Chapter 1

Isometries

The first three chapters of this book are dedicated to the study of isometries and their properties. Isometries, which are distance-preserving transformations from the plane to itself, appear as reflections, translations, glide reflections, and rotations. The proof of this profound and remarkable fact will result from our work in this and the next two chapters.

1.1 Transformations of the Plane

We denote points (resp. lines) in $\mathbb{R}^2$ by upper (resp. lower) case letters such as $A, B, C, \ldots$ (resp. $a, b, c, \ldots$). Lower case Greek letters such as $\alpha, \beta, \gamma, \ldots$ denote functions.

Definition 1 *A transformation of the plane is a function $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ with domain $\mathbb{R}^2$.*

Definition 2 *A transformation $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is injective (or one-to-one) if and only if the images of distinct points under α are distinct, i.e., if P and Q are distinct points in $\mathbb{R}^2$, then $\alpha(P) \neq \alpha(Q)$.*

Example 3 *The transformation $\beta\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ y \end{bmatrix}$ fails to be injective because $\beta\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \beta\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ while $\begin{bmatrix} -1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.*

Example 4 *The transformation $\gamma\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ 2x-y \end{bmatrix}$ is injective as the following argument (using the contrapositive of the implication in the definition) shows: Suppose that $\gamma\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \gamma\left(\begin{bmatrix} c \\ d \end{bmatrix}\right)$. Then*

$$\gamma\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+2b \\ 2a-b \end{bmatrix} = \begin{bmatrix} c+2d \\ 2c-d \end{bmatrix} = \gamma\left(\begin{bmatrix} c \\ d \end{bmatrix}\right)$$

and by equating x and y coordinates we obtain

$$\begin{aligned} a + 2b &= c + 2d \\ 2a - b &= 2c - d \end{aligned}.$$

A simple calculation now shows that $b = d$ and $a = c$ so that $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = \left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$.

Definition 5 *A transformation $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is surjective (or onto) if and only if every point lies in the image of α, i.e., given any point $Q \in \mathbb{R}^2$, there is some point $P \in \mathbb{R}^2$ such that $\alpha(P) = Q$.*

Example 6 *The transformation $\beta\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} x^2 \\ y \end{smallmatrix}\right]$ discussed in Example 3 fails to be surjective because there is no point $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathbb{R}^2$ for which $\beta\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$.*

Example 7 *The transformation $\gamma\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} x+2y \\ 2x-y \end{smallmatrix}\right]$ discussed in Example 4 is surjective as the following argument shows: Let $Q = \left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$ be any point in the plane. Are there choices for x and y such that $\gamma\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$? Equivalently, does $\left[\begin{smallmatrix} x+2y \\ 2x-y \end{smallmatrix}\right] = \left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$ for appropriate choices of x and y? The answer is yes if the system*

$$\begin{aligned} x + 2y &= c \\ 2x - y &= d \end{aligned}$$

has a solution, which indeed it does since the determinant $\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -1-4 = -5 \neq 0$. By solving simultaneously for x and y in terms of c and d we find that

$$\begin{aligned} x &= \tfrac{1}{5}c + \tfrac{2}{5}d \\ y &= \tfrac{2}{5}c - \tfrac{1}{5}d. \end{aligned}$$

Therefore $\gamma\left(\left[\begin{smallmatrix} \frac{1}{5}c+\frac{2}{5}d \\ \frac{2}{5}c-\frac{1}{5}d \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$ and γ is surjective by definition.

Definition 8 *A bijective transformation is both injective and surjective.*

Example 9 *The transformation γ discussed in Examples 4 and 7 is bijective; the transformation β discussed in Examples 3 and 6 is not.*

Distance-preserving transformations of the plane are fundamentally important in this course. The discussion below uses the following notation: If P and Q are distinct points, the symbols PQ, $\overline{PQ}$, $\overleftrightarrow{PQ}$ and P_Q denote distance between P and Q, the line segment connecting P and Q, the line through P and Q, and the circle centered at P through Q, respectively.

Definition 10 *An isometry is a distance-preserving transformation* $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$, *i.e., for all* $P, Q \in \mathbb{R}^2$, *if* $P' = \alpha(P)$ *and* $Q' = \alpha(Q)$, *then* $PQ = P'Q'$.

Example 11 *The identity transformation* $\iota : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by* $\iota(P) = P$ *is an isometry*

Proposition 12 *Isometries are injective.*

Proof. Let α be an isometry and let A and B be distinct points. Then $A' = \alpha(A)$ and $B' = \alpha(B)$ are distinct since $A'B' = AB > 0$. ■

Isometries are also surjective, but the proof requires the following lemma:

Lemma 13 *Three concurrent circles with non-collinear centers have a unique point of concurrency.*

Proof. We prove the contrapositive. Suppose that circles A_B, C_D and E_F are concurrent at two distinct points P and Q. Since all three circles share chord $\overline{PQ}$, their centers A, C and E are collinear since they lie on the perpendicular bisector of $\overline{PQ}$. ■

Theorem 14 *Isometries are surjective.*

Proof. Given an isometry α and an arbitrary point A, show there exists a point D such that $\alpha(D) = A$. If $\alpha(A) = A$, then $A = D$ and we're done, so assume that $B = A' = \alpha(A) \neq A$. Then $B' = \alpha(B)$ lies on circle B_A since $AB = A'B' = BB'$ (α is an isometry). Again, if $B' = A$, then $B = D$ and we're done, so assume that $B' \neq A$. Choose a point $C \in A_B \cap B_A$ and note that $\triangle ABC$ and $\triangle A'B'C'$ are equilateral as is (α is an isometry). Then $C' = \alpha(C)$ lies on B_A since $AB = AC = A'C' = BC'$. Again, if $C' = A$, then $D = C$ and we're done, so assume that $C' \neq A$ and consider $\triangle AB'C'$ whose vertices lie on B_A. By Lemma 13, A is the unique point of concurrency for circles A'_A, B'_A and C'_A (see Figure 1.1).

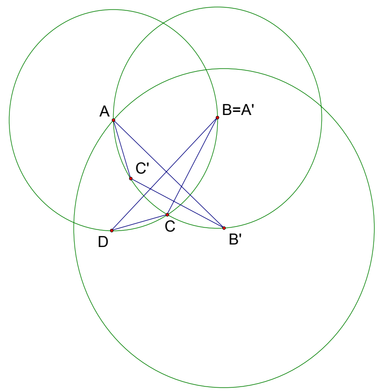


Figure 1.1.

Choose a point D on A_B such that $\triangle DBC \cong \triangle AB'C'$ with $\angle BCD \cong \angle B'C'A$. I claim that $D' = \alpha(D) = A$. Since A' and D lie on A_B and α is an isometry, $A'A = AD = A'D'$ and D' lies on A'_A. Since corresponding parts of congruent triangles are congruent (CPCTC) and α is an isometry, $B'A = BD = B'D'$ and D' lies on B'_A. Since CPCTC and α is an isometry, $C'A = CD = C'D'$ and D' lies on C'_A. Therefore D' is a point of concurrency for A'_A, B'_A and C'_A. We conclude that $D' = A$ by uniqueness. ■

Definition 15 *Let α be an isometry, let P be any point in the plane, and let Q be the unique point in the plane such that $\alpha(Q) = P$. The function inverse to α, denoted by α^{-1}, is defined by $\alpha^{-1}(P) = Q$.*

Proposition 16 *Let α and β be an isometries.*

1. *The composition $\alpha \circ \beta$ is an isometry.*
2. *$\alpha \circ \iota = \iota \circ \alpha = \alpha$, i.e., the identity transformation acts as an identity element.*
3. *α^{-1} is an isometry.*

Proof. The proof is left as an exercise for the reader. ■

As we shall see, the isometries include reflections in a fixed line, rotations about a fixed point, translations, and glide reflections. We begin our study of isometries with a look at the reflections. One of the fundamental results of this course is that every isometry can be written as a composition of three of fewer reflections, but that's getting ahead of the story. We conclude this section with one additional fact about isometries.

