Chapter 1

Geometry of Plane Curves

Definition 1 *A (parametrized) curve in* $\mathbb{R}^n$ *is a piece-wise differentiable function* $\boldsymbol{\alpha} : (a,b) \to \mathbb{R}^n$*. If* I *is any other subset of* $\mathbb{R}$*,* $\boldsymbol{\alpha} : I \to \mathbb{R}^n$ *is a curve provided that* $\boldsymbol{\alpha}$ *extends to a piece-wise differentiable function on some open interval* $(a,b) \supset I$*. The trace of a curve* $\boldsymbol{\alpha}$ *is the set of points* $\boldsymbol{\alpha}((a,b)) \subset \mathbb{R}^n$*. A subset* $S \subset \mathbb{R}^n$ *is parametrized by* $\boldsymbol{\alpha}$ *if and only if there exists set* $I \subseteq \mathbb{R}$ *such that* $\boldsymbol{\alpha} : I \to \mathbb{R}^n$ *is a curve for which* $\boldsymbol{\alpha}(I) = S$*. A one-to-one parametrization* α *assigns and orientation to a curve* C *thought of as the direction of increasing parameter, i.e., if* $s < t$*, the direction of increasing parameter runs from* $\alpha(s)$ *to* $\alpha(t)$*.*

A subset $S \subset \mathbb{R}^n$ may be parametrized in many ways.

Example 2 *Consider the circle* $C : x^2 + y^2 = 1$ *and let* $\omega > 0$*. The curve* $\boldsymbol{\alpha} : \left(0, \frac{2\pi}{\omega}\right] \to \mathbb{R}^2$ *given by*

$$\boldsymbol{\alpha}(t) = (\cos \omega t, \sin \omega t).$$

is a parametrization of C *for each choice of* $\omega \in \mathbb{R}$*. Recall that the velocity and acceleration are respectively given by*

$$\boldsymbol{\alpha}'(t) = (-\omega \sin \omega t, \omega \cos \omega t)$$

and

$$\boldsymbol{\alpha}''(t) = \left(-\omega^2 \cos 2\pi\omega t, -\omega^2 \sin 2\pi\omega t\right) = -\omega^2 \boldsymbol{\alpha}(t).$$

The speed is given by

$$v(t) = \|\boldsymbol{\alpha}'(t)\| = \sqrt{(-\omega \sin \omega t)^2 + (\omega \cos \omega t)^2} = \omega. \tag{1.1}$$

The various parametrizations obtained by varying ω change the velocity, acceleration and speed but have the same trace, i.e., $\boldsymbol{\alpha}\left(\left(0, \frac{2\pi}{\omega}\right]\right) = C$ *for each* ω.

Definition 3 *Let* $S \subset \mathbb{R}^n$ *and let* $\boldsymbol{\alpha} : (a,b) \to \mathbb{R}^n$ *be a parametrization of* S. *A curve* $\boldsymbol{\beta} : (c,d) \to \mathbb{R}^n$ *is a reparametrization of* $\boldsymbol{\alpha}$ *if there is a differentiable function* $h : (c,d) \to (a,b)$ *such that* $\boldsymbol{\beta} = \boldsymbol{\alpha} \circ h$. *Furthermore, if* $h'(t) > 0$ *(or* $h'(t) < 0$*) for all* $t \in (c,d)$ *we say that* $\boldsymbol{\beta}$ *is a positive (or negative) reparametrization of* $\boldsymbol{\alpha}$.

Intuitively, positive reparametrizations preserve orientation while negative reparametrizations reverse orientation.

Example 4 *Refer to Example 2 above and define* $h : (0, \pi] \to (0, 2\pi]$ *by* $h(t) = 2t$. *Consider curves* $\boldsymbol{\alpha}(t) = (\cos t, \sin t)$ *and* $\boldsymbol{\beta}(t) = (\cos 2t, \sin 2t)$, *which correspond to choices* $\omega = 1$ *and* $\omega = 2$, *respectively. Both orient* α *in the counterclockwise direction. Then* $(\boldsymbol{\alpha} \circ h)(t) = \boldsymbol{\alpha}(h(t)) = \boldsymbol{\alpha}(2t) = (\cos 2t, \sin 2t) = \boldsymbol{\beta}(t)$ *and* $h'(t) = 2 > 0$ *so* $\boldsymbol{\beta}$ *is a positive reparametrization of* $\boldsymbol{\alpha}$. *Note that this reparametrization preserves orientation.*

1.1 Some Classical Plane Curves

1.1.1 Cycloids

Consider two circular disks A and B with the same center and respective radii a and b. Assume that disk B is locked onto disk A so that both rotate in the same direction with the same angular velocity.

Definition 5 *The locus of points traced out by a point* P *on the boundary of disk* B *as disk* A *rolls without slipping along a straight line is called a*

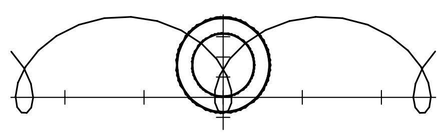
1. *Prolate cycloid if* $a < b$
2. *Cycloid if* $a = b$
3. *Curate cycloid if* $a > b$.

The standard parametrization of the cycloid generated by P as A rolls without slipping along the x-axis in the positive direction is constructed in two steps: When $t = 0$, assume that A is positioned with its center on the y-axis at $(0, a)$ and P is positioned at the point $(0, a - b)$. Assume that disk A rolls with constant angular velocity of one revolution every 2π seconds so that the common center is positioned at (at, a) at time t. First parametrize the circle of radius b centered at the origin so that orientation is clockwise, the position of P at time $t = 0$ is $(0, -b)$ and the angular velocity is constantly 1 revolution per 2π seconds, i.e.,

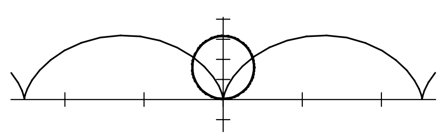
$$(b \sin(-t), -b \cos(-t)) = (-b \sin t, -b \cos t).$$

Second, at each time t, translate by the vector (at, a) and move the circle into correct position:

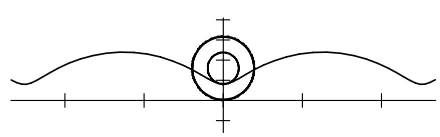
$$\begin{aligned} \boldsymbol{\gamma}(t) &= (-b \sin t, -b \cos t) + (at, a) \\ &= (at - b \sin t, a - b \cos t). \end{aligned}$$



A prolate cycloid with $a = 2$ and $b = 3$.



A cycloid with $a = b = 2$.



A curate cycloid with $a = 2$ and $b = 1$.

1.1.2 Cardioids

Consider two circular disks A and B with the same center and respective radii a and b. Assume that disk B is locked onto A so that both rotate in the same direction with the same angular velocity.

