1}$$

The parameter a determines three families of such curves:

- When $a = 0$, (1) defines the *light cone* $x = \pm t$.
- When $a \in \mathbb{R}^+$, (1) defines a *real hyperbolic circle of radius* a, which is the hyperbola $t^2 - x^2 = a^2$ inside the light cone.
- When $a = ib \in i\mathbb{R}^+$, (1) defines an *imaginary hyperbolic circle of radius* ib, which is the hyperbola $x^2 - t^2 = b^2$ outside the light cone.

- *Isotropic vectors* have zero Minkowski norm and live on the light-cone.
- *Time-like vectors* have positive real Minkowski norm and live inside the light-cone.
- *Space-like vectors* have positive imaginary Minkowski norm and live outside the light-cone.

Let $C : (t(u), x(u))$, $a \leq u \leq b$, be a parametrized curve in $\mathbb{R}_1^2$ and define the *hyperbolic arc length* s *of* C to be

$$s = \int_a^b \|(t'(u), x'(u))\| \, du = \int_a^b \sqrt{(t')^2 - (x')^2} du.$$

Hyperbolic Circles

Recall that Euclidean angle θ measures the arc length along the unit circle in $\mathbb{R}^2$ from $(1, 0)$ to $(\cos\theta, \sin\theta)$. Note that the area of the sector subtending the arc from $(\cos\theta, -\sin\theta)$ to $(\cos\theta, \sin\theta)$ is also θ. On the other hand, the arc length from $(1, 0)$ to $(\cosh\theta, \sinh\theta)$ along the real hyperbolic circle $t^2 - x^2 = 1$ is θi, but we'd like it to be θ. Thankfully, the area of the real hyperbolic sector subtending the hyperbolic arc from $(\cosh\theta, -\sinh\theta)$ to $(\cosh\theta, \sinh\theta)$ is exactly θ. So let's redefine Euclidean angle θ to be the area of the sector subtending the arc from $(\cos\theta, -\sin\theta)$ to $(\cos\theta, \sin\theta)$; then analogously, define the *hyperbolic angle* θ to be the area of the real hyperbolic sector subtending the real hyperbolic arc from $(\cosh\theta, -\sinh\theta)$ to $(\cosh\theta, \sinh\theta)$. Note that $(\cos\theta, \sin\theta)$ and $(-\cos\theta, -\sin\theta)$ are antipodes on the unit circle, and likewise $(\cosh\theta, \sinh\theta)$ and $(-\cosh\theta, -\sinh\theta)$ are antipodes on the real hyperbolic circle $t^2 - x^2 = 1$.

Parametrize the imaginary hyperbolic circle $C : x^2 - t^2 = b^2$ by $t = b\sinh\theta$, $x = b\cosh\theta$. Then the arc length function along C is

$$s(\theta) = \int_0^\theta \|(b\cosh u, b\sinh u)\| \, du = \sqrt{b^2} \int_0^\theta du = b\theta$$

so that $\theta = s/b$. Now substituting for θ in terms of s reparametrizes C by arc length and gives the position function

$$\mathbf{r}(s) = b\left(\sinh\left(\frac{s}{b}\right), \cosh\left(\frac{s}{b}\right)\right)$$

with velocity

$$\mathbf{v}(s) = \left(\cosh\left(\frac{s}{b}\right), \sinh\left(\frac{s}{b}\right)\right)$$

and constant speed

$$\|\mathbf{v}(s)\| = \sqrt{\cosh^2\left(\frac{s}{b}\right) - \sinh^2\left(\frac{s}{b}\right)} = 1.$$

Thus the unit tangent vector field $\mathbf{T}(s)$ along C is simply the velocity vector field $\mathbf{v}(s)$ along C, and the curvature κ of C is the magnitude of $\mathbf{T}'(s)$, i.e., the instantaneous rate at which $\mathbf{T}$ changes direction. Thus

$$\mathbf{T}'(s) = \mathbf{v}'(s) = \frac{1}{b}\left(\sinh\left(\frac{s}{b}\right), \cosh\left(\frac{s}{b}\right)\right)$$

and the curvature

$$\kappa = \|\mathbf{T}'(s)\| = \frac{1}{b}\sqrt{\sinh^2\left(\frac{s}{b}\right) - \cosh^2\left(\frac{s}{b}\right)} = \frac{i}{b} = -\frac{1}{bi}$$

is the *negative reciprocal of the imaginary radius*. Note that unlike Euclidean circular motion, in which the acceleration and position have opposite directions, the acceleration and position of imaginary hyperbolic circular motion *have the same direction:*

$$\mathbf{b}(s) = \mathbf{v}'(s) = \frac{1}{b^2}\mathbf{r}(s).$$

Nevertheless, hyperbolic and Euclidean circles have similar properties.