Definition 17 *A* <u>*collineation*</u> *is a transformation that maps lines to lines.*

Proposition 18 *Isometries are collineations.*

Proof. The proof is left as an exercise for the reader. ■

Exercises

1. Which of the following transformations are injective? Which are surjective? Which are bijective?

$$\alpha\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}x^3\\y\end{bmatrix}\quad \beta\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}\cos x\\\sin y\end{bmatrix}\quad \gamma\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}x^3-x\\y\end{bmatrix}$$
$$\delta\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}2x\\3y\end{bmatrix}\quad \varepsilon\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}-x\\x+3\end{bmatrix}\quad \eta\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}3y\\x+2\end{bmatrix}$$
$$\rho\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}\sqrt[3]{x}\\e^y\end{bmatrix}\quad \sigma\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}-x\\-y\end{bmatrix}\quad \tau\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}x+2\\y-3\end{bmatrix}$$

2. Prove that the composition of transformations is a transformation.

3. Prove that the composition of isometries is an isometry.

4. Prove that *the composition of functions is associative, i.e., if* α,β,γ are functions, *then* $\alpha\circ(\beta\circ\gamma)=(\alpha\circ\beta)\circ\gamma$. (Hint: Show that $[\alpha\circ(\beta\circ\gamma)](P)=[(\alpha\circ\beta)\circ\gamma](P)$ *for every element* P in the domain.)

5. Prove that the identity transformation ι is an identity element for the set of all transformations with respect to composition, i.e., if α is a transformation, then $\alpha\circ\iota=\iota\circ\alpha=\alpha$.

6. Prove that the inverse of a bijective transformation is a bijective transformation. (Remark: Exercises 2,4,5 and 6 show that the set of all bijective transformations is a group under composition.)

7. Prove that the inverse of an isometry is an isometry (Remark: Exercises 3,4,5 and 7 show that the set of all isometries is a group under composition.)

8. Let α and β be bijective transformations. Prove that $(\alpha\circ\beta)^{-1}=\beta^{-1}\circ\alpha^{-1}$, i.e., the inverse of a composition is the composition of the inverses in reverse order.

9. Prove Proposition 18: Isometries are collineations.

10. Let ℓ be the line with equation $aX + bY + c = 0$. Show that each of the following transformations α is a collineation by finding the equation of the image line $\ell' = \alpha(\ell)$.

    a. $\alpha\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y \end{bmatrix}$.

    b. $\alpha\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix}$.

    c. $\alpha\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ -y \end{bmatrix}$.

    d. $\alpha\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2y-x \\ x-2 \end{bmatrix}$.

11. Which of the transformations in Exercise 1 are collineations? For each collineation in Exercise 1, find the equation of the image of the line ℓ with equation $aX + bY + c = 0$.

12. Consider the collineation $\alpha\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3y \\ x-y \end{bmatrix}$ and the line ℓ' whose equation is $3X - Y + 2 = 0$. Find the equation of the line ℓ such that $\alpha(\ell) = \ell'$.

13. Find an example of a bijective transformation that is not a collineation.

1.2 Reflections

Definition 19 *A point P is a fixed point for a transformation α if and only if $\alpha(P) = P$.*

Definition 20 *Let ℓ be a line. The reflection in line ℓ is the transformation $\sigma_\ell : \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies the following two conditions:*

1. *Each point $P \in \ell$ is a fixed point for σ_ℓ and*
2. *If $P \notin \ell$ and $P' = \sigma_\ell(P)$, then ℓ is the perpendicular bisector of $\overline{PP'}$.*

Theorem 21 *Reflections are isometries.*

Proof. Let ℓ be any line and let σ_ℓ be the reflection in line ℓ. Let P and Q be distinct points in the plane and let $P' = \sigma_\ell(P)$ and $Q' = \sigma_\ell(Q)$. We shall consider several cases depending upon the position of P and Q relative to ℓ.

Case 1: Suppose that P and Q lie on ℓ. Then $P' = P$ and $Q' = Q$ so $P'Q' = PQ$

as required.

<u>*Case 2:*</u> Suppose that P lies on ℓ and Q lies off of ℓ. Then $P = P'$ and ℓ is the $\perp$ bisector of $\overline{QQ'}$.

<u>*Subcase 2a:*</u> If $\overleftrightarrow{PQ} \perp \ell$ then Q' lies on $\overleftrightarrow{PQ}$ and P is the midpoint of $\overline{QQ'}$ so that $PQ = PQ' = P'Q'$ as required.

<u>*Subcase 2b:*</u> Otherwise, let R be the point of intersection of ℓ with $\overline{QQ'}$. Observe that $\triangle PQR \cong \triangle PQ'R$ by SAS, where the angles considered here are the right angles (see Figure 1.2).

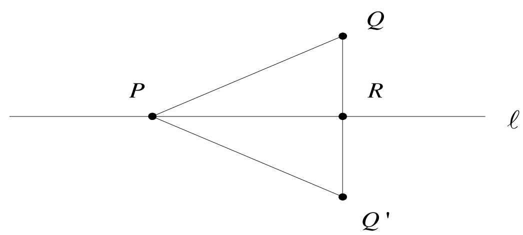


Figure 1.2.

Since corresponding parts of congruent triangles are congruent ($CPCTC$) we have $PQ = PQ' = P'Q'$ as required.

<u>*Case 3:*</u> Suppose that both P and Q lie off ℓ and on the same side of ℓ.

<u>*Subcase 3a:*</u> If $\overleftrightarrow{PQ} \perp \ell$, let R be the point of intersection of ℓ with $\overleftrightarrow{PQ}$. If $PR > QR$ then $PQ = PR - QR = P'R - Q'R = P'Q'$, and similarly if $QR > PR$.

<u>*Subcase 3b:*</u> Otherwise, let R be the point of intersection of ℓ with $\overleftrightarrow{PP'}$ and let S be the point of intersection of ℓ with $\overleftrightarrow{QQ'}$. Then $\triangle PRS \cong \triangle P'RS$ by SAS so that $PS = PS'$ by $CPCTC$ (see Figure 1.3).

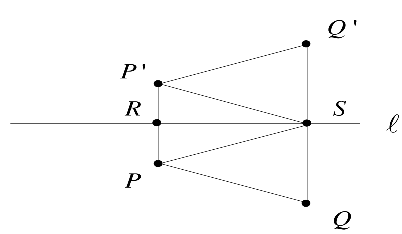


Figure 1.3.

Now $\overleftrightarrow{PP'} \perp \ell$ and $\overleftrightarrow{QQ'} \perp \ell$ so $\overleftrightarrow{PP'} \parallel \overleftrightarrow{QQ'}$. Lines $\overleftrightarrow{PS}$ and $\overleftrightarrow{P'S}$ are transversals

so $\angle QSP = \angle SPR = \angle SP'R = \angle Q'SP'$. Since $QS = Q'S$ we have $\triangle PQS \cong \triangle P'Q'S$ from which it follows that $PQ = P'Q'$ by $CPCTC$.

<u>*Case 4:*</u> Suppose that both P and Q lie off ℓ and on opposite sides of ℓ. (Proofs

of these subcases are left as exercises for the reader.)

<u>*Subcase 4a:*</u> If $\overleftrightarrow{PQ} \perp \ell$, let R be the point of intersection of ℓ with $\overleftrightarrow{PQ}$.

