Chapter 6

Some Special Curves on Surfaces

In this chapter we introduce three families of curves on surfaces, namely, principal, asymptotic and geodesic curves. Principal curves follow paths of maximal bending, asymptotic curves follow paths of zero bending and geodesics curves always go "straight ahead" in the surface.

6.1 Vector Fields Along Curves

Our discussion begins with an introduction to vector fields along curves.

Definition 135 Let \( \alpha : (a, b) \to \mathbb{R}^3 \) be a curve given componentwise by \( \alpha(t) = (x(t), y(t), z(t)) \). A vector field \( V \) along \( \alpha \) is a function defined on \( (a, b) \) such that \( V(t) \in \mathbb{R}^3_{\alpha(t)} \) for all \( t \in (a, b) \). Each vector field \( V \) along \( \alpha \) has the form

\[
V(t) = (V_1(t), V_2(t), V_3(t))_{\alpha(t)},
\]

where \( V_i : (a, b) \to \mathbb{R} \). A vector field \( V \) along \( \alpha \) is differentiable if each \( V_i \) is differentiable; the derivative of a differentiable vector field \( V \) along \( \alpha \) is the vector field \( V' \) along \( \alpha \) defined by

\[
V'(t) = (V'_1(t), V'_2(t), V'_3(t))_{\alpha(t)}.
\]

The velocity vector field along \( \alpha \) is defined by

\[
\alpha'(t) = (x'(t), y'(t), z'(t))_{\alpha(t)};
\]

the acceleration vector field along \( \alpha \) is defined by...
\[ \alpha''(t) = (x''(t), y''(t), z''(t)) \alpha(t). \]

All vector fields henceforth are assumed to be infinitely differentiable, i.e., smooth.

**Definition 136** Let \( \alpha : (a, b) \to \mathbb{R}^3 \) be a curve, let \( f : (a, b) \to \mathbb{R} \) be a differentiable function and let \( V, W \) be vector fields along \( \alpha \). For all \( t \in (a, b) \), define

a. \((V + W)(t) = V(t) + W(t)\)

b. \((fV)(t) = f(t)V(t)\)

c. \((V \cdot W)(t) = V(t) \cdot W(t).\)

The proof of the next Proposition is left for the reader.

**Proposition 137** Differentiation of vector fields has the following properties:

a. \((V + W)' = V' + W'\)

b. \((fV)' = f'V + fV'\)

c. \((V \cdot W)' = V' \cdot W + V \cdot W'.\)

### 6.2 Principal, Asymptotic & Geodesic Curves

Now suppose that the image of a curve \( \alpha \) is contained in the image of some regular patch \( x \). Define

**Definition 138** A tangent vector \( v_p \in M_p \) is principal if \( k(v_p) = S(v_p) \cdot v_p = k_1 \) or \( k_2 \); it is asymptotic if \( k(v_p) = S(v_p) \cdot v_p = 0 \). Directions in \( M_p \) that contain principal tangent vectors are called principal directions; directions in \( M_p \) that contain asymptotic tangent vectors are called asymptotic directions.

Note that in an asymptotic direction at \( p \), \( M \) does not bend away from \( M_p \) (at least instantaneously at \( p \)).
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Definition 139 A curve \( \alpha \subset M \) is a principal curve if \( \alpha' \) is a principal tangent vector field along \( \alpha \); \( \alpha \) is an asymptotic curve if \( \alpha'(t) \) is an asymptotic tangent vector field along \( \alpha \).

Principal curves travel in directions of extreme bending. If \( p \) is a nonumbilic point of \( M \), there are exactly two principal curves through \( p \) intersecting each other orthogonally at \( p \). Asymptotic curves, on the other hand, travel in directions of no bending. Since normal curvature \( k \) measures the component of acceleration normal to \( M \), normal curvature \( k = 0 \) implies that acceleration \( \alpha'' \) along an asymptotic curve \( \alpha \) is always tangent to \( M \).

Principal curves on a cylinder are circular crosssections and lines parallel to the axis; these lines are also asymptotic curves. Principal curves on the hyperbolic paraboloid \( z = xy \) are the parabolas \( z = \pm x^2 \); the asymptotic curves are the \( xy \)-coordinate axes \( xy = 0 \).