Exercises

1. Show that the length of the arc along the real hyperbolic unit circle from $(1, 0)$ to $(\cosh\theta, \sinh\theta)$ is θi.
2. Show that the area of the real hyperbolic unit sector with angle 2θ, i.e., the plane region bounded by the lines $x = \pm t\tanh\theta$ and $t^2 - x^2 = 1$, is exactly θ. (Your integration will require a hyperbolic trigonometric substitution.)
3. Given $a > 0$, parametrize the real hyperbolic circle $C : t^2 - x^2 = a^2$ by $t = a\cosh\theta$, $x = a\sinh\theta$. Reparametrize by arc length and show that the curvature $\kappa = \frac{1}{a}$ and that acceleration and position have opposite directions.

The Poincaré Group

Recall that an isometry (fixing the origin) is a norm preserving linear transformation. Plane Euclidean isometries are rotations ρ_θ about the origin through angle θ and reflections σ_θ in lines through the origin with inclination θ. Euclidean rotations fix circles centered at the origin and send lines through the origin to lines through the origin; reflections fix their reflecting lines point-wise. Rotations ρ_θ are represented by orthogonal matrices with determinant $+1$:

$$[\rho_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Reflections σ_θ are represented by orthogonal matrices with determinant -1:

$$[\sigma_\theta] = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{bmatrix};$$

and in particular,

$$[\sigma_0] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Euclidean isometries are represented by elements of the *orthogonal group* $O(2)$, which is the union of two disjoint components

$$[\rho_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ and } [\rho_\theta][\sigma_0] = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

The component

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \cos\theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

parametrizes the circle

$$C_1 : u_1^2 + u_2^2 = 2$$

in the 2-plane spanned by

$$\mathcal{B}_1 = \left\{ \mathbf{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \right\}.$$

Note that the trivial rotation

$$[\rho_0] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \sqrt{2}\mathbf{u}_1 + 0\mathbf{u}_2$$

is the point $\left(\sqrt{2}, 0\right)$ on this circle in the basis $\mathcal{B}_1$. Similarly, the component

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = \cos\theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

parametrize the circle

$$C_2 : u_3^2 + u_4^2 = 2$$

in the 2-plane spanned by

$$\mathcal{B}_2 = \left\{ \mathbf{u}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}, \ \mathbf{u}_4 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \right\}.$$

Since $\mathcal{B}_1 \cup \mathcal{B}_2$ is linearly independent in $\mathbb{R}^{2\times 2}$, $C_1 \cap C_2 = \varnothing$ and $C_1 \cup C_2 = O(2)$.

The situation in Minkowski space is similar but a bit more complicated. Here the isometries are hyperbolic rotations and hyperbolic reflections. The group of all such transformations, called the *Poincaré group* $O(1,1)$, has four connected components, which appear as the branches of two hyperbolas in $\mathbb{R}^{2\times 2}$. A *hyperbolic rotation* R_θ *through angle* θ is represented by the matrix

$$[R_\theta] = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$$

and is given in coordinates by

$$R_\theta(t,x) = (t\cosh\theta + x\sinh\theta, t\sinh\theta + x\cosh\theta).$$

Note that R_θ fixes hyperbolic circles: If $(\bar{t}, \bar{x}) = R_\theta(t,x)$, then

$$(\bar{t})^2 - (\bar{x})^2 = (t\cosh\theta + x\sinh\theta)^2 - (t\sinh\theta + x\cosh\theta)^2 = t^2 - x^2.$$