<u>*Subcase 4b:*</u> Otherwise, let R be the point of intersection of ℓ with $\overleftrightarrow{PP'}$, let S be the point of intersection of ℓ with $\overleftrightarrow{QQ'}$, and let T be the point of intersection of ℓ with $\overleftrightarrow{PQ}$. ■

When $\mathbb{R}^2$ comes equipped with a Cartesian system of coordinates, one can use analytic geometry to calculate the coordinates of $P' = \sigma_\ell(P)$ from the equation of ℓ and the coordinates of P. We now derive the formulas (called equations of σ_ℓ) for doing this. Let ℓ be a line with equation $aX+bY+c=0$, with $a^2+b^2>0$, and consider points $P = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ and $P' = \left[\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right]$ such that $P' = \sigma_\ell(P)$. Assume for the moment that P is off ℓ. By definition, $\overleftrightarrow{PP'} \perp \ell$. So if neither ℓ nor $\overleftrightarrow{PP'}$ is vertical, the product of their respective slopes is -1, i.e.,

$$-\frac{a}{b} \cdot \frac{y'-y}{x'-x} = -1$$

or

$$\frac{y'-y}{x'-x} = \frac{b}{a}.$$

Cross-multiplying gives

$$a(y'-y) = b(x'-x). \tag{1.1}$$

Note that equation (1.1) holds when ℓ is vertical or horizontal as well. If ℓ is vertical, its equation is $X+c=0$, in which case $a=1$ and $b=0$. But reflection in a vertical line preserves the y-coordinate so that $y=y'$. On the other hand, if ℓ is horizontal its equation is $Y+c=0$, in which case $a=0$ and $b=1$. But reflection in a horizontal line preserves the x-coordinate so that $x=x'$. But this is exactly what equation (1.1) gives in either case. Now the midpoint M of P and P' has coordinates

$$M = \begin{bmatrix} \frac{x+x'}{2} \\ \frac{y+y'}{2} \end{bmatrix}.$$

Since M lies on ℓ, its coordinates satisfy $aX+bY+c=0$, which is the equation of line ℓ. Therefore

$$a\left(\frac{x+x'}{2}\right) + b\left(\frac{y+y'}{2}\right) + c = 0. \tag{1.2}$$

Now rewrite equations (1.1) and (1.2) to obtain the following system of linear equations in x' and y' :

$$\begin{cases} bx' - ay' = s \\ ax' + by' = t \end{cases}.$$

where $s = bx - ay$ and $t = -2c - ax - by$. Write this system in matrix form as

$$\begin{bmatrix} b & -a \\ a & b \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix}. \tag{1.3}$$

Since $a^2 + b^2 > 0$, the coefficient matrix is invertible, we may solve for $\left[\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right]$ and obtain

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} b & -a \\ a & b \end{bmatrix}^{-1} \begin{bmatrix} s \\ t \end{bmatrix} = \frac{1}{a^2+b^2} \begin{bmatrix} b & a \\ -a & b \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \frac{1}{a^2+b^2} \begin{bmatrix} bs+at \\ bt-as \end{bmatrix}.$$

Substituting for s and t gives

$$\begin{aligned} x' &= \frac{1}{a^2+b^2}\left[b\,(bx-ay) + a\,(-2c-ax-by)\right] && (1.4)\\ &= \frac{1}{a^2+b^2}\left(b^2x - bay - 2ac - a^2x - aby\right) \\ &= \frac{1}{a^2+b^2}\left(b^2x + (a^2x - a^2x) - a^2x - 2aby - 2ac\right) \\ &= x - \frac{2a}{a^2+b^2}\,(ax+by+c)\,, \end{aligned}$$

and similarly

$$y' = y - \frac{2b}{a^2+b^2}\,(ax+by+c)\,. \qquad (1.5)$$

Finally, if $P = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ is on ℓ, then $ax + by + c = 0$ and equations (1.4) and (1.5) reduce to

$$\begin{array}{l} x' = x \\ y' = y \end{array},$$

in which case P is a fixed point as required by the definition of σ_ℓ. We have proved:

Theorem 22 *Let ℓ be a line with equation $aX + bY + c = 0$, where $a^2 + b^2 > 0$. The equations of σ_ℓ (the reflection in line ℓ) are:*

$$\begin{array}{l} x' = x - \frac{2a}{a^2+b^2}\,(ax+by+c) \\ y' = y - \frac{2b}{a^2+b^2}\,(ax+by+c) \end{array}. \qquad (1.6)$$

Remark 23 *The equations of σ_ℓ are not to be confused with the equation of line ℓ.*

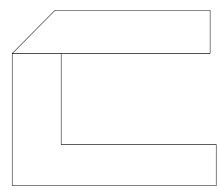
Exercises

1. Words such as `MOM` and `RADAR` that spell the same backward and forward, are called *palindromes*.

   a. When reflected in their vertical midlines, `MOM` remains `MOM` but the `R`'s and `D` in `RADAR` appear backward. Find at least five other palindromes like `MOM` that are preserved under reflection in their vertical midlines.

b. When reflected in their horizontal midlines, `MOM` becomes `WOW`, but `BOB` remains `BOB`. Find at least five other words like `BOB` that are preserved under reflection in their horizontal midlines.

2. What capital letters could be cut out of paper and given a single fold to produce the figure below?



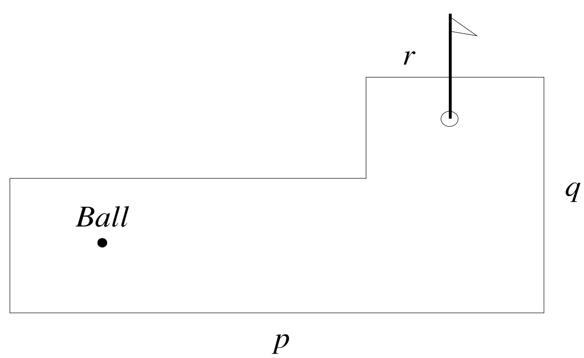
3. The diagram below shows a par 2 hole on a miniature golf course. Use a MIRA to construct the path the ball must follow to score a hole-in-one after banking the ball off

   a. wall p.

   b. wall q.

   c. walls p and q.

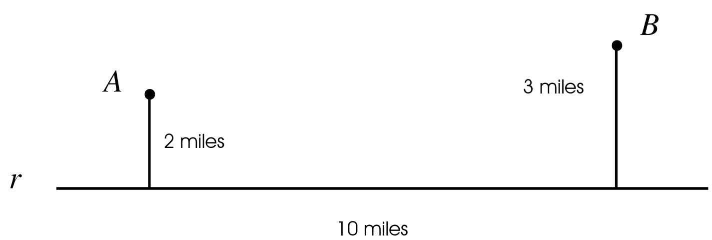
   d. walls p, q and r.



4. Two cities, located at points A and B in the diagram below, need to pipe water from the river, represented by line r. City A is 2 miles north of the

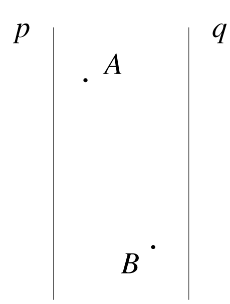
river; city B is 10 miles downstream from A and 3 miles north of the river. The State will build one pumping station along the river.

a. Use a MIRA to locate the point C along the river at which the pumping station should be built so that the minimum amount of pipe is used to connect city A to C and city B to C.



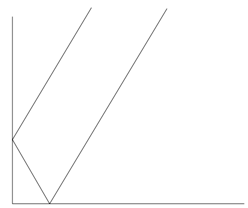
b. Having located point C, prove that if D is any point on r distinct from C, then $AD + DB > AC + CB$.

5. Given two parallel lines p and q in the diagram below, use a MIRA to construct the path of a ray of light issuing from A and passing through B after being reflected exactly twice in p and once in q.



6. Use the fact that the path followed by a the ray of light traveling from point A to point B is the line segment $\overline{AB}$ to prove the following statement: When a ray of light is reflected by a flat mirror, the angle of incidence equals the angle of reflection.

7. A ray of light is reflected by two perpendicular flat mirrors. Prove that the emerging ray is parallel to the initial incoming ray as indicated in the diagram below.



8. The Smiths, who range in height from 170 cm to 182 cm, wish to purchase a flat wall mirror that allows each Smith to view the full length of his or her image. Use the fact that each Smith's eyes are 10 cm below the top of his or her head to determine the minimum length of such a mirror.