Lemma 140 Let \( \alpha \) be a regular curve in \( M \) and let \( U \) be a surface normal.

a. \( \alpha \) is a principal curve if and only if
\[
-D_{\alpha'} U(\alpha) = \lambda \alpha'.
\]

b. If \( \alpha \) is principal curve, then the principal curvature
\[
k(\alpha') = \frac{\alpha'' \cdot U(\alpha)}{\|\alpha'\|^2}.
\]

Proof. For part a, note that \(-D_{\alpha'} U(\alpha) = \lambda \alpha'\) if and only if \( S(\alpha') = \lambda \alpha' \) if and only if \( \lambda(t) \) is a principal curvature at \( \alpha(t) \) for all \( t \) if and only if \( \alpha' \) is a principal tangent vector field along \( \alpha \). For part b, the vector field \( \alpha' \) consists of principal tangent vectors belonging to some principal curvature, say \( k_i \). Hence
\[
k_i = k(\alpha') = \frac{S(\alpha') \cdot \alpha'}{\|\alpha'\|^2} = \frac{\alpha'' \cdot U(\alpha)}{\|\alpha'\|^2},
\]
where the last equality is given by Lemma 101.

In this lemma, statement (a) gives a simple criterion for a curve to be a principal curve, while statement (b) gives the principal curvature along a curve known to be principal.
Definition 141 Let $\mathcal{M}$ be the image of a regular patch. A geodesic on $\mathcal{M}$ is a curve $\alpha : (a, b) \to \mathcal{M}$ whose acceleration vector field $\alpha''$ is always in the direction normal to $\mathcal{M}$, i.e., if $U$ is a surface normal, then

$$\alpha'' = \lambda U(\alpha).$$

Thus a geodesic is a curve that always goes “straight ahead” in the surface; its acceleration only serves to keep it in the surface. There is no component of the acceleration tangent to the surface. Let $\mathcal{M}_p\perp$ denote the orthogonal complement of $\mathcal{M}_p$ in $\mathbb{R}_p^3$.

Example 142 Suppose the image of a regular patch $\mathcal{M}$ contains a straight line segment $\alpha(t) = p + tv$, where $t \in (a, b)$. Then $\alpha''(t) = 0$ for all $t$ and in particular, $\alpha''(t) \cdot v_{\alpha(t)} = 0$ for all tangent vectors $v_{\alpha(t)} \in \mathcal{M}_{\alpha(t)}$. Hence $\alpha''(t) \in \mathcal{M}_{\alpha(t)}\perp$, for all $t$ and it follows that straight line segments on a regular patch are geodesics.

Example 143 Let $\mathcal{M}$ be the cylinder $x^2 + y^2 = 1$ thought of as the level surface for $g(x, y, z) = x^2 + y^2 - 1$ of height zero. The gradient is $\nabla g(x, y, z) = (2x, 2y, 0)$, so a surface normal $U$ on $\mathcal{M}$ is given by

$$U(x, y, z) = (x, y, 0),$$

where $p = (x, y, z) \in \mathcal{M}$. For each $a, b, c, d$, the curve

$$\alpha(t) = (\cos(at + b), \sin(at + b), ct + d)$$

is a geodesic in the cylinder because

$$\alpha''(t) = (-a^2 \cos(at + b), -a^2 \sin(at + b), 0)_{\alpha(t)} = -a^2 U(\alpha(t)),$$

for all $t$. Furthermore, the geodesic is a

1. (a) Helix if $a, c \neq 0$:

$$\alpha(t) = (\cos(at + b), \sin(at + b), ct + d)$$

(b) Line if $a = 0$ and $c \neq 0$:

$$\alpha(t) = (\cos(b), \sin(b), ct + d)$$
(c) Circle if $a \neq 0$ and $c = 0$:

$$\alpha(t) = (\cos(at + b), \sin(at + b), d)$$

(d) Point if $a, c = 0$:

$$\alpha(t) = (\cos(b), \sin(b), d).$$

**Example 144** Let $M$ be the sphere $x^2 + y^2 + z^2 = r^2$. Slice $M$ with a plane $\Gamma$ through the origin and obtain a great circle $C = M \cap \Gamma$. Choose any orthonormal basis $\{e_1, e_2\}$ for $\Gamma$ (thought of as a subspace of $\mathbb{R}^3$) and parametrize $C$ by

$$\alpha(t) = (r \cos at) e_1 + (r \sin at) e_2,$$

with $t \in \mathbb{R}$ and $a \neq 0$. As with the cylinder, there is a surface normal $U$ on $M$ given by