Furthermore, R_θ sends lines through the origin to lines through the origin since

$$R_\theta(a,0) = a(\cosh\theta, \sinh\theta).$$

Denote reflection in the t-axis by

$$S_0(t,x) = (t, -x)$$

and reflection in the x-axis by

$$S_\infty(t,x) = (-t, x).$$

Then S_0 and S_∞ fix hyperbolic circles since

$$t^2 - x^2 = t^2 - (-x)^2 = (-t)^2 - x^2.$$

Let l_θ be a line through the origin with inclination $\theta \neq \pm\pi/4$, and let R_θ be the hyperbolic rotation that rotates l_θ onto the t-axis if $-\pi/4 < \theta < \pi/4$, and rotates l_θ onto the x-axis if $\pi/4 < \theta < 3\pi/4$. The *hyperbolic reflection in line* l is the composition

$$S_m = \begin{cases} R_\theta^{-1} S_0 R_\theta, & \text{if } -\pi/4 < \theta < \pi/4 \\ R_\theta^{-1} S_\infty R_\theta, & \text{if } \pi/4 < \theta < 3\pi/4. \end{cases}$$

Hyperbolic reflections fix hyperbolic circles and fix their reflecting lines point-wise. Somewhat surprisingly perhaps, there are no hyperbolic reflections in the lines $x = \pm t$ (see Exercise 7 below). The reflections S_0 and S_∞ are represented by the matrices

$$[S_0] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } [S_\infty] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Minkowski isometries are represented by elements of the Poincaré group $O(1,1)$, which is the union of four mutually disjoint components:

$$[R_\theta] = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}, \qquad [R_\theta][S_0][S_\infty] = \begin{bmatrix} -\cosh\theta & -\sinh\theta \\ -\sinh\theta & -\cosh\theta \end{bmatrix},$$

$$[R_\theta][S_0] = \begin{bmatrix} \cosh\theta & -\sinh\theta \\ \sinh\theta & -\cosh\theta \end{bmatrix}, \qquad [R_\theta][S_\infty] = \begin{bmatrix} -\cosh\theta & \sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix}.$$

The components

$$\begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} = \cosh\theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sinh\theta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} -\cosh\theta & -\sinh\theta \\ -\sinh\theta & -\cosh\theta \end{bmatrix} = -\cosh\theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \sinh\theta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

form the two branches of the hyperbola

$$H_1 : u_1^2 - u_4^2 = 2$$

in the 2-plane spanned by

$$\mathcal{B}_1' = \left\{ \mathbf{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \ \mathbf{u}_4 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \right\}.$$

The trivial hyperbolic rotation

$$[R_0] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \sqrt{2}\mathbf{u}_1 + 0\mathbf{u}_4$$

is the point $\left(\sqrt{2}, 0\right)$ on this hyperbola in the basis $\mathcal{B}_1$. The components

$$\begin{bmatrix} \cosh\theta & -\sinh\theta \\ \sinh\theta & -\cosh\theta \end{bmatrix} = \cosh\theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \sinh\theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} -\cosh\theta & \sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix} = -\cosh\theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \sinh\theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

form the two branches of the hyperbola

$$H_2 : u_3^2 - u_2^2 = 2$$

in the 2-plane spanned by

$$\mathcal{B}_2' = \left\{ \mathbf{u}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \right\}.$$

Since $\mathcal{B}_1' \cup \mathcal{B}_2'$ is linearly independent in $\mathbb{R}^{2\times 2}$, $H_1 \cap H_2 = \varnothing$ and $H_1 \cup H_2 = O(1,1)$.

Exercises

4. Find the matrices $\left[R_\theta^{-1}\right]$, $\left[R_\theta^{-1}S_0R_\theta\right]$, and $\left[R_\theta^{-1}S_\infty R_\theta\right]$.
5. Prove that a hyperbolic reflection fixes its reflecting line point-wise.
6. Prove that S_0 and S_∞ are the only hyperbolic reflections that are also Euclidean reflections.
7. Prove that a Minkowski norm preserving linear transformation that fixes the line $x = t$ point-wise is the identity transformation. Prove the analogous statement for the line $x = -t$.

Special Relativity

The speed of light $c \approx 3 \times 10^8$ m/sec.

- An *event* is a point (t, x) in space-time.
- The *world-line* of a particle P is a parametrized curve $\mathbf{r}(t) = (ct, x(t))$.
- The *relative velocity* of P along its world line is $\mathbf{v}(t) = (c, x'(t))$.
- The *ordinary velocity* of P is $x'(t)$.
- The *relative speed* of P along its world line is $\|\mathbf{v}\| = \sqrt{c^2 - (x')^2}$.
- The *ordinary speed* of P is $|x'|$.