Definition 6 *The locus of points traced out by a point P on the boundary of disk B as disk A rolls without slipping along a circle C of radius c is called a*

1. *Prolate cardioid if $a < b$*
2. *Cardioid if $a = b$*
3. *Curate cardioid if $a > b$.*

A cardioid can be parametrized in much the same way that the cycloid was parametrized above, but with one critical difference: The rotation of disk A about the origin as it rolls along C contributes to the total turning angle. When $t = 0$, position C with center at the origin, position A with center at $(a + c, 0)$ and position point P at $(a + b + c, 0)$. Roll circle A with constant angular velocity of one revolution every 2π seconds. With B translated to the origin, point P has position $(b, 0)$ when $t = 0$. This time disk A rolls ct units along C in t seconds. Now if A were rolling along a straight line, the radius from the common center of B and A to P would turn counterclockwise through an angle ct/a in t seconds. So provisionally parametrize B by

$$(b \cos(ct/a), b \sin(ct/a))$$

For the moment, imagine sliding circle A (without rolling) ct units along C. Then the radius from the common center of B and A to P would turn counterclockwise through an angle t. So the total turning angle as A rolls along C is $t + ct/a$ and the desired parametrization of B is

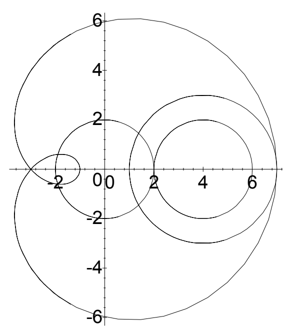
$$\boldsymbol{\beta}(t) = (b \cos(t + ct/a), b \sin(t + ct/a)).$$

Now as A rolls counterclockwise along C, the center of disk A traces out a circle of radius $a + c$ centered at the origin. Parametrize the trajectory of the center of circle A at time t by

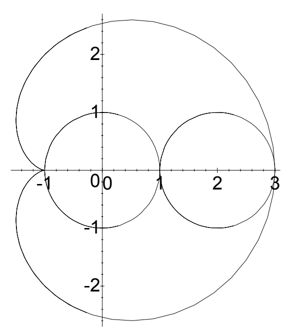
$$\boldsymbol{\alpha}(t) = ((a + c) \cos t, (a + c) \sin t).$$

To obtain the position of point P at time t, translate by the vector $\boldsymbol{\alpha}(t)$ at time t and obtain

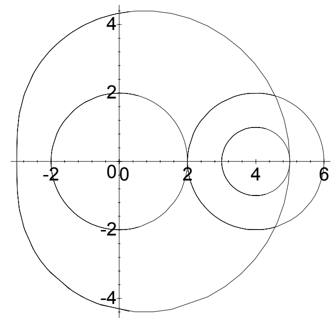
$$\boldsymbol{\gamma}(t) = \boldsymbol{\beta}(t) + \boldsymbol{\alpha}(t) = (b\cos(t + ct/a) + (a + c)\cos t, b\sin(t + ct/a) + (a + c)\sin t).$$



A prolate cardioid with $a = c = 2$ and $b = 3$.



A cardioid with $a = b = c = 1$.



A curate cardioid with $a = c = 2$and $b = 1$.

Exercise 1.1.1 *Following the construction of the cardioid, parametrize the trace of a point P if circle A rolls without slipping along the inside of circle C. Use mathematica to draw the related curate and prolate objects.*

1.1.3 The Tractrix

Let $a > 0$. A fox at position $(a, 0)$ on the x-axis spies a rabbit running up the y-axis at constant speed. The instant the rabbit passes through the origin, the fox gives chase. Assume that the distance between the rabbit and the fox remains constantly a and that the fox's nose always points along the line tangent to his trajectory which passes through the position of the rabbit (see text p. 61). The trajectory of the fox is that part of the *tractrix* in the upper half-plane. The lower part of the tractrix is the reflection of the fox's trajectory in the x-axis.

The tractrix is fundamentally important in geometry. The surface of revolution obtained by revolving the tractrix about the y-axis is called a *pseudosphere.* Interestingly, the sum of the angles in a triangle on a pseudosphere is strictly less than 180°. So the pseudosphere gives us a setting in which to study non-Euclidean geometry. More about that later.

From the diagram on p. 61 of the text, we see that the right triangle with base x and hypotenuse a has altitude $\sqrt{a^2 - x^2}$. Hence the slope of the line tangent to the tractrix at (x, y) is

$$\frac{dy}{dx} = -\frac{\sqrt{a^2 - x^2}}{x},$$

subject to the initial condition $y(a) = 0$. Separating variables gives

$$y = -\int \frac{\sqrt{a^2 - x^2}}{x}\, dx.$$

Using the trigonometric substitution $x = a\sin(t)$ we obtain

$$y = a\int \sin(t) - \csc(t)\ dt = a\ln|\csc(t) + \cot(t)| - a\cos(t) + C$$

Since $\sin(t) = \frac{x}{a}$, when $x = a$ we have $t = \frac{\pi}{2}$. Hence $C = 0$ and

$$y = a\ln|\csc(t) + \cot(t)| - a\cos(t).$$

This gives the following parametrization of the tractrix:

$$\alpha(t) = (a\sin(t), a\ln|\csc(t) + \cot(t)| - a\cos(t)),\ \text{with } t \in (0, \pi).$$

To obtain the parametrization in the text, note that

$$\ln|\csc(t) + \cot(t)| = \ln\left|\frac{1 + \cos(t)}{\sin(t)}\right|^{(-2)\cdot(-\frac{1}{2})} = -\frac{1}{2}\ln\left(\frac{\sin^2(t)}{(1 + \cos(t))^2}\right)$$

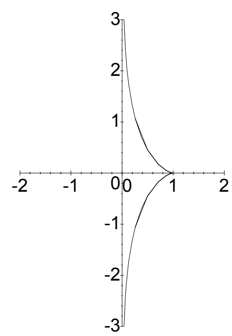
$$= -\frac{1}{2}\ln\left(\frac{(1 - \cos(t)}{(1 + \cos(t)}\right) = -\ln\sqrt{\frac{(1 - \cos(t)}{(1 + \cos(t)}} = -\ln(\tan(\frac{t}{2})).$$

So we may rewrite the parametrization α as:

$$\alpha(t) = a\left(\sin(t), -\cos(t) - \ln\left(\tan\left(\frac{t}{2}\right)\right)\right),\ \text{with } t \in (0, \pi).$$

Since the tractrix is symmetrical with respect to the x-axis, reversing the sign of the y-component simply reverses the direction of the parameter. Hence we may reparametrize the tractrix by

$$\beta(t) = a\left(\sin(t), \ln\left(\tan\left(\frac{t}{2}\right)\right) + \cos(t)\right),\ \text{with } t \in (0, \pi).$$



The tractrix with $a = 1$.

Exercise 1.1.2 *A rabbit starts at the origin and runs up the y-axis with constant speed a. At the same instant a fox, running with constant speed $b > a$, starts at point $(c, 0)$ and pursues the rabbit (see diagram below). Parametrize the trajectory of the fox and and determine how far he must run to catch the rabbit.*

1.2 Arc Length and Curvature

We are interested in the "intrinsically geometric" properties of a curve $\boldsymbol{\alpha}$, i.e., those quantities that are independent of the choice of parametrization. Two such quantities are *arc length* and *curvature.*

1.2.1 Arc Length

Definition 7 *Let $\boldsymbol{\alpha} : [a, b] \to \mathbb{R}^n$ be a curve. Define the length of $\boldsymbol{\alpha}$ by*

$$Length[a,b]\,[\boldsymbol{\alpha}] = Length(\boldsymbol{\alpha}) = \int_a^b \left\| \boldsymbol{\alpha}'(t) \right\| \, dt.$$

Example 8 *Referring again to Example 2, we see from 1.1 that*

$$Length\,[\boldsymbol{\alpha}] = \omega \int_0^{2\pi/\omega} dt = 2\pi,$$

for every choice of ω*.*

This suggests the following:

Theorem 9 *Let* $\boldsymbol{\beta} : [c,d] \to \mathbb{R}^n$ *be a reparametrization of* $\boldsymbol{\alpha} : [a,b] \to \mathbb{R}^n$*. Then*

$$Length\,[\boldsymbol{\beta}] = Length\,[\boldsymbol{\alpha}]\,,$$

i.e. arc length is independent of the choice of parametrization.