$$U(x, y, z) = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)_{(x,y,z)},$$

where $(x, y, z) \in M$. Then for all $t$,

$$\alpha''(t) = (-a^2 r \cos(at) e_1 - a^2 r \sin(at) e_2)_{\alpha(t)} = -a^2 \alpha(t)_{\alpha(t)} = -a^2 (U \circ \alpha)(t),$$

so the great circle $C$ is a geodesic. Since $\Gamma$ and $e_1$ were arbitrarily chosen, every great circle is a geodesic. Furthermore, if $\alpha(t) = r e_1$ is a constant curve, then $\alpha''(t) = 0 \in M^\perp_{re_1}$ for all $t$, and the point $r e_1$ is a geodesic. We conclude that points and great circles on a sphere are geodesics.

**Proposition 145** Geodesic on the image of a regular patch have constant speed.

**Proof.** Let $M$ be the image of a regular patch and let $\alpha: (a, b) \rightarrow M$ be a geodesic. Then $\alpha'(t) \in M_{\alpha(t)}$ and $\alpha''(t) \in M^\perp_{\alpha(t)}$, so that $\alpha'(t) \cdot \alpha''(t) = 0$. Therefore

$$\frac{d}{dt} \|\alpha'(t)\|^2 = \frac{d}{dt} [\alpha'(t) \cdot \alpha'(t)] = 2 \alpha'(t) \cdot \alpha''(t) = 0,$$

so the speed is constant. $\blacksquare$

Here is a simple geometric way to find examples of geodesics. Let $U$ be a surface normal on $M$. We say that a plane $\Pi$ is orthogonal to $M$ at $p$ if $U(p) \subset \Pi$. 
Proposition 146 If $\Pi$ is a plane everywhere orthogonal to $\mathcal{M}$ and $\alpha$ is a unit speed curve in $\mathcal{M} \cap \Pi$, then $\alpha$ is a geodesic of $\mathcal{M}$.

Proof. Since $\alpha$ has constant speed, $\alpha' \cdot \alpha' = c$ so that $\alpha' \cdot \alpha'' = 0$. Furthermore, $\alpha', \alpha'' \subset \Pi$ since $\alpha \subset \Pi$. By assumption, $U(\alpha(t)) \subset \Pi$ for all $t$. Therefore $\alpha''(t) = \lambda(t) U(\alpha(t))$ and is $\alpha$ is a geodesic.

We could have used this proposition to find the geodesics in all of the examples above except the helices on the cylinder. Note that meridians on a surface of revolution are geodesics since they are cut from $\mathcal{M}$ by a plane passing through the axis.

Exercise 6.2.1 Let $\alpha(t) = (x(t), y(t))$ be a constant speed curve in the upper half plane $y > 0$ with no self-intersections and let $\mathcal{M}$ be the surface of revolution obtained by revolving the trace of $\alpha$ about the $x$-axis. For each $t \in (a, b)$ define $\beta_t : \mathbb{R} \rightarrow \mathcal{M}$ by

$$\beta_t(\theta) = (x(t), y(t) \cos \theta, y(t) \sin \theta).$$

The circles $\beta_t$ are called the parallels on $\mathcal{M}$. Show that a parallel $\beta_t$ is a geodesic if and only if the slope $\frac{y'(t)}{x'(t)}$ of the line tangent to $\alpha$ at $\alpha(t)$ is zero.

In summary, let’s compare the essential properties of principal, asymptotic and geodesic curves.

If a curve $\alpha$ is

- principal: $k(\alpha') \in \{k_1, k_2\}$, $S(\alpha') = \alpha' \lambda$
- asymptotic: $k(\alpha') = 0$, $S(\alpha') \cdot \alpha' = 0$, $\alpha''$ is tangent
- geodesic: $\alpha''$ is normal.

### 6.3 Existence and uniqueness of Geodesics

Intuitively, given a point $p$ on a surface $\mathcal{M}$ and any tangent vector $v_p$ there should be a geodesic in $\mathcal{M}$ passing through $p$ with initial velocity $v_p$. After all, a bicycler riding on the surface and passing through $p$ should be able to continue traveling “straight ahead” with constant speed $\|v\|$ and thereby
follow a geodesic in \( \mathcal{M} \). Our main theorem asserts that this intuitive idea is indeed correct and that the geodesic with these properties is in some sense unique.