Physical Assumption 1: *The ordinary speed of a particle cannot exceed the speed of light.*

Hence $(x')^2 \le c^2$ and

$$\|\mathbf{v}\|^2 = c^2 - (x')^2 \ge 0.$$

Thus vectors tangent to the world-line of a particle in motion are either time-like or isotropic.

Physical Assumption 2: *A particle traveling at the speed of light has zero mass.*

- The world-line of a particle at rest is a horizontal line inside the light cone.
- The world-line of a particle with non-zero mass is a curve inside the light cone.
- The world-line of a particle with non-zero mass and constant speed is a line inside the light cone.
- The world-line of a photon is a line on the light cone.

Consider the world-line $\mathbf{r}(t) = (ct, x(t))$ of a particle P with non-zero mass and constant ordinary velocity v in the positive x-direction. The relative velocity of P is (c, v) and its relative speed is $\sqrt{c^2 - v^2}$. The arc length function s for the world-line of P is

$$s(t) = \int_0^t \|(c, v)\|\, du = \sqrt{c^2 - v^2}\int_0^t du = t\sqrt{c^2 - v^2}; \tag{2}$$

hence

$$t = \frac{s}{\sqrt{c^2 - v^2}} = \frac{s/c}{\sqrt{1 - v^2/c^2}}.$$

The *proper elapsed time* of P is the quantity

$$\frac{s}{c} = t\sqrt{1 - v^2/c^2}.$$

If P is at rest, for example, its proper elapsed time is t.

Lorentz Transformations

"Lorentz transformations" are special elements of the Poincaré group that change coordinates from reference frame $\bar{K}\,(c\bar{t},\bar{x})$ to reference frame $K\,(ct,x)$ or vice versa as $\bar{K}$ moves along a straight line with constant velocity relative to K.

Physical Assumption 3: *The speed of light c is the same in every frame of reference.*

Assume that $\bar{K}$ moves in the positive x direction in K with constant speed v. A *Lorentz transformation* is a hyperbolic change of coordinates

$$\begin{bmatrix} ct \\ x \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} c\bar{t} \\ \bar{x} \end{bmatrix}. \tag{3}$$

Note that

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

lies in the component of $I \in O(1,1)$ since $A \to I$ as $v \to 0$. Hence A is a hyperbolic rotation

$$R_\theta = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$$

and equation (3) becomes

$$\begin{bmatrix} ct \\ x \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} c\bar{t} \\ \bar{x} \end{bmatrix} \tag{4}$$

Example. Let $\theta = \ln 2$; then $\cosh\theta = \frac{5}{4}$ and $\sinh\theta = \frac{3}{4}$. Thus

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix}.$$

This particular hyperbolic rotation moves the point $(4,0)$ "counterclockwise" along the hyperbola $t^2 - x^2 = 16$ to the point $(5,3)$, and the point $(-3,5)$ "clockwise" along the hyperbola $x^2 - t^2 = 16$ to the point $(0,4)$. The flows along these two hyperbolas asymptotically approach the light cone $x = t$ in the second quadrant (see Figure 1).

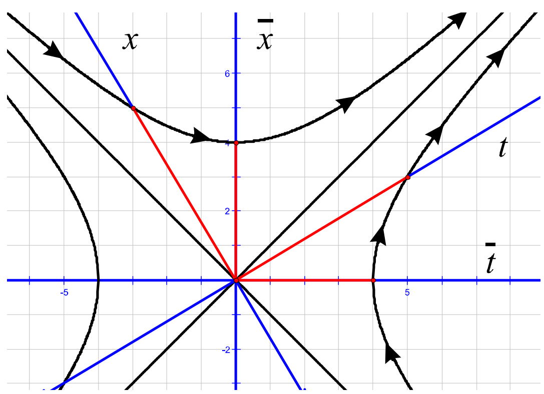

Figure 1. Hyperbolic rotations fix hyperbolas.

