ELEMENTARY DIFFERENTIAL GEOMETRY

MATH 457 — TEST 1 SOLUTIONS

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March 1, 2013 Name ______________________________

Instructions: Please turn off and stow all cell phone, pagers, and other electronic devices. Calculators may be used, but not the calculator feature on your cell phone. Write all solutions in the space provided. Each problem is worth 10 points.

1. Consider the following parametrization of the unit circle:

$$\overrightarrow{x}(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right),\ t \in \mathbb{R}.$$

Prove that the parameter $t = \tan\theta$, where θ is the angle between the vectors $\overrightarrow{i} = (1,0)$ and $\overrightarrow{x}(t) + \overrightarrow{i}$.

Proof. First note that $\overrightarrow{x} + \overrightarrow{i} = \left(\frac{1-t^2}{1+t^2} + 1, \frac{2t}{1+t^2}\right) = \left(\frac{2}{1+t^2}, \frac{2t}{1+t^2}\right)$; then $comp_{\overrightarrow{i}}\left(\overrightarrow{x} + \overrightarrow{i}\right) = \left(\overrightarrow{x} + \overrightarrow{i}\right) \cdot \overrightarrow{i} = \frac{2}{1+t^2}$. Let $\overrightarrow{j} = (0,1)$; then $comp_{\overrightarrow{j}}\left(\overrightarrow{x} + \overrightarrow{i}\right) = \overrightarrow{x} \cdot \overrightarrow{j} = \frac{2t}{1+t^2}$. Thus

$$\tan\theta = \frac{\left(\overrightarrow{x} + \overrightarrow{i}\right) \cdot \overrightarrow{j}}{\left(\overrightarrow{x} + \overrightarrow{i}\right) \cdot \overrightarrow{i}} = \frac{2t}{1+t^2} \cdot \frac{1+t^2}{2} = t.$$

■

2. Consider the plane curve C parametrized by $\overrightarrow{x}(t) = \left(t^2, t^3 - t\right),\ t \in \mathbb{R}$. Show that C has exactly one point $\overrightarrow{p}$ of self-intersection and find the angle between the two tangent vectors at $\overrightarrow{p}$.

Proof. First note that $\overrightarrow{x}(1) = \overrightarrow{x}(-1) = (1,0)$ is a point of self-intersection. To prove that $(1,0)$ is the unique point of self-intersection, let $\overrightarrow{x}(t)$ be any point of self-intersection. Then there exists $s \neq t$ such that $\left(s^2, s^3 - s\right) = \left(t^2, t^3 - t\right)$; thus $t^2 = s^2$ and $t^3 - t = s^3 - s$. Since $s \neq t$, either $s \neq 0$ or $t \neq 0$. Assume $s \neq 0$. Then $\left(\frac{t}{s}\right)^2 = 1$ and $s \neq t$ implies $t = -s$, and consequently, $s^3 - s = t^3 - t = (-s)^3 - (-s) = -s^3 + s$. Hence $2s^3 - 2s = 0$ and $s(s-1)(s+1) = 0$. Since $s \neq 0$ we have $s = \pm 1$, and it follows that $\overrightarrow{x}(1) = \overrightarrow{x}(-1) = (1,0)$ is the unique point of self-intersection. ■

To compute the angle of self-intersection, note that $\overrightarrow{x}'(t) = \left(2t, 3t^2 - 1\right)$ so that

$$\cos\theta = \frac{\overrightarrow{x}'(-1) \cdot \overrightarrow{x}'(1)}{\left\|\overrightarrow{x}'(-1)\right\| \left\|\overrightarrow{x}'(1)\right\|} = \frac{(-2,2) \cdot (2,2)}{\left\|\overrightarrow{x}'(-1)\right\| \left\|\overrightarrow{x}'(1)\right\|} = 0;$$

thus $\theta = \frac{\pi}{2}$.

3. Consider the catenary $\overrightarrow{x}(t) = (t, \cosh t),\ t \in \mathbb{R}$. Find its evolute $\overrightarrow{E}(t)$ and involute $\overrightarrow{i}(t)$ with $\overrightarrow{i}(0) = (0,1)$, and show that $\overrightarrow{E}(t)$ and $\overrightarrow{i}(t)$ are non-regular on $\mathbb{R}$.