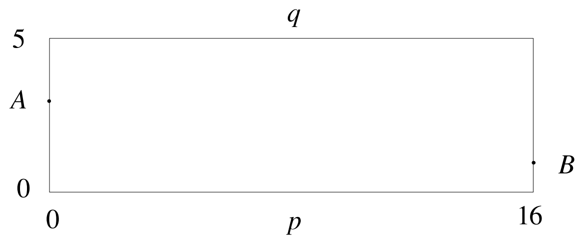
9. Graph the line ℓ with equation $X + 2Y - 6 = 0$ on graph paper. Plot the point $P = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$ and use a MIRA to locate and mark its image P'. Visually estimate the coordinates $\begin{bmatrix} x' \\ y' \end{bmatrix}$ of the image point P' and record your estimates. Using formulas (1.6), write down the equations for the reflection σ_ℓ and use them to compute the coordinates of the image point P'. Compare these analytical calculations with your visual estimates of the coordinates.

10. Fill in the missing entry in each row of the following table:

Equation of ℓ	Point P	$\sigma_\ell(P)$
$X = 0$	$\begin{bmatrix} x \\ y \end{bmatrix}$	$*$
$Y = 0$	$*$	$\begin{bmatrix} x \\ y \end{bmatrix}$
$Y = X$	$*$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$
$Y = X$	$\begin{bmatrix} x \\ y \end{bmatrix}$	$*$
$X = 2$	$\begin{bmatrix} -2 \\ 3 \end{bmatrix}$	$*$
$Y = -3$	$\begin{bmatrix} -4 \\ -1 \end{bmatrix}$	$*$

Equation of ℓ	Point P	$\sigma_\ell(P)$
$Y = -3$	$\begin{bmatrix} x \\ y \end{bmatrix}$	$*$
$*$	$\begin{bmatrix} 5 \\ 3 \end{bmatrix}$	$\begin{bmatrix} -8 \\ 3 \end{bmatrix}$
$*$	$\begin{bmatrix} 0 \\ 3 \end{bmatrix}$	$\begin{bmatrix} -3 \\ 0 \end{bmatrix}$
$*$	$\begin{bmatrix} -y \\ -x \end{bmatrix}$	$\begin{bmatrix} x \\ y \end{bmatrix}$
$Y = 2X$	$*$	$\begin{bmatrix} 4 \\ 3 \end{bmatrix}$
$2Y = 3X + 5$	$\begin{bmatrix} x \\ y \end{bmatrix}$	

11. Horizontal lines p and q in the diagram have respective equations $Y = 0$ and $Y = 5$.



a. Use a MIRA to construct the shortest path from point $A\left[\begin{smallmatrix}0\\3\end{smallmatrix}\right]$ to point $B\left[\begin{smallmatrix}16\\1\end{smallmatrix}\right]$ that first touches q and then p.

b. Determine the coordinates of the point on q and the point on p touched by the path constructed in part a.

c. Find the length of the path from A to B constructed in part a.

12. For each of the following pairs of points P and P', determine the equation of the reflecting line ℓ such that $P' = \sigma_\ell(P)$.

    a. $P\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, $P'\left[\begin{smallmatrix}-1\\-1\end{smallmatrix}\right]$

    b. $P\left[\begin{smallmatrix}2\\6\end{smallmatrix}\right]$, $P'\left[\begin{smallmatrix}4\\8\end{smallmatrix}\right]$

13. The equation of line ℓ is $Y = 2X - 5$. Find the coordinates of the images of $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1\\-3\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}2\\4\end{smallmatrix}\right]$ under reflection in line ℓ.

14. The equation of line m is $X - 2Y + 3 = 0$. Find the coordinates of the images of $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}4\\-1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}-3\\5\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}3\\6\end{smallmatrix}\right]$ under reflection in line m.

15. The equation of line p is $2X + 3Y + 4 = 0$; the equation of line q is $X - 2Y + 3 = 0$. Find the equation of the line $r = \sigma_q(p)$.

16. Prove subcases 4a and 4b in the proof of Theorem 21.

17. For any line ℓ, prove that $\sigma_\ell^{-1} = \sigma_\ell$.

1.3 Translations

A translation of the plane is an isometry whose effect is the same as sliding the plane in a direction parallel to some line for some finite distance.

Definition 24 *Let P and Q be points. The translation from P to Q is the transformation $\tau_{P,Q} : \mathbb{R}^2 \to \mathbb{R}^2$ with the following properties:*

1. $Q = \tau_{P,Q}(P)$.
2. If $P = Q$, then $\tau_{P,Q} = \iota$.
3. If $P \neq Q$, *let* A *be any point on* $\overleftrightarrow{PQ}$ *and let* B *be any point off* $\overleftrightarrow{PQ}$; *let* $A' = \tau_{P,Q}(A)$ *and let* $B' = \tau_{P,Q}(B)$. *Then quadrilaterals* $\square PQB'B$ *and* $\square AA'B'B$ *are parallelograms (see Figure 1.4).*

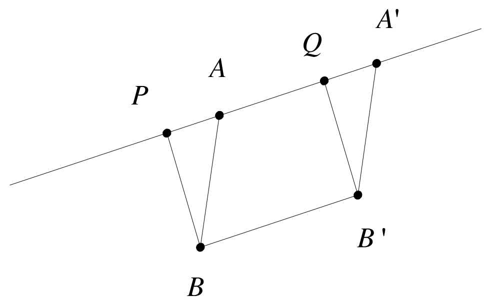


Figure 1.4.

When $P \neq Q$ one can think of a translation in the following way: If A is any point and ℓ is the line through A parallel to $\overleftrightarrow{PQ}$, then $B = \tau_{P,Q}(A)$ is the point on ℓ whose directed distance from A is PQ.

In the discussion below, the symbol $\overrightarrow{PQ}$ denotes the ray from P through Q.

Definition 25 *Let* P *and* Q *be distinct points. The* <u>*vector*</u> <u>$\mathbf{PQ}$</u> *is the quantity with magnitude* PQ *and direction* $\overrightarrow{PQ}$. *The vector* $\mathbf{PP}$, *called the* <u>*zero vector*</u>, *has magnitude* 0 *and arbitrary direction. If* $P = \begin{bmatrix} a \\ b \end{bmatrix}$ *and* $Q = \begin{bmatrix} c \\ d \end{bmatrix}$, *the quantities* $c - a$ *and* $d - b$ *are called the* <u>*x-component*</u> *and* <u>*y-component*</u> *of* $\mathbf{PQ}$, *respectively.*

Since a vector $\mathbf{P} = \begin{bmatrix} x \\ y \end{bmatrix}$ is uniquely determined by its components, we identify the vector $\mathbf{P}$ with the point $P = \begin{bmatrix} x \\ y \end{bmatrix}$ and write $\mathbf{P} = P$.

Definition 26 *The* <u>*vector sum*</u> *of* $\mathbf{PQ} = \begin{bmatrix} u \\ v \end{bmatrix}$ *and* $\mathbf{RS} = \begin{bmatrix} x \\ y \end{bmatrix}$ *is the vector*

$$\mathbf{PQ} + \mathbf{RS} = \begin{bmatrix} u + x \\ v + y \end{bmatrix}.$$

Definition 27 *Let* P *and* Q *be points. The* <u>*translation by vector* $\mathbf{PQ}$</u> *is the transformation* $\tau_{\mathbf{PQ}} : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by*

$$\tau_{\mathbf{PQ}}(R) = \mathbf{R} + \mathbf{PQ}.$$

Vector $\mathbf{PQ}$ *is called* <u>*the vector of* $\tau_{\mathbf{PQ}}$</u>.

The next theorem relates the two formulations of translation defined above; each has its advantages.

Theorem 28 $\tau_{P,Q} = \tau_{\mathbf{PQ}}$ *for all points* P *and* Q.