Proof. By assumption, there is a differentiable function $h : (c,d) \to (a,b)$ such that $\boldsymbol{\beta} = \boldsymbol{\alpha} \circ h$. Assume that $h'(t) > 0$; the case $h'(t) < 0$ is left to the reader. By the chain rule,

$$\begin{aligned} \|\boldsymbol{\beta}'(t)\| &= \left\| \frac{d}{dt}\boldsymbol{\alpha}(h(t)) \right\| = \|h'(t)\boldsymbol{\alpha}'(h(t))\| = |h'(t)|\,\|\boldsymbol{\alpha}'(h(t))\| \\ &= h'(t)\,\|\boldsymbol{\alpha}'(h(t))\|\,. \end{aligned}$$

Let $u = h(t)$, then $du = h'(t)dt$. Extend h continuously to $h : [c,d] \to [a,b]$ so that $h\,(c) = a$ and $h\,(d) = b$. Then $u(c) = h(c) = a$ and $u(d) = h(d) = b$ and by a change ofvariables we have

$$Length\,[\boldsymbol{\beta}] = \int_c^d \left\| \boldsymbol{\alpha}'(h(t)) \right\|\; h'(t)dt = \int_a^b \|\boldsymbol{\alpha}'(u)\|\; du = Length\,[\boldsymbol{\alpha}]\,.$$

■

Definition 10 *Let* $\boldsymbol{\alpha} : (a,b) \to \mathbb{R}^n$ *be a curve and let* $c \in (a,b)$*. The* <u>*arc length function of* $\boldsymbol{\alpha}$</u> *with initial point* c *is defined by*

$$s(t) = \int_c^t \|\boldsymbol{\alpha}'(u)\|\; du.$$

Note that if $d \in (a,b)$*, then*

$$Length[c,d]\,[\boldsymbol{\alpha}] = \int_c^d \|\boldsymbol{\alpha}'(u)\|\; du = s(d) - s(c).$$

Definition 11 *A curve* $\boldsymbol{\alpha} : (a,b) \to \mathbb{R}^n$ *is* <u>*regular*</u> *if* $\boldsymbol{\alpha}'(t) \neq \mathbf{0}$ *for all* t*.*

Thus the velocity of a regular curve never vanishes.

Example 12 *All curves in Example 2 have constant speed and are therefore regular.*

Regular curves never have cusps since velocity must vanish in order change directions abruptly. Here is an example of a non-regular curve:

Example 13 *The semicubical parabola is given parametrically by*

$$\mathbf{sc}(t) = (t^2, t^3),$$

with $t \in (-\infty, \infty)$ *(see text p. 21). This curve is non-regular since its velocity* $\mathbf{sc}'(t) = (2t, 3t^2)$ *vanishes when* $t = 0$*. Note the cusp in the trace at the origin.*

Theorem 14 *Every regular curve can be reparametrized with unit speed.*

Proof. Let $\boldsymbol{\alpha} : (a, b) \to \mathbb{R}^n$ be a regular curve and consider any arc length function of $\boldsymbol{\alpha}$

$$s(t) = \int_c^t \|\boldsymbol{\alpha}'(u)\| \, du,$$

where $c \in (a, b)$. By the Second Fundamental Theorem of Calculus (Smith & Minton, p. 368),

$$s'(t) = \|\boldsymbol{\alpha}'(t)\| .$$

Since $\boldsymbol{\alpha}$ is regular by assumption, $s'(t) > 0$ for all t, which implies that s is strictly increasing and hence one-to-one. By the Inverse Function Theorem (Smith & MInton, p. 491) s has a differentiable inverse u and $u'(t) = \frac{1}{s'(u(t))} = \frac{1}{\|\boldsymbol{\alpha}'(u(t))\|}$. Consider the reparametrization $\boldsymbol{\beta}$ given by

$$\boldsymbol{\beta}(t) = \boldsymbol{\alpha}(u(t)).$$

By the chain rule,

$$\boldsymbol{\beta}'(t) = u'(t)\boldsymbol{\alpha}'(u(t))$$

so that

$$\|\boldsymbol{\beta}'(t)\| = \|u'(t)\boldsymbol{\alpha}'(u(t))\| = u'(t) \|\boldsymbol{\alpha}'(u(t))\| = \frac{1}{\|\boldsymbol{\alpha}'(u(t))\|} \|\boldsymbol{\alpha}'(u(t))\| = 1.$$

■

If $\boldsymbol{\alpha} : (a, b) \to \mathbb{R}^n$ is parametrized with unit speed, then its arc length function starting at a is

$$s(t) = \int_a^t \|\boldsymbol{\alpha}'(u)\| \ du = \int_a^t du = t - a.$$

So the arc length along the trace of $\boldsymbol{\alpha}$ from $\boldsymbol{\alpha}(a)$ to $\boldsymbol{\alpha}(t)$ is exactly the change in the parameter t. Consequently, we refer to unit speed curves as being *parametrized by arc length* and often use s as the parameter instead of t.

Example 15 *Consider the circle $x^2 + y^2 = a^2$ parametrized by $\boldsymbol{\alpha}(t) = (a \cos t, a \sin t)$, where $t \in (0, 2\pi]$. To reparametrize $\boldsymbol{\alpha}$ by arc length we compose it with the inverse of its arc length function at* 0.

$$s = s(t) = \int_0^t \sqrt{(-a \sin u)^2 + (a \cos u)^2} dt = a \int_0^t \ du = at.$$

So its inverse is just

$$t = t(s) = \frac{s}{a}$$

and the desired reparametrization is

$$\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(t(s)) = (a \cos \frac{s}{a}, a \sin \frac{s}{a}).$$

Exercise 1.2.1 *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *be points in* $\mathbb{R}^n$*. Initially parametrize the line segment from* $\mathbf{p}$ *to* $\mathbf{q}$ *by*

$$\boldsymbol{\alpha}(t) = (1 - t)\mathbf{p} + t\mathbf{q},$$

where $t \in (0, 1)$. Reparametrize this line segment by arc length.

1.2.2 Signed Curvature

Let $\boldsymbol{\alpha} : (a, b) \to \mathbb{R}^2$ be a regular plane curve. The complex structure on $\mathbb{R}^2$ is the linear map $\mathbf{J} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mathbf{J}(p_1, p_2) = (-p_2, p_1),$$

which rotates the vector (p_1, p_2) counterclockwise 90°.

Definition 16 *The (signed) curvature $\kappa_2[\boldsymbol{\alpha}]$ of $\boldsymbol{\alpha}$ is given by*

$$\kappa_2[\boldsymbol{\alpha}](t) = \frac{\boldsymbol{\alpha}''(t) \bullet \mathbf{J}(\boldsymbol{\alpha}'(t))}{\left\|\boldsymbol{\alpha}'(t)\right\|^3} . \tag{1.2}$$

The reciprocal of the curvature $\frac{1}{\kappa_2[\boldsymbol{\alpha}](t)}$ is the radius of curvature of $\boldsymbol{\alpha}$.

For a more familiar form, let $\boldsymbol{\alpha}(t) = (x(t), y(t))$; then $\boldsymbol{\alpha}'(t) = (x'(t), y'(t))$, $\mathbf{J}(\boldsymbol{\alpha}'(t)) = (-y'(t), x'(t))$, and $\boldsymbol{\alpha}''(t) = (x''(t), y''(t))$. So (1.2) can be rewritten as

$$\kappa_2[\boldsymbol{\alpha}](t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{\left([x'(t)]^2 + [y'(t)]^2\right)^{3/2}} .$$

This formulation may differ in sign from the formula derived in Calculus III (c.f. Smith & Minton, p. 888 with $x = t$, $y = f(t)$) whose numerator is enclosed in absolute values. Consequently, curvature defined in Calculus III is strictly non-negative.