Proof. First, $\overrightarrow{x}'(t) = (1, \sinh t)$, $s'(t) = \left\|\overrightarrow{x}'(t)\right\| = \sqrt{1 + \sinh^2 t} = \sqrt{\cosh^2 t} = \cosh t$, and $\overrightarrow{x}''(t) = (0, \cosh t)$ so that

$$\kappa_g(t) = \frac{\cosh t}{\left(1 + \sinh^2 t\right)^{3/2}} = \frac{\cosh t}{\cosh^3 t} = \frac{1}{\cosh^2 t},$$

$$\overrightarrow{T}(t) = \frac{(1, \sinh t)}{\cosh t} = (\operatorname{sech} t, \tanh t) \quad \text{and} \quad \overrightarrow{U}(t) = \frac{(-\sinh t, 1)}{\cosh t}.$$

Therefore

$$E(t) = (t, \cosh t) + \cosh^2 t \frac{(-\sinh t, 1)}{\cosh t} = (t - \cosh t \sinh t, 2\cosh t).$$

Furthermore,

$$E'(t) = \left(1 - \sinh^2 t - \cosh^2 t, 2\sinh t\right)\Big|_{t=0} = (0,0)$$

and $E(t)$ is non-regular on $\mathbb{R}$. Second, $s(t) = \int \cosh t \; dt = \sinh t + C$ implies

$$\overrightarrow{i}(t) = (t, \cosh t) + (C - \sinh t)(\operatorname{sech} t, \tanh t),$$

and $\overrightarrow{i}(0) = (0,1) = (0,1) + C(1,0) = (C,1)$ implies $C = 0$. Therefore

$$\overrightarrow{i}(t) = (t, \cosh t) - \sinh t(\operatorname{sech} t, \tanh t) = (t - \tanh t, \cosh t - \sinh t \tanh t).$$

Furthermore,

$$\overrightarrow{i}'(t) = \left(1 - \operatorname{sech}^2 t, \sinh t - \cosh t \tanh t - \sinh t \operatorname{sech}^2 t\right)\Big|_{t=0} = (0,0)$$

and $\overrightarrow{i}(t)$ is non-regular on $\mathbb{R}$. ■

4. Consider the graph of a simple closed curve C in polar coordinates $r = f(\theta)$ with $\theta \in [\theta_1, \theta_2]$. Recall that the arc length l of C and area A of the region enclosed by C are given by

$$l = \int_{\theta_1}^{\theta_2} \sqrt{[r'(\theta)]^2 + [r(\theta)]^2} d\theta \text{ and } A = \int_{\theta_1}^{\theta_2} \frac{1}{2}[r(\theta)]^2 \, d\theta.$$

Compute the arc length l and the area A of the region enclosed by the cardioid $r(\theta) = 1 - \cos\theta$, $\theta \in [0, 2\pi]$, and verify that the isoperimetric inequality holds for the cardioid.

First,

$$\begin{aligned} l &= \int_0^{2\pi} \sqrt{\sin^2 t + (1-\cos t)^2} dt = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t - 2\cos t + 1} dt \\ &= \int_0^{2\pi} \sqrt{2 - 2\cos t} dt = \int_0^{2\pi} \sqrt{4\sin^2\left(\frac{t}{2}\right)} dt = 2\int_0^{2\pi} \sin\left(\frac{t}{2}\right) \, dt \\ &= -4\cos\left(\frac{t}{2}\right)\Bigg|_0^{2\pi} = -4(-1-1) = 8 \text{ units.} \end{aligned}$$

Second,

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} (1-\cos t)^2 \, dt = \frac{1}{2}\int_0^{2\pi} \left(1 - 2\cos t + \cos^2 t\right) \, dt = \frac{1}{2}\int_0^{2\pi} \left(1 - 2\cos t + \frac{1}{2} + \frac{1}{2}\cos 2t\right) \, dt \\ &= \frac{1}{2}\left(t - 2\sin t + \frac{1}{2}t + \frac{1}{4}\sin 2t\right)\Bigg|_0^{2\pi} = \frac{1}{2}\left(2\pi + \frac{1}{2}(2\pi)\right) = \frac{3\pi}{2} \text{ units}^2. \end{aligned}$$

Third,

$$l^2 - 4\pi A = 8^2 - 4\pi\left(\frac{3\pi}{2}\right) = 64 - 6\pi^2 > 64 - 6(3.2)^2 = 2.56 > 0,$$

which verifies the isoperimetric inequality.