Proof. If $P = Q$, then $\tau_{P,Q} = \iota = \tau_{\mathbf{PQ}}$. So assume that $P \neq Q$. If A is any point on $\overleftrightarrow{PQ}$ and B is any point off $\overleftrightarrow{PQ}$, let $A' = \tau_{P,Q}(A)$ and $B' = \tau_{P,Q}(B)$. Then $\square PQA'A$ and $\square PQB'B$ are parallelograms by definition, in which case $\mathbf{AA'} = \mathbf{BB'} = \mathbf{PQ}$. Therefore, $\tau_{\mathbf{PQ}}(A) = \mathbf{A} + \mathbf{PQ} = \mathbf{A} + \mathbf{AA'} = \mathbf{A'} = A' = \tau_{P,Q}(A)$ and $\tau_{P,Q}(B) = \mathbf{B} + \mathbf{PQ} = \mathbf{B} + \mathbf{BB'} = \mathbf{B'} = B' = \tau_{P,Q}(B)$. ■

Corollary 29 *If* $\tau_{\mathbf{PQ}}(R) = S$, *then* $\tau_{\mathbf{PQ}} = \tau_{\mathbf{RS}}$.

Proof. If $P = Q$, then $\tau_{\mathbf{PQ}} = \iota$ and $R = S$; hence $\tau_{\mathbf{RS}} = \iota$. If $P \neq Q$, then $\tau_{\mathbf{PQ}}(R) = S = \tau_{P,Q}(R)$ by Theorem 28. But $\square PQSR$ is a parallelogram by definition of $\tau_{P,Q}$, hence $\mathbf{PQ} = \mathbf{RS}$ and $\tau_{\mathbf{PQ}} = \tau_{\mathbf{RS}}$. ■

Corollary 29 tells us that a translation is uniquely determined by any point and its image. Consequently, we shall often refer to a general translation τ without specific reference to a point P and its image Q or to a vector $\mathbf{PQ}$. When we need the vector of τ, for example, we simply evaluate τ at any point $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ and obtain the image point $\left[\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right]$; the desired components are $x' - x$ and $y' - y$. Furthermore, given $\mathbf{PQ} = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = \left[\begin{smallmatrix} x'-x \\ y'-y \end{smallmatrix}\right]$, we immediately obtain the equations of the translation by vector $\mathbf{PQ}$:

Proposition 30 *Let* $\mathbf{PQ} = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$. *The equations for the translation* $\tau_{\mathbf{PQ}}$ *are*

$$\begin{array}{l} x' = x + a \\ y' = y + b \end{array}. \tag{1.7}$$

Example 31 *Let* $P = \left[\begin{smallmatrix} 4 \\ 5 \end{smallmatrix}\right]$ *and* $Q = \left[\begin{smallmatrix} -1 \\ 3 \end{smallmatrix}\right]$. *Then* $\mathbf{PQ} = \left[\begin{smallmatrix} -5 \\ -2 \end{smallmatrix}\right]$ *and the equations for* $\tau_{\mathbf{PQ}}$ *are*

$$\begin{array}{l} x' = x - 5 \\ y' = y - 2 \end{array}.$$

In particular, $\tau_{\mathbf{PQ}}\left(\left[\begin{smallmatrix} 7 \\ -5 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 2 \\ -7 \end{smallmatrix}\right]$.

Theorem 32 *Translations are isometries.*

Proof. Let τ be a translation, let P and Q be arbitrary points, let $P' = \tau(P)$ and $Q' = \tau(Q)$. If $P = Q$, then $P' = Q'$ and $PQ = 0 = P'Q'$. So assume that $P \neq Q$. Then $\square PP'Q'Q$ is a parallelogram by definition, in which case

$PQ = P'Q'$ as required. The case of collinear points P, Q, P', Q' is left as an exercise for the reader. ■

Although function composition is not commutative in general, the composition of translations is commutative. Intuitively, this says that you will arrive at the same destination either by a move through directed distance d_1 parallel to line ℓ_1 followed by a move through directed distance d_2 parallel to line ℓ_2, or by a move through directed distance d_2 parallel to ℓ_2 followed by a move through directed distance d_1 parallel to ℓ_1. The paths to your destination follow the two routes along the edges a parallelogram from one vertex to its diagonal opposite. This fact is the second part of the next proposition, whose proof is left as an exercise.

Proposition 33 *Let P, Q, R, and S be arbitrary points.*

1. The composition of translations is a translation. In fact,

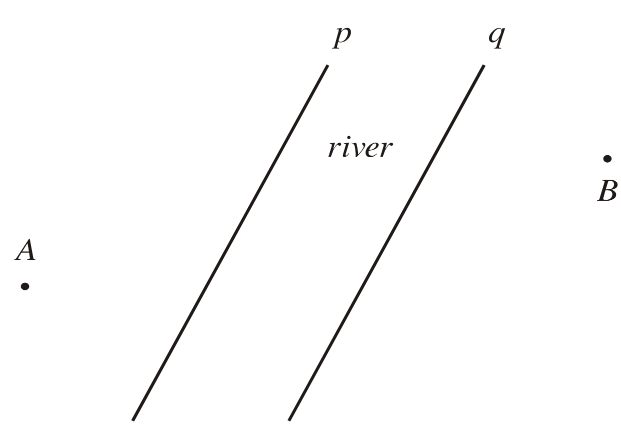
$$\tau_{\mathbf{RS}} \circ \tau_{\mathbf{PQ}} = \tau_{\mathbf{PQ}+\mathbf{RS}}.$$

2. The composition of translations commutative:

$$\tau_{\mathbf{RS}} \circ \tau_{\mathbf{PQ}} = \tau_{\mathbf{PQ}} \circ \tau_{\mathbf{RS}}.$$

Exercises

1. A river with parallel banks p and q is to be spanned by a bridge at right angles to p and q.



a. Using an overhead transparency to perform a translation and a MIRA, locate the bridge that minimizes the distance from city A to city B.

b. Let $\overline{PQ}$ denote the bridge at right angles to p and q constructed in part a. Prove that if $\overline{RS}$ is any other bridge spanning river r distinct from and parallel to $\overline{PQ}$, then $AR + RS + SB > AP + PQ + QB$.

2. Let τ be the translation such that $\tau\left[\begin{smallmatrix}-1\\3\end{smallmatrix}\right] = \left[\begin{smallmatrix}5\\2\end{smallmatrix}\right]$.

   a. Find the vector of τ.

   b. Find the equations of τ.

   c. Find $\tau\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$, $\tau\left[\begin{smallmatrix}3\\-7\end{smallmatrix}\right]$, and $\tau\left[\begin{smallmatrix}-5\\-2\end{smallmatrix}\right]$.

   d. Find x and y such that $\tau\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$.

3. Let τ be the translation such that $\tau\left[\begin{smallmatrix}4\\6\end{smallmatrix}\right] = \left[\begin{smallmatrix}7\\10\end{smallmatrix}\right]$.

   a. Find the vector of τ.

   b. Find the equations of τ.

   c. Find $\tau\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$, $\tau\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$, and $\tau\left[\begin{smallmatrix}-3\\-4\end{smallmatrix}\right]$.

   d. Find x and y such that $\tau\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$.

4. Let $P = \left[\begin{smallmatrix}4\\-1\end{smallmatrix}\right]$ and $Q = \left[\begin{smallmatrix}-3\\5\end{smallmatrix}\right]$.

   a. Find the vector of $\tau_{P,Q}$.

   b. Find the equations of $\tau_{P,Q}$.

   c. Find $\tau_{P,Q}\left[\begin{smallmatrix}3\\6\end{smallmatrix}\right]$, $\tau_{P,Q}\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$, and $\tau_{P,Q}\left[\begin{smallmatrix}-3\\-4\end{smallmatrix}\right]$.

   d. Let ℓ be the line with equation $2X + 3Y + 4 = 0$. Find the equation of the line $\ell' = \tau_{P,Q}(\ell)$.

5. Complete the proof of the statement in Theorem 32: Let τ be a translation, let P and Q be distinct points, let $P' = \tau(P)$ and let $Q' = \tau(Q)$. If P, Q, P' and Q' are collinear, prove that $PQ = P'Q'$.