Example 17 *Consider the circle $x^2 + y^2 = a^2$ parametrized by $\boldsymbol{\alpha}(t) = (a\cos t, a\sin t)$, where $t \in (0, 2\pi)$.*

$$\boldsymbol{\alpha}'(t) = (-a\sin t, a\cos t)$$

$$\mathbf{J}(\boldsymbol{\alpha}'(t)) = (-a\cos t, -a\sin t)$$
$$\boldsymbol{\alpha}''(t) = (-a\cos(t), -a\sin(t))$$
$$\kappa_2[\boldsymbol{\alpha}](t) = \frac{a^2\cos^2 t + a^2\sin^2 t}{a^3} = \frac{1}{a}.$$

Thus the radius of curvature of a circle is exactly its radius a.

Exercise 1.2.2 *Let $\mathbf{p}$ and $\mathbf{q}$ be points in $\mathbb{R}^n$. Parametrize the line through $\mathbf{p}$ and $\mathbf{q}$ by*

$$\boldsymbol{\alpha}(t) = (1-t)\mathbf{p} + t\mathbf{q},$$

where $t \in (-\infty, \infty)$. Show that the curvature is identically zero.

Exercise 1.2.3 *Find a parametrization $\boldsymbol{\alpha} : [0, 2\pi] \to \mathbb{R}^2$ of the ellipse $16x^2 + 4y^2 = 64$ such that $\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}(2\pi) = (2, 0)$. Using your parametrization, find the extreme (maximum and minimum) values of the curvature on the interval $[0, 2\pi]$.*

Recall that the velocity $\boldsymbol{\alpha}'(t)$ is tangent to trace of $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(t)$. When $\boldsymbol{\alpha}$ is regular, we can normalize the velocity.

Definition 18 *Let $\boldsymbol{\alpha} : I \to \mathbb{R}^2$ be a regular curve. The unit tangent vector at $\boldsymbol{\alpha}(t)$ is*

$$\mathbf{T}(t) = \frac{\boldsymbol{\alpha}'(t)}{\|\boldsymbol{\alpha}'(t)\|} .$$

The unit normal vector at $\boldsymbol{\alpha}(t)$ is

$$\mathbf{N}(t) = \mathbf{J}(\mathbf{T}(t)).$$

The unit tangent and unit normal vectors can be used to interpret the sign of the curvature as follows:

1. $\boldsymbol{\alpha}$ has positive curvature at $\boldsymbol{\alpha}(t)$ iff $\mathbf{T}(t)$ is instantaneously turning counterclockwise iff $\mathbf{T}(t)$ instantaneously turning towards $\mathbf{N}(t)$.
2. $\boldsymbol{\alpha}$ has negative curvature at $\boldsymbol{\alpha}(t)$ iff $\mathbf{T}(t)$ is instantaneously turning clockwise iff $\mathbf{T}(t)$ instantaneously turning away from $\mathbf{N}(t)$.

Since the notion of clockwise/counterclockwise is not meaningful in space, signed curvature only makes sense in the plane. We now observe that up to sign, curvature is independent of the parametrization. Let λ be a non-zero real number and define

$$sign(\lambda) = \frac{\lambda}{|\lambda|} = \pm 1.$$

Theorem 19 *Let $\boldsymbol{\alpha} : (a,b) \to \mathbb{R}^2$ be a regular curve and let $\boldsymbol{\beta} :(c,d) \to \mathbb{R}^2$ be a reparametrization. Write $\boldsymbol{\beta} = \boldsymbol{\alpha} \circ h$, with $h : (c,d) \to (a,b)$ is differentiable. Then*

$$\kappa_2[\boldsymbol{\beta}](t) = sign(h'(t))\ \kappa_2[\boldsymbol{\alpha}](h(t)),$$

wherever $sign(h'(t))$ is defined.

Proof. By the chain rule, $\boldsymbol{\beta}'(t){=}h'(t)\ \boldsymbol{\alpha}'(h(t))$. Since $\mathbf{J}$ is a linear map, $\mathbf{J}(\boldsymbol{\beta}'(t)) = h'(t)\ \mathbf{J}(\boldsymbol{\alpha}'(h(t)))$. By the product rule along with the chain rule, $\boldsymbol{\beta}''(t){=}h''(t)\ \boldsymbol{\alpha}'(h(t)) + [h'(t)]^2 \boldsymbol{\alpha}''(h(t))$. Since $\mathbf{v} \bullet \mathbf{J}(\mathbf{v}) = 0$ we have

$$\kappa_2[\boldsymbol{\beta}](t) = \frac{\{h''(t)\ \boldsymbol{\alpha}'(h(t)) + [h'(t)]^2 \boldsymbol{\alpha}''(h(t))\} \bullet h'(t)\ \mathbf{J}(\boldsymbol{\alpha}'(h(t)))}{\|h'(t)\ \boldsymbol{\alpha}'(h(t))\|^3}$$

$$= \frac{[h'(t)]^3}{|h'(t)|^3} \frac{\boldsymbol{\alpha}''(h(t)) \bullet \mathbf{J}(\boldsymbol{\alpha}'(h(t)))}{\|\boldsymbol{\alpha}'(h(t))\|^3} = sign(h'(t))\ \kappa_2[\boldsymbol{\alpha}](h(t)).$$

■

This explains the absolute value in the numerator of the curvature formula from Calculus III. One needs absolute values if the value of curvature is to be strictly *independent* of the parametrization. However, in the special setting of plane curves, it is useful to know how a curve bends relative to the unit normal.

Corollary 20 *(The First Frenet Formula) If $\boldsymbol{\beta}$ is a unit speed plane curve, then*

$$\mathbf{T}' = \kappa_2[\boldsymbol{\beta}]\ \mathbf{N}.$$

Proof. Since $\boldsymbol{\beta}$ has unit speed, $\mathbf{T} = \boldsymbol{\beta}'$ and $\mathbf{N} = \mathbf{J}(\boldsymbol{\beta}')$. Differentiating $\boldsymbol{\beta}' \bullet \boldsymbol{\beta}' = 1$ we obtain $\boldsymbol{\beta}'' \bullet \boldsymbol{\beta}' = 0$, in which case $\boldsymbol{\beta}''$ and $\boldsymbol{\beta}'$ are orthogonal. Hence $\boldsymbol{\beta}''$ is parallel to $\mathbf{J}(\boldsymbol{\beta}')$ and $\boldsymbol{\beta}'' = \lambda\ \mathbf{J}(\boldsymbol{\beta}')$ for some scalar function λ. Substitute for $\boldsymbol{\beta}''$ in (1.2) and solve for λ :

$$\kappa_2[\boldsymbol{\beta}] = \frac{\lambda \mathbf{J}(\boldsymbol{\beta}')\ \bullet\ \mathbf{J}(\boldsymbol{\beta}')}{\left\|\boldsymbol{\beta}'\right\|^3}$$

$$\lambda = \kappa_2[\boldsymbol{\beta}] \frac{\left(\left\|\boldsymbol{\beta}'\right\|^2\right)^{3/2}}{\left\|\mathbf{J}(\boldsymbol{\beta}')\right\|^2} = \kappa_2[\boldsymbol{\beta}].$$

Therefore $\mathbf{T}' = \boldsymbol{\beta}'' = \kappa_2[\boldsymbol{\beta}]\ \mathbf{J}(\boldsymbol{\beta}') = \kappa_2[\boldsymbol{\beta}]\ \mathbf{N}$. ■

Let $\boldsymbol{\alpha} : I \to \mathbb{R}^2$ be a regular curve. Curvature measures how fast $\mathbf{T}$ is turning per unit of arc length along α. Since $\mathbf{T}(t)$ is a unit vector, there is an angle $\theta[\boldsymbol{\alpha}](t)$ such that $\mathbf{T}(t) = (\cos\theta[\boldsymbol{\alpha}](t), \sin\theta[\boldsymbol{\alpha}](t))$.