**Definition 147** A patch \( \mathbf{x} \) is orthogonal if \( F = \mathbf{x}_u \cdot \mathbf{x}_v = 0 \).

Let \( \mathcal{M} \) be the image of an orthogonal patch \( \mathbf{x} \) and let \( \alpha = \mathbf{x} (u(t), v(t)) \) be a curve. Then \( \alpha' = u' \mathbf{x}_u + v' \mathbf{x}_v \) and

\[
\alpha'' = u'' \mathbf{x}_u + \left( u' \right)^2 \mathbf{x}_{uu} + 2u'v' \mathbf{x}_{uv} + v'' \mathbf{x}_v + \left( v' \right)^2 \mathbf{x}_{vv}.
\]

Let \( \mathbf{U} \) be a surface normal; we need to write \( \mathbf{x}_{uu}, \mathbf{x}_{uv} \) and \( \mathbf{x}_{vv} \) in the basis \( \{ \mathbf{x}_u, \mathbf{x}_v, \mathbf{U} \} \), i.e.,

\[
\begin{align*}
\mathbf{x}_{uu} &= \Gamma_{uu}^u \mathbf{x}_u + \Gamma_{uu}^v \mathbf{x}_v + \ell \mathbf{U} \\
\mathbf{x}_{uv} &= \Gamma_{uv}^u \mathbf{x}_u + \Gamma_{uv}^v \mathbf{x}_v + m \mathbf{U} \\
\mathbf{x}_{vv} &= \Gamma_{vv}^u \mathbf{x}_u + \Gamma_{vv}^v \mathbf{x}_v + n \mathbf{U}
\end{align*}
\]

for appropriate choice of coefficients \( \Gamma_{jk}^i, i, j, k \in \{ u, v \} \) and \( \ell = \mathbf{x}_{uu} \cdot \mathbf{U}, m = \mathbf{x}_{uv} \cdot \mathbf{U} \) and \( n = \mathbf{x}_{vv} \cdot \mathbf{U} \). The coefficients \( \Gamma_{jk}^i \) are known as Christoffel Symbols. To compute \( \Gamma_{uu}^u \), first note that

\[
\mathbf{x}_{uu} \cdot \mathbf{x}_u = \Gamma_{uu}^u \mathbf{x}_u \cdot \mathbf{x}_u + \Gamma_{uu}^v \mathbf{x}_v \cdot \mathbf{x}_u + \ell \mathbf{U} \cdot \mathbf{x}_u
= \Gamma_{uu}^u \mathbf{E}
\]

On the other hand, differentiating \( E = \mathbf{x}_u \cdot \mathbf{x}_u \) with respect to \( u \) gives

\[
E_u = \mathbf{x}_{uu} \cdot \mathbf{x}_u + \mathbf{x}_u \cdot \mathbf{x}_{uu} = 2 \mathbf{x}_{uu} \cdot \mathbf{x}_u.
\]

Therefore

\[
\mathbf{x}_{uu} \cdot \mathbf{x}_u = \frac{1}{2} E_u = \Gamma_{uu}^u \mathbf{E},
\]

which gives

\[
\Gamma_{uu}^u = \frac{E_u}{2E}.
\]

Similarly,

\[
\begin{align*}
\mathbf{x}_{uv} \cdot \mathbf{x}_u &= \frac{1}{2} E_v = \Gamma_{uv}^u \mathbf{E}, \\
\mathbf{x}_{uv} \cdot \mathbf{x}_v &= \frac{1}{2} G_u = \Gamma_{uv}^v \mathbf{G}, \\
\mathbf{x}_{vv} \cdot \mathbf{x}_v &= \frac{1}{2} G_v = \Gamma_{vv}^v \mathbf{G}
\end{align*}
\]
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give
\[ \Gamma_{uv}^u = \frac{E_v}{2E}, \quad \Gamma_{uv}^v = \frac{G_u}{2G}, \quad \text{and} \quad \Gamma_{vv}^v = \frac{G_v}{2G}. \]

For the two remaining coefficients, differentiate \( x_u \cdot x_v = 0 \) with respect to \( u \) and \( v \). With respect to \( u \) we get
\[ x_{uu} \cdot x_v + x_u \cdot x_{vu} = 0 \]
or
\[ x_{uu} \cdot x_v = -x_{uv} \cdot x_u = -\frac{1}{2} E_v = \Gamma_{uu}^v G, \]
which gives
\[ \Gamma_{uu}^v = -\frac{E_v}{2G} \]
and similarly with respect to \( v \) we obtain
\[ \Gamma_{vv}^u = -\frac{G_u}{2E}. \]