6. Prove that the composition of translations is a translation.

7. Prove the composition of translations is commutative.

8. Let ℓ be a line containing distinct points P and Q. Prove that $\tau_{P,Q}(\ell) = \ell$.

9. Let A and B be points. Prove that $\tau_{\mathbf{AB}}^{-1} = \tau_{\mathbf{BA}}$.

10. Let ℓ and m be the lines with respective equations $X + Y - 2 = 0$ and $X + Y + 8 = 0$.

a. Compose the equations of σ_m and σ_ℓ and show that the composition $\sigma_m \circ \sigma_\ell$ is a translation τ.

b. Compare the norm of the vector of τ with the distance between ℓ and m.

1.4 Glide Reflections

In this section we introduce glide reflections, which are certain compositions of reflections with translations. In exercise 3 of Section 1 above, you proved that the composition of isometries is an isometry. Thus glide reflections are automatically isometries. Before we begin, let's consider an example of reflection composed with a translation that fails to be a glide reflection in the sense defined below.

Example 34 *Consider the composition $\sigma_y \circ \tau_{\mathbf{PQ}}$ where $\tau_{\mathbf{PQ}}$ is the translation with vector $\mathbf{PQ} = \left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$ and σ_y is reflection in the y-axis. The equations for $\tau_{\mathbf{PQ}}$ are $x' = x + 2$ and $y' = y$; the equations for σ_y are $x' = -x$ and $y' = y$. To obtain the equations for the composition $\sigma_y \circ \tau_{\mathbf{PQ}}$, compose equations as follows: $x' = -(x+2) = -x - 2$ and $y' = y$. As the reader can easily check, these are exactly the equations for reflection in line $\ell : X + 1 = 0$. Thus $\sigma_y \circ \tau_{\mathbf{PQ}} = \sigma_\ell$.*

Note that the composition $\alpha = \sigma_y \circ \tau_{\mathbf{PQ}}$ in Example 34 "fixes" *every* line parallel to the translation vector. This distinguishes α from a glide reflection, which fixes exactly one line parallel to the translation vector.

Definition 35 *A transformation α <u>fixes</u> a set s if and only if $\alpha(s) = s$. A transformation α fixes a set s <u>pointwise</u> if and only if $\alpha(S) = S$ each point $S \in s$.*

Example 36 *Let c be a line. The reflection σ_c fixes c pointwise. If $\mathbf{PQ}$ is a non-zero vector parallel to c, the translation $\tau_{\mathbf{PQ}}$ fixes c but not pointwise. The composition $\sigma_c \circ \tau_{\mathbf{PQ}}$ fixes line c but not lines distinct from and parallel to c.*

Definition 37 *A transformation $\gamma : \mathbb{R}^2 \to \mathbb{R}^2$ is a <u>glide reflection</u> if and only if there exists a non-identity translation τ and a reflection σ_c such that*

1. $\gamma = \sigma_c \circ \tau$ *and*

2. $\tau(c) = c$.

The vector of τ is called the <u>*glide vector of γ*</u>; *the line c is called the <u>axis of γ</u>.*

In your mind, picture the footprints you leave when walking in the sand. Imagine that your footprint pattern extends infinitely far in either direction.

Imagine a line c positioned midway between your left and right footprints. In your mind, slide the entire pattern one-half step in a direction parallel to c then reflect in line c. The image pattern exactly superimposes on the original pattern. This transformation is an example of a glide reflection with axis c (see Figure 1.5).

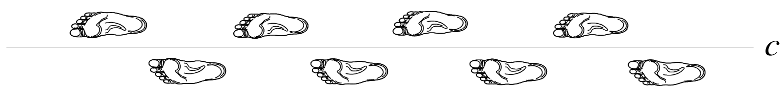


Figure 1.5: Footprints fixed by a glide reflection.

As mentioned above, the following proposition is an immediate consequence of exercise 3 in Section 1:

Proposition 38 *Glide reflections are isometries.*

Alternatively, we can think of a glide reflection with axis c as a reflection in line c followed by a translation parallel to c.

Proposition 39 *If τ is a non-identity translation fixing line c, then $\sigma_c \circ \tau = \tau \circ \sigma_c$.*

Proof. Left as an exercise for the reader. ■

Since a glide reflection is the composition of a reflection in some line c with a non-identity translation whose vector is parallel to c, the equations of a glide reflection are easy to obtain.

Proposition 40 *Let γ be a glide reflection with axis $aX + bY + c = 0$ and glide vector $\begin{bmatrix} d \\ e \end{bmatrix}$, where $ad + be = 0$. Then the equations of γ are given by:*

$$\begin{array}{l} x' = x - \frac{2a}{a^2+b^2}\left(ax + by + c\right) + d \\ y' = y - \frac{2b}{a^2+b^2}\left(ax + by + c\right) + e \end{array}.$$

Proof. Composing the equations for a reflection given in Theorem 22 with those for a translation given in Proposition 30 gives the result. ■

Example 41 *Let m be the line with equation $3X - 4Y + 1 = 0$; let τ be the translation with vector $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$. Since $ad + be = (3)(4) + (-4)(3) = 0$, the vector of τ is parallel to m and $\gamma = \sigma_m \circ \tau$ is a glide reflection whose equations are*

$$\begin{array}{l} x' = \frac{1}{25}\left(7x + 24y + 94\right) \\ y' = \frac{1}{25}\left(24x - 7y + 83\right) \end{array}.$$

Let $P = \begin{bmatrix} 0 \\ 19 \end{bmatrix}$ *and* $P' = \begin{bmatrix} 22 \\ -2 \end{bmatrix} = \gamma\left(\begin{bmatrix} 0 \\ 19 \end{bmatrix}\right)$. *Then* $M = \begin{bmatrix} 11 \\ \frac{17}{2} \end{bmatrix}$, *which is the midpoint of* P *and* P', *lies on* m. *This is no accident, as the next proposition shows.*

Proposition 42 *Let* γ *be a glide reflection with axis* c.

1. γ has no fixed points.

2. The midpoint of point P and its image $\gamma(P)$ lies on c.

3. γ interchanges the halfplanes of c.

4. γ fixes exactly one line, its axis c.

Proof. Let $\gamma = \sigma_c \circ \tau$ be a glide reflection and let P be any point. Then $\tau \neq \iota$ and $\tau(c) = c$. Let $Q = \tau(P)$ and $P' = \sigma_c(Q)$; then $\overleftrightarrow{PQ} \| c$. If P lies on c then $P' = Q$ is a point on c distinct from P, and the midpoint of P and P' lies on c. But if P lies off c, P and P' lie on opposite sides of c and γ interchanges the halfplanes of its axis c. Furthermore, $\overleftrightarrow{PP'}$ intersects c at some point M (see Figure 1.6).

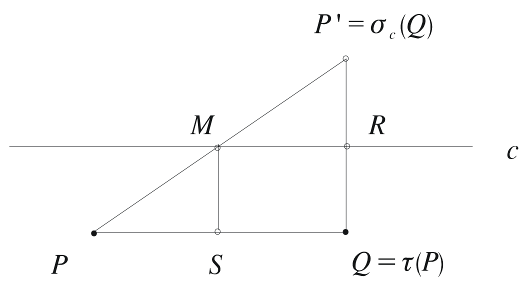
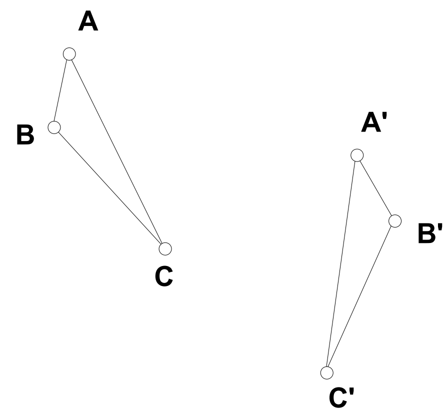


Figure 1.6.