Definition 21 *Let $\boldsymbol{\alpha} : I \to \mathbb{R}^2$ be a regular curve. Define the turning angle of $\boldsymbol{\alpha}$ to be the real valued function $\theta[\boldsymbol{\alpha}] : I \to [0, 2\pi]$ determined for each t by*

$$\mathbf{T}(t) = (\cos\theta[\boldsymbol{\alpha}](t), \sin\theta[\boldsymbol{\alpha}](t)). \tag{1.3}$$

Theorem 22 *Let $\boldsymbol{\beta}$ be a unit speed plane curve. Then the curvature is the rate of change of turning angle with respect to arc length, i.e.,*

$$\kappa_2[\boldsymbol{\beta}](s) = \frac{d}{ds}\ \theta[\boldsymbol{\beta}](s).$$

Proof. Differentiate (1.3) and apply Corollary 20. ■

1.3 Implicitly Defined Plane Curves

Definition 23 *Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a function. The set of zeros of F is the set*

$$\left\{(x, y) \in \mathbb{R}^2 \mid F(x, y) = 0\right\}.$$

When F is differentiable, we refer to this set as an <u>*implicitly defined plane curve*</u> *and denote it by $F(x, y) = 0$.*

An implicitly defined plane curve is also referred to as a *level curve* (of height zero) of F. It may or may not be possible to parametrize such a curve. However,

Theorem 24 *Let $F : \mathbb{R}^2 \to \mathbb{R}$ be differentiable. If $\mathbf{p} \in \mathbb{R}^2$ satisfies $F(\mathbf{p}) = 0$ and $\nabla F(\mathbf{p}) \neq \mathbf{0}$, then the implicitly defined curve $F(x, y) = 0$ admits a local parametrization near $\mathbf{p}$, i.e., there is an open disc $D \subset \mathbb{R}^2$ centered at $\mathbf{p}$ and a parametrized curve $\boldsymbol{\alpha} : (c, d) \to \mathbb{R}^2$ such that*

$$\boldsymbol{\alpha}((c, d)) = \{(x, y) \in D \mid F(x, y) = 0\}.$$

Proof. Let (a, b) be the xy-coordinates of $\mathbf{p} \in \mathbf{R}^2$. By assumption, $\nabla F(\mathbf{p}) = \left(\frac{\partial F}{\partial x}(\mathbf{p}), \frac{\partial F}{\partial y}(\mathbf{p})\right) \neq \mathbf{0}$ so that either $\frac{\partial F}{\partial x}(\mathbf{p}) \neq 0$ or $\frac{\partial F}{\partial y}(\mathbf{p}) \neq 0$.. Suppose that $\frac{\partial F}{\partial y}(\mathbf{p}) \neq 0$. Then by the Implicit Function Theorem, $F(x, y) = 0$ defines y as a differentiable function of x in some open interval (c, d) containing a, i.e., $b = y(a)$ and $F(x, y(x)) = 0$, for all $x \in (c, d)$. Hence the graph of y, which is the set $\{(x, y(x)) \mid x \in (c, d)\}$, is contained in the level curve of F through (a, b) and the function $\boldsymbol{\alpha}(t) = (t, y(t))$ with $t \in (a, b)$ is a parametrization. ■

Example 25 *Let $r > 0$ and consider the circle $F(x, y) = x^2 + y^2 - r^2 = 0$. The gradient $\nabla F(x, y) = (2x, 2y)$. So at the point $P(0, r)$ on this circle we have $\frac{\partial F}{\partial x}(0, r) = 2(0) = 0$ and $\frac{\partial F}{\partial y}(0, r) = 2(r) \neq 0$. Theorem 24 guarantees the existence of a parametrization of the form $\boldsymbol{\alpha}(t) = (t, y(t))$ near the point $(0, r)$. Observe that $y = \sqrt{r^2 - x^2}$ is a differentiable function of x in the open interval $(-r, r)$, which contains 0. So a parametrization that satisfies the conclusion of Theorem 24 is $\boldsymbol{\alpha}(t) = (t, \sqrt{a^2 - t^2})$ with $t \in (-a, a)$. Note that $\boldsymbol{\alpha}$ is a local parametrization; its trace is the semi-circle in the upper half-plane.*

Definition 26 *Let* $S \subset \mathbb{R}^2$, *let* $\alpha : (a,b) \to \mathbb{R}^2$ *be a curve and let* $F : \mathbb{R}^2 \to \mathbb{R}$ *be a differentiable function. If* S *is the trace of* α, *then* α *is a parametric form of* S. *If* S *is the set of zeros of* F, *then the equation* $F(x,y) = 0$ *is called the implicit form of* S.

Exercise 1.3.1 *The lemniscate of Bernoulli, defined by*

$$F(x,y) = x^4 + y^4 + 2x^2y^2 - 4x^2 + 4y^2 = 0$$

has a double point at the origin (see Figure 3.1) and consequently fails to define either x *as an implicit function of* y *or* y *as an implicit function of* x *near* 0. *Thus the lemniscate of Bernoulli fails to admit a local parametrization near* $(0,0)$. *Explain why this is consistent with Theorem 24.*

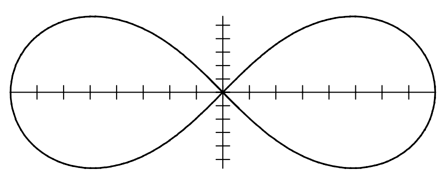


Figure 3.1: The Lemniscate of Bernoulli.

Theorem 24 is an *existence* theorem; it gives no clue about how to find explicit local parametrizations. Since the curvature is defined in terms of a parametrization, one might wonder whether or not it's possible to compute the curvature of an implicitly defined plane curves in general. Indeed, if F is at least twice differentiable, the answer is affirmative.