5. Prove that the vertices of the ellipse $\overrightarrow{x}(t) = (a\cos t, b\sin t)$ with $0 < a < b$ and $t \in [0, 2\pi]$ are $(a,0)$, $(0,b)$, $(-a,0)$, and $(0,-b)$.

Proof. First, $\overrightarrow{x}'(t) = (-a\sin t, b\cos t)$ and $\overrightarrow{x}''(t) = (-a\cos t, -b\sin t)$ so that

$$\kappa_g(t) = \frac{ab}{\left(a^2\sin^2 t + b^2\cos^2 t\right)^{3/2}}.$$

To find the critical points, we set

$$\kappa_g'(t) = \frac{-3ab\left(2a^2\sin t\cos t - 2b^2\sin t\cos t\right)}{2\left(a^2\sin^2 t + b^2\cos^2 t\right)^3} = \frac{-3ab\left(a^2-b^2\right)\sin 2t}{2\left(a^2\sin^2 t + b^2\cos^2 t\right)^3} = 0$$

and obtain

$$\sin 2t = 0.$$

Thus $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and the corresponding vertices are $(a,0)$, $(0,b)$, $(-a,0)$, and $(0,-b)$. ■

6. Calculate the curvature and torsion functions for the twisted cubic $\overrightarrow{x}(t) = \left(t, t^2, t^3\right)$, $t \in \mathbb{R}$.
First, $\overrightarrow{x}'(t) = \left(1, 2t, 3t^2\right)$, $\overrightarrow{x}''(t) = (0, 2, 6t)$, and $\overrightarrow{x}'''(t) = (0,0,6)$. Then $\|\overrightarrow{x}'(t) \times \overrightarrow{x}''(t)\| = \left\|\left(6t^2, -6t, 2\right)\right\| = 2\left(9t^4 + 9t^2 + 1\right)^{1/2}$ and $(\overrightarrow{x}'(t) \times \overrightarrow{x}''(t)) \cdot \overrightarrow{x}'''(t) = 12$. Thus

$$\kappa(t) = 2\frac{\left(9t^4 + 9t^2 + 1\right)^{1/2}}{\left(9t^4 + 4t^2 + 1\right)^{3/2}} \text{ and } \tau(t) = \frac{12}{4\left(9t^4 + 9t^2 + 1\right)} = \frac{3}{9t^4 + 9t^2 + 1}.$$

Problems 7-9 refer to the helix: $\overrightarrow{x}(t) = \left(2\cos t, 2\sin t, \sqrt{5}t\right)$, $t \in \mathbb{R}$.

7. Compute the Frenet frame $\left\{\overrightarrow{T}(t), \overrightarrow{P}(t), \overrightarrow{B}(t)\right\}$.
First, $\overrightarrow{x}'(t) = \left(-2\sin t, 2\cos t, \sqrt{5}\right)$ and $\|\overrightarrow{x}'(t)\| = 3$ implies

$$\overrightarrow{T}(t) = \left(-\frac{2}{3}\sin t, \frac{2}{3}\cos t, \frac{\sqrt{5}}{3}\right).$$

Next $\overrightarrow{T}'(t) = \left(-\frac{2}{3}\cos t, -\frac{2}{3}\sin t, 0\right)$ and $\left\|\overrightarrow{T}'(t)\right\| = \frac{2}{3}$ implies

$$\overrightarrow{P}(t) = (-\cos t, -\sin t, 0).$$

Finally,

$$\overrightarrow{B}(t) = \overrightarrow{T}(t) \times \overrightarrow{P}(t) = \left(\frac{\sqrt{5}}{3}\sin t, -\frac{\sqrt{5}}{3}\cos t, \frac{2}{3}\right).$$