Let S be the foot of the perpendicular from M to $\overleftrightarrow{PQ}$. To show that M is the midpoint of $\overline{PP'}$ it suffices to prove that $\triangle P'RM \cong \triangle MSP$, in which case $PM = MP'$ since $CPCTC$. By definition of a reflection, c is the perpendicular bisector of $\overline{QP'}$; hence $R = c \cap \overline{QP'}$ is the midpoint of $\overline{QP'}$. Then $\overline{MS} \cong \overline{RQ}$ and $\angle MPS \cong \angle P'MR$ since these angles are corresponding. Also $\angle P'RM$ and $\angle MSP$ are right angles so $\triangle P'RM \cong \triangle MSP$ by AAS. Finally, to prove part (d), let ℓ be a line fixed by γ and let P be a point on ℓ. Then $P' = \gamma(P)$ lies on ℓ and $\ell = \overleftrightarrow{PP'}$. Now by part (2), the midpoint M of P and P' lies on c. Similarly, if $M' = \gamma(M)$, then $\ell = \overleftrightarrow{MM'}$ and the midpoint N of M and M' lies on c. Thus $M, N \in \ell \cap c$ and $c = \overleftrightarrow{MN} = \overleftrightarrow{MM'} = \ell$. ■

Exercises

1. A glide reflection γ maps $\triangle ABC$ onto $\triangle A'B'C'$ in the diagram below. Use a MIRA to construct the axis and glide vector of γ.



2. Let c be the line with equation $X - 2Y + 3 = 0$; let $P = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ and $Q = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$.
   a. Prove that $\gamma = \sigma_c \circ \tau_{\mathbf{PQ}}$ is a glide reflection and find the equations of γ.

   b. Find the image of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ -2 \end{bmatrix}$ under γ.

   c. Write the equations for the composite transformation $\tau_{\mathbf{PQ}} \circ \sigma_c$ and compare with the equations found in part a.

3. Prove Proposition 39: Let τ be a non-identity translation that fixes line c. Then $\sigma_c \circ \tau = \tau \circ \sigma_c$.

4. If $\gamma = \sigma_c \circ \tau$ is a glide reflection with axis c, prove that
   a. $\gamma^{-1} = \sigma_c \circ \tau^{-1}$.

   b. γ^{-1} is a glide reflection with axis c.

5. Let γ be a glide reflection with axis c and let τ be a translation that fixes c. Use Exercise 3 to prove that $\tau \circ \gamma = \gamma \circ \tau$.

6. Let γ be a glide reflection with axis c and glide vector $\mathbf{v}$. Given any point M on c, construct a point P such that M is the midpoint of P and $\gamma(P)$.

1.5 Halfturns

In this section we consider halfturns, which have the same effect as 180° rotations of the plane about some fixed point. Halfturns play an important role in our theoretical discussion to follow.

Definition 43 *Let C be a point. A transformation $\varphi_C : \mathbb{R}^2 \to \mathbb{R}^2$ is a halfturn about center C if and only if each point P and its image $P' = \varphi_C(P)$ are related in one of the following ways:*

1. *If $P = C$ then $P = P'$.*

2. *If $P \neq C$ then C is the midpoint of $\overline{PP'}$.*

Condition (1) implies that a halfturn fixes its center of rotation.

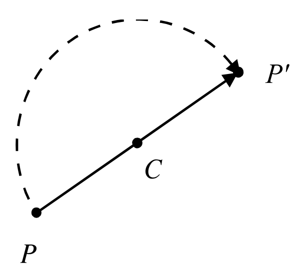


Figure 1.7: A halfturn about C.

Remark 44 *Some authors refer to the halfturn about point C as "the reflection through point C." In this course we use the "halfturn" terminology exclusively.*

Theorem 45 *Halfturns are isometries.*

Proof. Let φ_C be a halfturn about a point C, let P and Q be distinct points, let $P' = \varphi_C(P)$ and let $Q' = \varphi_C(Q)$. Suppose that either $P = C$ or $Q = C$. If $Q = C$, then $Q' = Q$ by condition (1) in the definition. But $PC = P'C$ by condition (2) hence $PQ = PC = P'C = P'Q'$, as desired, and similarly for $P = C$. On the other hand, suppose that C is distinct from both P and Q. By condition (2), $PC = P'C$ and $QC = Q'C$. Furthermore, $\angle PCQ \cong \angle P'CQ'$, since vertical angles are congruent, and it follows by SAS that $\triangle PCQ \cong \triangle P'CQ'$ (see Figure 1.8). Therefore $PQ = P'Q'$ since $CPCTC$. ■

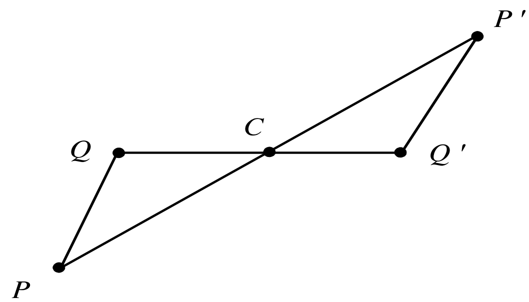


Figure 1.8.

We now derive the equations of a halfturn about the point $C = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$. Let $P = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ and $P' = \left[\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right]$, where $P' = \varphi_C(P)$. If $P \neq C$, then by definition C is the midpoint P and P' so that

$$a = \frac{x + x'}{2} \text{ and } b = \frac{y + y'}{2}.$$

These equations simplify to

$$\begin{array}{c} x' = 2a - x \\ y' = 2b - y \end{array} . \tag{1.8}$$

On the other hand, evaluating the equations in (1.8) at the point $C = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ gives

$$\begin{array}{c} x' = a \\ y' = b \end{array} .$$

This says that the center of rotation C is fixed by the transformation whose equations are given in (1.8), which proves:

Theorem 46 *Let $C = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ be a point in the plane. The equations of the halfturn φ_C are given by*

$$\begin{array}{c} x' = 2a - x \\ y' = 2b - y \end{array} .$$

Example 47 *Let O denote the origin; the equations for the halfturn about the origin φ_O are*

$$\begin{array}{c} x' = -x \\ y' = -y \end{array} .$$

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is odd if and only if $f(-x) = -f(x)$. An example of such a function is $f(x) = \sin(x)$. Let f be odd and consider a point

$P = \left[\begin{smallmatrix} x \\ f(x) \end{smallmatrix}\right]$ on the graph of f. The image of P under the halfturn φ_O is

$$\varphi_O\left(\begin{bmatrix} x \\ f(x) \end{bmatrix}\right) = \begin{bmatrix} -x \\ -f(x) \end{bmatrix} = \begin{bmatrix} -x \\ f(-x) \end{bmatrix},$$

which is also a point on the graph of f. Thus φ_O fixes the graph of f.

Exercises

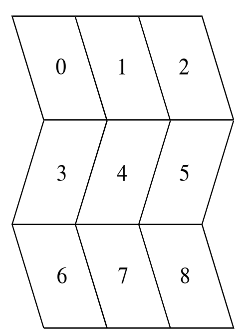
1. People in distress on a deserted island sometimes write `SOS` in the sand.

   a. Why is this signal particularly effective when viewed from searching aircraft?

   b. The word `SWIMS`, like `SOS`, reads the same after a halfturn about its centroid. Find at least three other words that are preserved under a halfturn about their centroids.

2. Try it! Plot the graph of $y = \sin(x)$ on graph paper, pierce the graph paper at the origin with your compass point and push the compass point into your writing surface. This provides a point around which you can rotate your graph paper. Now physically rotate your graph paper 180° and observe that the graph of $y = \sin(x)$ is fixed by φ_O in the sense defined above.

3. The parallelogram "0" in the figure below is mapped to each of the other eight parallelograms by a reflection, a translation, a glide reflection or a halfturn. Indicate which of these apply in each case (more than one may apply in some cases).



4. Find the coordinates for the center of the halfturn whose equations are $x' = -x + 3$ and $y' = -y - 8$.

5. Let $P = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$; let ℓ be the line with equation $5X - Y + 7 = 0$.
   a. Find the equations of φ_P.