Theorem 27 *Let* $F : \mathbb{R}^2 \to \mathbb{R}$ *be at least twice differentiable and let* $\mathbf{p}$ *be a point such that* $F(\mathbf{p}) = 0$. *If* $\nabla F(\mathbf{p}) \neq \mathbf{0}$, *the signed curvature of* $F(x,y) = 0$ *at* $\mathbf{p}$ *is given by*

$$\kappa_2[F](\mathbf{p}) = \frac{F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2}{(F_x^2 + F_y^2)^{3/2}}. \tag{1.4}$$

Proof. Recall that $\nabla F(\mathbf{p})$ is normal to the level curve $F(x,y) = 0$. Hence $\mathbf{J}(\nabla F) = (-F_y, F_x)$ is a differentiable tangent vector field along the curve $F(x,y) = 0$. Think of this as the velocity $\boldsymbol{\alpha}'$ of an unspecified parametrization

$\boldsymbol{\alpha}(t) = (x(t), y(t))$, i.e., $\boldsymbol{\alpha}'(t) = (-F_y(\boldsymbol{\alpha}(t)), F_x(\boldsymbol{\alpha}(t)))$. Then by the chain rule and equality of mixed partials,

$$\begin{aligned}
\boldsymbol{\alpha}'' &= \left(-\nabla F_y(\boldsymbol{\alpha}(t)) \bullet \boldsymbol{\alpha}'(t), \nabla F_x(\boldsymbol{\alpha}(t)) \bullet \boldsymbol{\alpha}'(t)\right) \\
&= \left(-(F_{yx}, F_{yy}) \bullet (-F_y, F_x), (F_{xx}, F_{xy}) \bullet (-F_y, F_x)\right) \\
&= (F_y F_{xy} - F_x F_{yy}, F_x F_{xy} - F_y F_{xx})
\end{aligned}$$

so that

$$\begin{aligned}
\boldsymbol{\alpha}'' \bullet \mathbf{J}(\boldsymbol{\alpha}') &= (F_y F_{xy} - F_x F_{yy}, F_x F_{xy} - F_y F_{xx}) \bullet (-F_x, -F_y) \\
&= F_{xx} F_y^2 - 2F_{xy} F_x F_y + F_{yy} F_x^2
\end{aligned}$$

and the result follows. ■

Exercise 1.3.2 *Refer to Exercise 1.3.1. Use formula 1.4 to compute the curvature* $\kappa_2[F](\mathbf{p})$ *of the lemniscate of Bernoulli at each point where* $\nabla F(\mathbf{p}) \neq 0$.

1.4 The Local Theory of Plane Curves

In this section we observe that the curvature function $\kappa_2[\boldsymbol{\alpha}]$ encodes all of the intrinsic geometry of a regular plane curve $\boldsymbol{\alpha}$, i.e., the curve $\boldsymbol{\alpha}$ can be completely recovered from $\kappa_2[\boldsymbol{\alpha}]$ and initial data $\boldsymbol{\alpha}(t_0)$ and $\boldsymbol{\alpha}'(t_0)$. Recall the following facts from Linear Algebra:

Theorem 28 *Let* A *be an* 2×2 *matrix. The following statements are all equivalent:*

1. A is orthogonal.
2. $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^2$, i.e., multiplication by A preserves Euclidean norm.
3. $A\mathbf{x} \bullet A\mathbf{y} = \mathbf{x} \bullet \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, i.e., multiplication by A preserves Euclidean inner product.

Lemma 29 *Let* $A : \mathbb{R}^2 \to \mathbb{R}^2$ *be an orthogonal transformation. Then*

$$A \circ J = \det(A) J \circ A.$$

Proof. For all non-zero $\mathbf{p} \in \mathbb{R}^2$,

$$(J \circ A)(\mathbf{p}) \bullet A(\mathbf{p}) = J(A(\mathbf{p})) \bullet A(\mathbf{p}) = 0.$$

On the other hand, $A(J(\mathbf{p})) \bullet A(\mathbf{p}) = J(\mathbf{p}) \bullet \mathbf{p}$ by Theorem 28, hence

$$(A \circ J)(\mathbf{p}) \bullet A(\mathbf{p}) = A(J(\mathbf{p})) \bullet A(\mathbf{p}) = J(\mathbf{p}) \bullet \mathbf{p} = 0.$$

Therefore both $(J \circ A)(\mathbf{p})$ and $(A \circ J)(\mathbf{p})$ are orthogonal to $A(\mathbf{p})$ in $\mathbb{R}^2$, which implies

$$(A \circ J)(\mathbf{p}) = \lambda (J \circ A)(\mathbf{p}) \tag{1.5}$$

for some $\lambda \in \mathbb{R}$. Now

$$\det[A] \det[J] = \det([A][J]) = \det[A \circ J] = \det[\lambda (J \circ A)]$$

$$= \lambda^2 \det[J \circ A] = \lambda^2 \det([J][A]) = \lambda^2 \det[J] \det[A]$$

and it follows that

$$\lambda = \pm 1. \tag{1.6}$$

We must show that $\lambda = \det(A)$. Consider the basis $\mathcal{B} = \{\mathbf{p},\, J(\mathbf{p})\}$ and write

$$A(\mathbf{p}) = a\mathbf{p} + bJ(\mathbf{p}),$$

for appropriate scalars $a, b \in \mathbb{R}$. By (1.5) and (1.6) we have

$$A(J(\mathbf{p})) = \lambda J(A(\mathbf{p})) = \lambda J[a\mathbf{p} + bJ(\mathbf{p})] = \lambda \left[aJ(\mathbf{p}) + bJ^2(\mathbf{p})\right]$$

$$= -\lambda b\mathbf{p} + \lambda a J(\mathbf{p}).$$

Hence

$$[A]_{\mathcal{B}} = \begin{bmatrix} a & -\lambda b \\ b & \lambda a \end{bmatrix}.$$

Since $\det A = \pm 1$we have

$$\pm 1 = \det A = \lambda(a^2 + b^2)$$

and by (1.6)

$$a^2 + b^2 = 1.$$

Therefore

$$\det(A) = \lambda$$

and the result follows from (1.5). ■

Our next theorem establishes the fact that Euclidean motions preserve curvature $\kappa_2[\boldsymbol{\alpha}]$ up to sign. Since the proof assumes familiarity with the derivative of maps $F : \mathbb{R}^2 \to \mathbb{R}^2$, let's take a moment to review the basic facts we need here. A careful development of these ideas will come later.

Given $F : \mathbb{R}^2 \to \mathbb{R}^2$, express F in terms of real valued component functions f_1 and f_2 by $F(x,y) = (f_1(x,y),\ f_2(x,y))$. Recall that if F is differentiable, then each component function $f_i : \mathbb{R}^2 \to \mathbb{R}$ is differentiable and the derivative of F is the matrix valued function $F' : \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$ given by

$$F'(x,y) = \begin{bmatrix} \nabla f_1(x,y) \\ \nabla f_2(x,y) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \end{bmatrix}.$$

Note that the derivative F' associates each point $\mathbf{p}$ with a linear map $dF_{\mathbf{p}} : \mathbb{R}^2 \to \mathbb{R}^2$, called the *differential of* F, whose matrix in the standard basis $[dF_{\mathbf{p}}] = F'(\mathbf{p})$. The map $dF_{\mathbf{p}}$ is the linear map that best approximates the change in F near $\mathbf{p}$.

Suppose that $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^2$ and $F : \mathbb{R}^2 \to \mathbb{R}^2$ are differentiable with $\boldsymbol{\alpha}(t) = (x(t), y(t))$ and $F(x,y) = (f_1(x,y),\ f_2(x,y))$. Represent $\boldsymbol{\alpha}' : \mathbb{R} \to \mathbb{R}^2$ as the matrix valued function $\boldsymbol{\alpha}'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$. Then by the Chain Rule, $F \circ \boldsymbol{\alpha}$ is differentiable and

$$\frac{d}{dt}(F \circ \boldsymbol{\alpha})(t) = F'(\boldsymbol{\alpha}(t))\,\boldsymbol{\alpha}'(t) = \begin{bmatrix} \frac{\partial f_1}{\partial x}(\boldsymbol{\alpha}(t)) & \frac{\partial f_1}{\partial y}(\boldsymbol{\alpha}(t)) \\ \frac{\partial f_2}{\partial x}(\boldsymbol{\alpha}(t)) & \frac{\partial f_2}{\partial y}(\boldsymbol{\alpha}(t)) \end{bmatrix} \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}.$$