8. Find the equation of the osculating plane at $t = 0$.
Since

$$\overrightarrow{B}(0) = \left(0, -\frac{\sqrt{5}}{3}, \frac{2}{3}\right),$$

the equation of osculating plane is

$$\left(0, -\sqrt{5}, 2\right) \cdot (x-2, y, z) = 0$$

or

$$\sqrt{5}y = 2z.$$

9. Find the center c and radius R of the osculating circle at $t = 0$.
First, we compute the radius $R(t)$: By the first Frenet formula,

$$T'(t) = s'(t)\,\kappa(t)\,\overrightarrow{P}(t)$$

$$\left(-\frac{2}{3}\cos t, -\frac{2}{3}\sin t, 0\right) = 3\kappa(t)(-\cos t, -\sin t, 0).$$

Thus

$$\kappa(t) = \frac{2}{9} \text{ and } R(0) = \frac{9}{2}.$$

The center of the osculating circle lies on the concave side along the normal line at $\overrightarrow{x}(0) = (2,0,0)$. Since $\overrightarrow{P}(0)$ points to the concave side in the normal direction,

$$c = (2,0,0) + \frac{9}{2}\overrightarrow{P}(0) = \left(2 - \frac{9}{2}, 0, 0\right) = \left(-\frac{5}{2}, 0, 0\right).$$

10. Prove that if $\overrightarrow{x}(s) = (x_1(s), x_2(s))$ is a plane curve parametrized by arc length such that $\overrightarrow{x}(0) = \left(\frac{1}{A}, 0\right)$, $\overrightarrow{T}(0) = (0,1)$, and $\kappa_g(s) = A$, where A is a positive constant, then $\overrightarrow{x}(s) = \left(\frac{1}{A}\cos As, \frac{1}{A}\sin s\right)$. (Hint: The components of the Frenet formula is a system of ODEs.)

Proof. Since s is the arc length parameter, $\overrightarrow{x}$ has unit speed and $\overrightarrow{T} = (x_1', {x_2}')$. Therefore $\overrightarrow{U} = (-x_2', x_1')$ and the components of the Frenet formula $\overrightarrow{T}' = \kappa_g \overrightarrow{U}$ give the system of ODEs

$$x_1'' = -Ax_2' \text{ and } x_2'' = Ax_1'. \tag{1}$$

By differentiating the first equation and substituting for x_2'' we obtain

$$x_1''' + A^2 x_1' = 0,$$

and similarly, differentiating the second equation and substituting gives

$$x_2''' + A^2 x_2' = 0.$$

The general solutions of these second order equations in x_i' are

$$x_i' = B_i \cos(As) + C_i \sin(As), \tag{2}$$

and integrating we obtain the general solutions

$$x_i = \frac{B_i}{A}\sin(As) - \frac{C_i}{A}\cos(As) + D_i.$$

Now $\overrightarrow{x}(0) = \left(\frac{1}{A}, 0\right) = \left(-\frac{C_1}{A} + D_1, -\frac{C_2}{A} + D_2\right)$ implies $C_1 = AD_1 - 1$ and $C_2 = AD_2$. Also $\overrightarrow{T}(0) = (0,1) = (B_1, B_2)$ implies $B_1 = 0$ and $B_2 = 1$. Now differentiating (2) we obtain

$$x_i'' = -AB_i \sin(As) + AC_i \cos(As).$$

From (1) we have

$$-AB_1 \sin(As) + AC_1 \cos(As) = -AB_2 \cos(As) - AC_2 \sin(As),$$

and at $s = 0$ we have

$$C_1 = -1 \Rightarrow D_1 = 0$$

Similarly, from (1) we have

$$-AB_2 \sin(As) + AC_2 \cos(As) = AB_1 \cos(As) + AC_1 \sin(As),$$

and at $s = 0$ we have

$$C_2 = 0 \Rightarrow D_2 = 0.$$

Therefore

$$\overrightarrow{x}(s) = \left(\frac{1}{A}\cos(As), \frac{1}{A}\sin(As)\right).$$

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