Lorentz transformations change coordinates of

- *time-like vectors (inside the light cone) from $\bar{K}$-coordinates to K-coordinates and*
- *space-like vectors (outside the light cone) from K-coordinates to $\bar{K}$-coordinates.*

Let's investigate the motion of a particle P with non-zero mass positioned at the origin $\bar{O}$ in the moving frame $\bar{K}$ as it moves in the positive x-direction in frame K with constant speed v. Since P is at rest in frame $\bar{K}$, its world line in $\bar{K}$ is the parametrized curve $(c\bar{t},0)$ contained in the $\bar{t}$ axis. But when viewed from frame

K, its world line has positive slope inside the light cone and is parameterized by $(ct, x) = (c\bar{t}\cosh\theta, c\bar{t}\sinh\theta)$ via equation (4). Now dividing second components by first components gives

$$\tanh\theta = \frac{x}{ct} = \frac{vt}{ct} = \frac{v}{c}. \tag{5}$$

Now using the fact that $\cosh\theta > 0$, solve for $\cosh\theta$ in the identity

$$1 = \cosh^2\theta - \sinh^2\theta = \cosh^2\theta\left(1 - \tanh^2\theta\right)$$

and obtain

$$\cosh\theta = \frac{1}{\sqrt{1 - \tanh^2\theta}} = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Combining this with equation (5) gives

$$\sinh\theta = \frac{v/c}{\sqrt{1 - v^2/c^2}},$$

and substituting in (4) we obtain

$$t = \frac{1}{\sqrt{1 - v^2/c^2}}\left(\bar{t} + \left(v/c^2\right)\bar{x}\right)$$
$$x = \frac{1}{\sqrt{1 - v^2/c^2}}\left(v\bar{t} + \bar{x}\right).$$

In matrix form this is

$$\begin{bmatrix} t \\ x \end{bmatrix} = \frac{1}{\sqrt{1 - v^2/c^2}}\begin{bmatrix} 1 & v/c^2 \\ v & 1 \end{bmatrix}\begin{bmatrix} \bar{t} \\ \bar{x} \end{bmatrix}. \tag{6}$$

Physical Implications:

Suppose that velocity $v << c$, i.e., v is small relative to the speed of light. Then v/c is negligible and the Lorentz transformations in (6) reduce to

$$\begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix}\begin{bmatrix} \bar{t} \\ \bar{x} \end{bmatrix}$$

or equivalently

$$t = \bar{t} \text{ and } x = v\bar{t} + \bar{x}.$$

These are the *Galilean transformations* of classical physics in which relativistic effects are not apparent. However, when relative speeds v are near c, the Lorentz transformations produce some surprising and dramatic relativistic effects.

Length Contraction:

A spaceship flying through space along a line with constant speed v flies by the international space station. At instant $\bar{t}$ in the moving frame $\bar{K}\,(c\bar{t}, \bar{x})$ of the spaceship, the ship's captain observes that the endpoints of the space station are positioned at $\bar{x}_1$ and $\bar{x}_2$ on the $\bar{x}$-axis; thus its ordinary length measured by the ship's captain is $\Delta\bar{x} = \bar{x}_2 - \bar{x}_1$. Thinking of these measurements as events, their $\bar{K}$-coordinates are $(\bar{t}, \bar{x}_1)$ and $(\bar{t}, \bar{x}_2)$, and we can use equation (6) to change coordinates and calculate the ordinary length $\Delta x = x_2 - x_1$ in the fixed reference frame $K\,(ct, x)$ of the space station. According to (6), the relationship between the lengths $\Delta\bar{x}$ and Δx at instant $\bar{t}$ is

$$\Delta x = x_2 - x_1 = \frac{v\bar{t} + \bar{x}_2}{\sqrt{1 - v^2/c^2}} - \frac{v\bar{t} + \bar{x}_1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - v^2/c^2}}\Delta\bar{x},$$

or equivalently,

$$\Delta\bar{x} = \sqrt{1 - v^2/c^2}\,\Delta x.$$

Since $\sqrt{1-v^2/c^2} < 1$, ordinary length in frame K appears to contract when viewed from from $\bar{K}$. For example, if $v = .73c$, then

$$\sqrt{1-.73^2} \approx \sqrt{.47} \approx .69;$$

if $\Delta x = \sqrt{34} \approx 5.83$, then

$$\Delta \bar{x} \approx (.69)(5.83) \approx 4.$$

So the ordinary length of the space station measured by the spaceship captain appears to be about 31% less than the ordinary length measured by the space station manager.