Example 30 *Let F be a linear map. Then $F(x,y) = (ax+by,\ cx+dy)$ is differentiable and for all* $\mathbf{p}$ *and*

$$F'(\mathbf{p}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = [dF_{\mathbf{p}}] = [F].$$

This simply says that F is the linear map that best approximates its change near $\mathbf{p}$. *Now if $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^2$ is differentiable, then by the chain rule*

$$\frac{d}{dt}(F \circ \boldsymbol{\alpha})(t) = F'(\boldsymbol{\alpha}(t))\,\boldsymbol{\alpha}'(t) = [F] \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = F(\boldsymbol{\alpha}'(t)).$$

Theorem 31 *Let* $\boldsymbol{\alpha} : (a,b) \to \mathbb{R}^2$ *be a regular curve and let* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *be a Euclidean motion given by* $F(\mathbf{p}) = A(\mathbf{p}) + \mathbf{q}$, *where* A *is orthogonal and* $\mathbf{p}, \mathbf{q} \in \mathbb{R}^2$. *If* $\boldsymbol{\gamma} = F \circ \boldsymbol{\alpha}$, *then for all* $t \in (a,b)$,

$$\|\boldsymbol{\gamma}'(t)\| = \|\boldsymbol{\alpha}'(t)\|$$

and

$$\kappa_2[\boldsymbol{\gamma}](t) = \det A \cdot \kappa_2[\boldsymbol{\alpha}](t).$$

Thus speed is invariant under Euclidean motions. Curvature is invariant under orientation-preserving Euclidean motions and reverses sign under orientation-reversing Euclidean motions.

Proof. For all $t \in (a,b)$,

$$\boldsymbol{\gamma}(t) = F(\boldsymbol{\alpha}(t)) = A(\boldsymbol{\alpha}(t)) + \mathbf{q}.$$

Then by the chain rule,

$$\boldsymbol{\gamma}'(t) = \frac{d}{dt}\left(A(\boldsymbol{\alpha}(t)) + \mathbf{q}\right) = A'(\boldsymbol{\alpha}(t))\,[\boldsymbol{\alpha}'(t)] = [A]\,[\boldsymbol{\alpha}'(t)] = A(\boldsymbol{\alpha}'(t)). \quad (1.7)$$

By Theorem 28,

$$\|\boldsymbol{\gamma}'(t)\| = \|A(\boldsymbol{\alpha}'(t))\| = \|\boldsymbol{\alpha}'(t)\|.$$

Furthermore, $\boldsymbol{\gamma}''(t) = A(\boldsymbol{\alpha}''(t))$ by the same computation as in (1.7). So again by Theorem 28,

$$\begin{aligned}
\kappa_2[\boldsymbol{\gamma}](t) &= \frac{\boldsymbol{\gamma}''(t) \bullet J(\boldsymbol{\gamma}'(t))}{\|\boldsymbol{\gamma}'(t)\|^3} = \frac{A(\boldsymbol{\alpha}''(t)) \bullet J\left[A(\boldsymbol{\alpha}'(t))\right]}{\|A(\boldsymbol{\alpha}'(t))\|^3} \\
&= \det(A)\frac{A(\boldsymbol{\alpha}''(t)) \bullet A\left[J(\boldsymbol{\alpha}'(t))\right]}{\|A(\boldsymbol{\alpha}'(t))\|^3} = \det(A)\frac{\boldsymbol{\alpha}''(t) \bullet J(\boldsymbol{\alpha}'(t))}{\|\boldsymbol{\alpha}'(t)\|^3} \\
&= \det(A) \cdot \kappa_2[\boldsymbol{\alpha}](t).
\end{aligned}$$

■

The Fundamental Theorem of Plane Curves is the converse of Theorem 31.

Theorem 32 ***(The Fundamental Theorem of the Local Theory of Plane Curves)***
Let $\boldsymbol{\alpha}, \boldsymbol{\gamma} : (a,b) \to \mathbb{R}^2$ *be regular plane curves such that* $\kappa_2[\boldsymbol{\alpha}](t) = \kappa_2[\boldsymbol{\gamma}](t)$ *for all* $t \in (a,b)$. *Then there is an orientation-preserving Euclidean motion* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *such that* $\boldsymbol{\gamma} = F \circ \boldsymbol{\alpha}$.

Proof. By Theorem 14, we may assume without loss of generality that $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ are unit-speed curves. Fix a point $s_0 \in (a,b)$ and let

$$\mathbf{q} = \boldsymbol{\gamma}(s_0) - \boldsymbol{\alpha}(s_0).$$

Composing $\boldsymbol{\alpha}$ with $\tau_{\mathbf{q}}$ translates $\boldsymbol{\alpha}$ so that

$$(\tau_{\mathbf{q}} \circ \boldsymbol{\alpha})(s_0) = \tau_{\mathbf{q}}(\boldsymbol{\alpha}(s_0)) = \boldsymbol{\alpha}(s_0) + \boldsymbol{\gamma}(s_0) - \boldsymbol{\alpha}(s_0) = \boldsymbol{\gamma}(s_0).$$

Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear rotation such that $A(\mathbf{T}_{\boldsymbol{\alpha}}(s_0)) = \mathbf{T}_{\gamma}(s_0)$ and define the orientation-preserving Euclidean motion $F : \mathbb{R}^2 \to \mathbb{R}^2$ as the composition of the affine rotation $\tau_{\boldsymbol{\gamma}(s_0)} \circ A \circ \tau^{-1}_{\boldsymbol{\gamma}(s_0)}$ with the translation $\tau_{\mathbf{q}}$, i.e.,

$$\begin{aligned} F(\mathbf{p}) &= (\tau_{\boldsymbol{\gamma}(s_0)} \circ A \circ \tau^{-1}_{\boldsymbol{\gamma}(s_0)} \circ \tau_{\mathbf{q}})(\mathbf{p}) = \left(\tau_{\boldsymbol{\gamma}(s_0)} \circ A \circ \tau^{-1}_{\boldsymbol{\gamma}(s_0)}\right)(\mathbf{p}+\mathbf{q}) \\ &= \left(\tau_{\boldsymbol{\gamma}(s_0)} \circ A\right)[\mathbf{p} - \boldsymbol{\alpha}(s_0)] = \tau_{\boldsymbol{\gamma}(s_0)}\left[A(\mathbf{p}) - A(\boldsymbol{\alpha}(s_0))\right] \\ &= A(\mathbf{p}) + [\boldsymbol{\gamma}(s_0) - (A \circ \boldsymbol{\alpha})(s_0)]. \end{aligned}$$

Then $F \circ \boldsymbol{\alpha}$ is the curve obtained by translating $\boldsymbol{\alpha}$ by $\mathbf{q}$ and then rotating $\tau_{\mathbf{q}} \circ \boldsymbol{\alpha}$ about the point $\boldsymbol{\gamma}(s_0)$ so that

$$(F \circ \boldsymbol{\alpha})(s_0) = A(\boldsymbol{\alpha}(s_0)) + [\boldsymbol{\gamma}(s_0) - A(\boldsymbol{\alpha}(s_0))] = \boldsymbol{\gamma}(s_0) \tag{1.8}$$

and

$$(A \circ \mathbf{T}_{\boldsymbol{\alpha}})(s_0) = \mathbf{T}_{\gamma}(s_0) \text{ and } (A \circ \mathbf{N}_{\boldsymbol{\alpha}})(s_0) = \mathbf{N}_{\gamma}(s_0). \tag{1.9}$$