Therefore
\[
\alpha'' = u'' x_u + (u')^2 \left( \frac{E_u}{2E} x_u - \frac{E_v}{2G} x_v + fU \right) + 2u' \nu' \left( \frac{E_v}{2E} x_u + \frac{G_u}{2G} x_v + mU \right) + (\nu')^2 \left( -\frac{G_u}{2E} x_u + \frac{G_v}{2G} x_v + nU \right) + \nu'' x_v
\]
\[ = \left[ u'' + (u')^2 \frac{E_u}{2E} + u' \nu' \frac{E_v}{E} - (\nu')^2 \frac{G_u}{2E} \right] x_u \]
\[ + \left[ \nu'' - (u')^2 \frac{E_v}{2G} + u' \nu' \frac{G_u}{G} + (\nu')^2 \frac{G_v}{2G} \right] x_v \]
\[ + \left[ (u')^2 \ell + 2u' \nu' m + (\nu')^2 n \right] U. \]

This proves:

**Theorem 148** \( \alpha \) is a geodesic if and only if
\[
u'' - (u')^2 \frac{E_v}{2G} + u' \nu' \frac{G_u}{G} + (\nu')^2 \frac{G_v}{2G} = 0.
\]

\[
u'' - (u')^2 \frac{E_u}{2E} + u' \nu' \frac{E_v}{E} - (\nu')^2 \frac{G_u}{2E} = 0.
\]
6.3. EXISTENCE AND UNIQUENESS OF GEODESICS

The system of second order differential equations above are known as the Geodesic Equations. All geodesics appear as solutions of these equations.

Given initial data consisting of a point \( p \) on \( M \) and a tangent vector \( v_p \in M_p \), the theory of ordinary differential equations guarantees the existence and uniqueness of a geodesic on \( M \) through \( p \) with velocity \( v_p \). We state this formally as:

**Theorem 149** Let \( M \) be the image of a regular patch, let \( p \in M \), and let \( v_p \in M_p \). Then there exists an open interval \((a,b)\) containing 0 and a geodesic \( \alpha:(a,b) \to M \) such that \( \alpha(0) = p \) and \( \alpha'(0) = v_p \). The geodesic \( \alpha \) is unique in the following sense: If \( \beta:(c,d) \to M \) is any other geodesic such that \( \beta(0) = p \) and \( \beta'(0) = v_p \), then \((c,d) \subset (a,b)\) and \( \alpha(t) = \beta(t) \), for all \( t \in (c,d) \).

**Definition 150** Let \( \alpha:(a,b) \to M \) be a geodesic with initial conditions \( \alpha(0) = p \) and \( \alpha'(0) = v_p \). We say that \( \alpha \) is maximal if every other geodesic \( \beta:(c,d) \to M \) with initial conditions \( \beta(0) = p \) and \( \beta'(0) = v_p \) satisfies \((c,d) \subset (a,b)\) and \( \alpha(t) = \beta(t) \), for all \( t \in (c,d) \).

In light Theorem 149, we now see that the geodesics on the sphere are exactly the constant points and great circles; on the cylinder the geodesics are straight lines, circles, helices, and constant points.

The connection between distance on a surface and its geodesics is the content of our next theorem, which we state without proof:

**Theorem 151** Let \( \alpha \) be a curve on the image of a regular patch \( M \) whose trace joins two points \( p \) and \( q \). If \( \text{length} [\alpha] \leq \text{length} [\beta] \) for every other curve \( \beta \) on \( M \) whose trace joins two points \( p \) and \( q \), then \( \alpha \) is a geodesic.

This theorem does not imply that a geodesic joining two given points always exists. Here is a simple counterexample:

**Example 152** Let \( M \) be the plane with the origin removed, i.e., \( M = \mathbb{R}^2 - \{(0,0)\} \). The distance from \( p = (1,0) \) to \( q = (-1,0) \) is easily seen to be 2, however, every curve \( \beta \) whose trace joins \( p \) and \( q \) has length strictly greater than 2. Furthermore, the difference \( 2 - \text{length} [\beta] \) can be made arbitrarily small by appropriately choosing \( \beta \). Therefore no distance minimizing path in \( \mathbb{R}^2 - \{(0,0)\} \) joining \( p \) and \( q \) exists and it follows that no geodesic joining \( p \) and \( q \) exists.