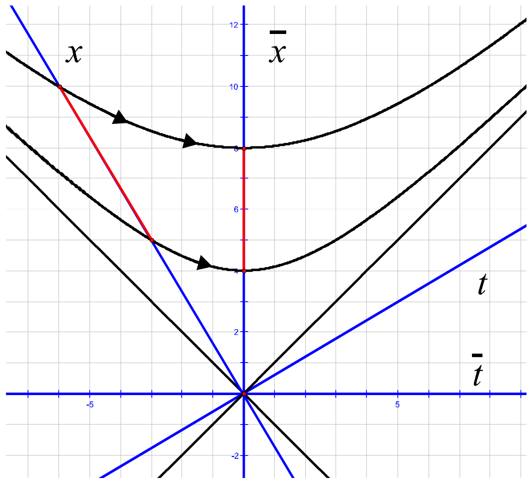

Figure 2. Length contracts as imaginary hyperbolic rotation angle increases.

In summary, *to an observer in a reference frame moving along a straight line with constant speed v relative to a fixed reference frame, the ordinary length of an object at rest in the fixed frame appears to be shorter than it does to an observer in the fixed frame by a factor of* $\sqrt{1-v^2/c^2}$. And indeed, $\Delta \bar{x} \to 0$ as $v \to c$. This phenomenon is called the *Lorentz length contraction.*

Time Dilation:

Now suppose a clock on board the spaceship is positioned at the origin $\bar{O}$ in the moving frame $\bar{K}$ of the spaceship. As the spaceship passes the space station, the captain takes two clock readings $\bar{t}_1$ and $\bar{t}_2$ and determines the elapsed time to be $\Delta \bar{t} = \bar{t}_2 - \bar{t}_1$. Thinking of these two readings as events, their $\bar{K}$-coordinates are $(\bar{t}_1, 0)$ and $(\bar{t}_2, 0)$, and the relationship between the elapsed time $\Delta \bar{t}$ measured in the moving frame and the elapsed time Δt in the fixed frame given by (6) is

$$\Delta t = \frac{\bar{t}_2}{\sqrt{1-v^2/c^2}} - \frac{\bar{t}_1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-v^2/c^2}} \Delta \bar{t}.$$

Since $\frac{1}{\sqrt{1-v^2/c^2}} > 1$, elapsed time in frame $\bar{K}$ appears to dilate when viewed from K. For example, if $v = .73c$, then

$$\frac{1}{\sqrt{1-.73^2}} \approx \frac{1}{\sqrt{.47}} \approx 1.46;$$

if $\Delta \bar{t} = 4$, then

$$\Delta t \approx (1.46)(4) = 5.84.$$

So as far as the space station manager can tell, space station clocks appear to run about 46% faster than clocks on board the passing spaceship.

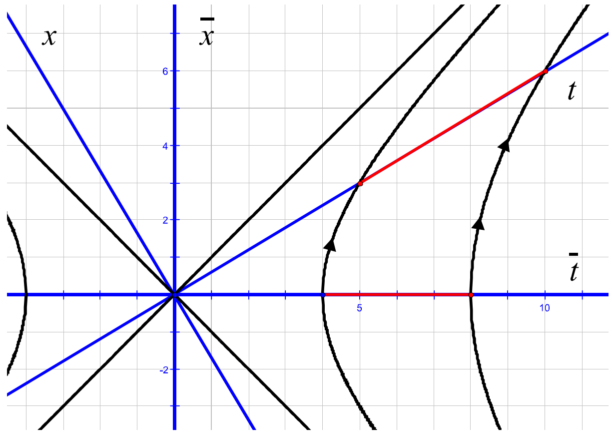

Figure 3. Time dilates as real hyperbolic angle increases.

In summary, *to an observer in a fixed reference frame, the elapsed time measured in a reference frame moving along a straight line with constant speed* v *appears to dilate by a factor of* $1/\sqrt{1 - v^2/c^2}$. And indeed, $\Delta t \to \infty$ as $v \to c$. This phenomenon is called the *Lorentz time dilation*.

Moral: Live fast; live long (relatively speaking...)!

Exercises

8. Consider a particle P positioned at the origin $\bar{O}$ in a frame $\bar{K}$ moving relative to a fixed frame K in the positive x-direction. Prove that the world line of P in K lies inside the light cone.

9. Compute the factors of length contraction and time dilation when $v = .9c$ and $v = .99c$.

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