The theorem will be proved if we can show that $\boldsymbol{\gamma} = F \circ \boldsymbol{\alpha}$. For $s \in (a,b)$, define the real-valued function

$$f(s) = \left\|(A \circ \mathbf{T}_{\boldsymbol{\alpha}})(s) - \mathbf{T}_{\gamma}(s)\right\|^2.$$

Then

$$\begin{aligned} f'(s) &= \frac{d}{ds}\{[(A \circ \mathbf{T}_{\boldsymbol{\alpha}})(s) - \mathbf{T}_{\gamma}(s)] \bullet [(A \circ \mathbf{T}_{\boldsymbol{\alpha}})(s) - \mathbf{T}_{\gamma}(s)]\} \qquad (1.10) \\ &= 2\,[(A \circ \mathbf{T}_{\boldsymbol{\alpha}})(s) - \mathbf{T}_{\gamma}(s)] \bullet \frac{d}{ds}[(A \circ \mathbf{T}_{\boldsymbol{\alpha}})(s) - \mathbf{T}_{\gamma}(s)] \\ &= 2\,[(A \circ \mathbf{T}_{\boldsymbol{\alpha}})(s) - \mathbf{T}_{\gamma}(s)] \bullet \left[(A \circ \mathbf{T}_{\boldsymbol{\alpha}})'(s) - \mathbf{T}'_{\gamma}(s)\right] \\ &= 2[(A \circ \mathbf{T}_{\boldsymbol{\alpha}})(s) \bullet (A \circ \mathbf{T}_{\boldsymbol{\alpha}})'(s) - (A \circ \mathbf{T}_{\boldsymbol{\alpha}})(s) \bullet \mathbf{T}'_{\gamma}(s) \\ &\quad - \mathbf{T}_{\gamma}(s) \bullet (A \circ \mathbf{T}_{\boldsymbol{\alpha}})'(s) + \mathbf{T}_{\gamma}(s) \bullet \mathbf{T}'_{\gamma}(s)]. \end{aligned}$$

Since $\boldsymbol{\alpha}$ is a unit-speed curve, so is $A \circ \boldsymbol{\alpha}$ by Theorem 31. Therefore

$$(A \circ \mathbf{T}_{\boldsymbol{\alpha}})(s) = A(\boldsymbol{\alpha}'(s)) = (A \circ \boldsymbol{\alpha})'(s) = \mathbf{T}_{A\circ\boldsymbol{\alpha}}(s)$$

and

$$\begin{aligned}(A \circ \mathbf{N}_{\boldsymbol{\alpha}})(s) &= A(\mathbf{N}_{\boldsymbol{\alpha}}(s)) = A\left[J(\mathbf{T}_{\boldsymbol{\alpha}}(s))\right] \\ &= J\left[A(\mathbf{T}_{\boldsymbol{\alpha}}(s))\right] = J(\mathbf{T}_{A\circ\boldsymbol{\alpha}}(s)) \\ &= \mathbf{N}_{A\circ\boldsymbol{\alpha}}(s).\end{aligned}$$

Furthermore $\kappa_2 = \kappa_2[\boldsymbol{\alpha}](s) = \kappa_2[A\circ\boldsymbol{\alpha}](s)$ and by assumption $\kappa_2[\boldsymbol{\alpha}](s) = \kappa_2[\boldsymbol{\gamma}](s)$. So by Corollary 20 (the *first Frenet formula*) we have

$$(A \circ \mathbf{T}_{\boldsymbol{\alpha}})(s) \bullet (A \circ \mathbf{T}_{\boldsymbol{\alpha}})'(s) = \mathbf{T}_{A\circ\boldsymbol{\alpha}}(s) \bullet \mathbf{T}'_{A\circ\boldsymbol{\alpha}}(s) = \mathbf{T}_{A\circ\boldsymbol{\alpha}}(s) \bullet \kappa_2 \mathbf{N}_{A\circ\boldsymbol{\alpha}}(s) = 0$$

and

$$\mathbf{T}_{\gamma}(s) \bullet \mathbf{T}'_{\gamma}(s) = \mathbf{T}_{\gamma}(s) \bullet \kappa_2 \mathbf{N}_{\gamma}(s) = 0.$$

So (1.10) reduces to

$$f'(s) = -2\left[\mathbf{T}_{A\circ\boldsymbol{\alpha}}(s) \bullet \mathbf{T}'_{\gamma}(s) + \mathbf{T}_{\gamma}(s) \bullet \mathbf{T}'_{A\circ\boldsymbol{\alpha}}(s)\right]. \tag{1.11}$$

Again by Corollary 20 and Theorem 28, (1.11) becomes

$$\begin{aligned}f'(s) &= -2\left\{\mathbf{T}_{A\circ\boldsymbol{\alpha}}(s) \bullet \kappa_2 \mathbf{N}_{\gamma}(s) + \mathbf{T}_{\gamma}(s) \bullet \kappa_2 \mathbf{N}_{A\circ\boldsymbol{\alpha}}(s)\right\} \\ &= -2\kappa_2\left\{\mathbf{T}_{A\circ\boldsymbol{\alpha}}(s) \bullet \mathbf{N}_{\gamma}(s) + \mathbf{T}_{\gamma}(s) \bullet J(\mathbf{T}_{A\circ\boldsymbol{\alpha}}(s))\right\} \\ &= -2\kappa_2\left\{\mathbf{T}_{A\circ\boldsymbol{\alpha}}(s) \bullet \mathbf{N}_{\gamma}(s) + J(\mathbf{T}_{\gamma}(s)) \bullet J\left[J(\mathbf{T}_{A\circ\boldsymbol{\alpha}}(s))\right]\right\} \\ &= -2\kappa_2\left\{\mathbf{T}_{A\circ\boldsymbol{\alpha}}(s) \bullet \mathbf{N}_{\gamma}(s) - \mathbf{T}_{A\circ\boldsymbol{\alpha}}(s) \bullet \mathbf{N}_{\gamma}(s)\right\} \\ &= 0.\end{aligned}$$

Therefore

$$f(s) = \textit{constant}.$$

But by the condition on F in (1.9) we have

$$f(s_0) = \left\|(A \circ \mathbf{T}_{\boldsymbol{\alpha}})(s_0) - \mathbf{T}_{\gamma}(s_0)\right\|^2 = 0,$$

in which case

$$f(s) = 0$$

for all s so that $A \circ \mathbf{T}_{\boldsymbol{\alpha}} = \mathbf{T}_{\gamma}$. But this implies that

$$\begin{aligned}
(F \circ \boldsymbol{\alpha})'(s) &= \frac{d}{ds}\left[F(\boldsymbol{\alpha}(s))\right] = \frac{d}{ds}\left[A(\boldsymbol{\alpha}(s)) + \boldsymbol{\gamma}(s_0) - A(\boldsymbol{\alpha}(s_0))\right] \\
&= A(\boldsymbol{\alpha}'(s)) = (A \circ \mathbf{T}_{\boldsymbol{\alpha}})(s) = \mathbf{T}_{\gamma}(s) \\
&= \boldsymbol{\gamma}'(s).
\end{aligned}$$

Now integrate both sides and obtain

$$(F \circ \boldsymbol{\alpha})(s) = \boldsymbol{\gamma}(s) + \mathbf{r}$$

for some constant vector $\mathbf{r} \in \mathbb{R}^2$. By definition of F in (1.8) we have

$$\mathbf{r} = (F \circ \boldsymbol{\alpha})(s_0) - \boldsymbol{\gamma}(s_0) = 0$$

so that

$$(F \circ \boldsymbol{\alpha})(s) = \boldsymbol{\gamma}(s)$$

for all s, which completes the proof. ■