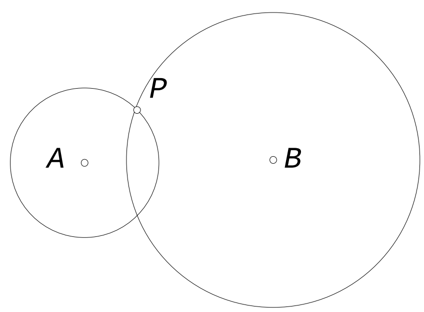
   b. Find the image of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and under φ_P.

   c. Find the equation of the line $\ell' = \varphi_P(\ell)$. On graph paper, plot point P and draw lines ℓ and ℓ'.

6. Repeat Exercise 5 with $P = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

7. $P = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $Q = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$. Find the equations of the composition $\varphi_Q \circ \varphi_P$ and inspect them carefully. These are the equations of an isometry we discussed earlier in the course. Can you identify which?

8. In the diagram below, use a MIRA to find a line through P intersecting circles A_P and B_P in chords of equal length.



9. Let $P = \begin{bmatrix} a \\ b \end{bmatrix}$ and $Q = \begin{bmatrix} c \\ d \end{bmatrix}$ be distinct points. Find the equations of $\varphi_Q \circ \varphi_P$ and prove that the composition $\varphi_Q \circ \varphi_P$ is a translation τ. Find the vector of τ.

10. For any point P, prove that $\varphi_P^{-1} = \varphi_P$.

11. Let ℓ and m be the lines with respective equations $X + Y - 2 = 0$ and $X - Y + 8 = 0$.

    a. Compose the equations of σ_m and σ_ℓ and show that the composition $\sigma_m \circ \sigma_\ell$ is a halfturn φ_C.

    b. Find the center C of this halfturn and the coordinates of the point $P = \ell \cap m$. What do you observe?

12. Prove that if $\varphi_A \circ \varphi_B = \varphi_B \circ \varphi_A$, then $A = B$.

1.6 Properties of Translations and Halfturns

In this section we study some important properties of translations and halfturns and some fascinating relationships between them.

Definition 48 *A collineation α is a* <u>dilatation</u> *if and only if $\alpha(\ell)\|\ell$ for every line ℓ.*

Theorem 49 *Translations are dilatations.*

Proof. Let τ be a translation and let ℓ be a line. Let A and B be distinct points on line ℓ; let $A' = \tau(A)$ and $B' = \tau(B)$. Then $\tau(\ell) = \overleftrightarrow{A'B'}$ since τ is a collineation by Proposition 18 and $\tau = \tau_{\mathbf{AA'}} = \tau_{\mathbf{BB'}}$ by Corollary 29. If A, B, A' and B' are collinear, then $\overleftrightarrow{AB} = \overleftrightarrow{A'B'}$. If A, B, A' and B' are non-collinear, then B lies off $\overleftrightarrow{AA'}$ and $\square AA'B'B$ is a parallelogram by definition of translation, and it follows that $\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$ (see Figure 1.9).

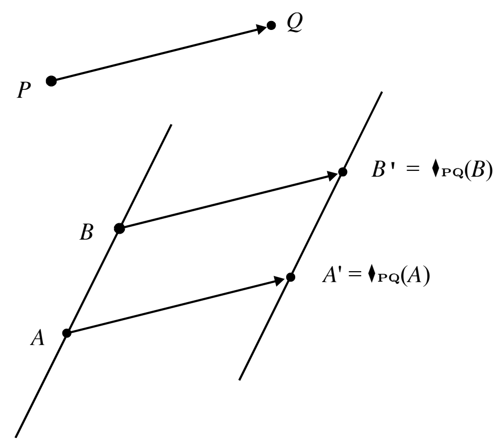


Figure 1.9.

■

Theorem 50 *A halfturn is a dilatation.*

Proof. Left as an exercise for the reader. ■

Theorem 51 *If P and Q are distinct points, then $\tau_{\mathbf{PQ}}$ fixes every line parallel to $\overleftrightarrow{PQ}$.*

Proof. Let ℓ be any line parallel to $\overleftrightarrow{PQ}$; let A be any point on line ℓ and let $A' = \tau_{\mathbf{PQ}}(A)$. If $\ell \neq \overleftrightarrow{PQ}$, then $\square PQA'A$ is a parallelogram by definition of translation and $\overleftrightarrow{PQ}\|\overleftrightarrow{AA'}$. Therefore A' lies on ℓ and ℓ is fixed by $\tau_{\mathbf{PQ}}$. The case with $\ell = \overleftrightarrow{PQ}$ is left as an exercise for the reader. ■

Definition 52 *A non-identity transformation α is an* <u>involution</u> *if and only if $\alpha^2 = \iota$.*

Note that an involution α has the property that $\alpha^{-1} = \alpha$.

Proposition 53 *A halfturn is an involution.*

Proof. Let C be a point and consider the halfturn φ_C. Then $\varphi_C^2(C) = \varphi_C(\varphi_C(C)) = \varphi_C(C) = C$. If $P \neq C$, let $P' = \varphi_C(P)$; then C is the midpoint of P and P'. Therefore $\varphi_C^2(P) = \varphi_C(\varphi_C(P)) = \varphi_C(P') = P$ so that $\varphi_C^2 = \iota$ as claimed. ■

Proposition 54 *A reflection is an involution.*

Proof. Left as an exercise for the reader. ■

Proposition 55 *A line ℓ is fixed by the halfturn φ_C if and only if C lies on ℓ.*

Proof. Let φ_C be a halfturn and let ℓ be a line. If C lies on ℓ, consider a point P on ℓ distinct from C. Let $P' = \varphi_C(P)$; by definition, C is the midpoint of $\overline{PP'}$. Hence P' is on ℓ and ℓ is fixed by $\varphi_{C.}$ Conversely, suppose that C lies off ℓ and consider any point P on ℓ. Then $P' = \varphi_C(P)$ lies off ℓ since otherwise the midpoint of $\overline{PP'}$, which is C, would lie on ℓ. Therefore ℓ is not fixed by φ_C.

■

The next theorem highlights a close relationship between halfturns and translations.

Theorem 56 *The composition of two halfturns with distinct centers is a translation. Furthermore, if B is the midpoint of A and C, then*

$$\varphi_B \circ \varphi_A = \tau_{\mathbf{AC}} = \varphi_C \circ \varphi_B.$$

Proof. Let $A = \left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right]$, and $C = \left[\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right]$. Since B is the midpoint of A and C, we have $AC = \mathbf{2}AB = \mathbf{2}BC$ so that

$$\mathbf{AC} = \begin{bmatrix} 2(b_1 - a_1) \\ 2(b_2 - a_2) \end{bmatrix} = \begin{bmatrix} 2(c_1 - b_1) \\ 2(c_2 - b_2) \end{bmatrix}$$

and the equations for $\tau_{\mathbf{AC}}$ can be written as

$$\begin{array}{l} x' = x + 2(b_1 - a_1) = x + 2(c_1 - b_1) \\ y' = y + 2(b_2 - a_2) = y + 2(c_2 - b_2) \end{array}.$$

Now

$$\begin{aligned}(\varphi_B \circ \varphi_A)\left(\begin{bmatrix}x\\y\end{bmatrix}\right) &= \varphi_B\left(\varphi_A\left(\begin{bmatrix}x\\y\end{bmatrix}\right)\right) = \varphi_B\left(\begin{bmatrix}2a_1 - x\\2a_2 - y\end{bmatrix}\right) \\ &= \begin{bmatrix}2b_1 - (2a_1 - x)\\2b_2 - (2a_2 - y)\end{bmatrix} = \begin{bmatrix}x + 2(b_1 - a_1)\\y + 2(b_2 - a_2)\end{bmatrix} \\ &= \tau_{\mathbf{AC}}\left(\begin{bmatrix}x\\y\end{bmatrix}\right).\end{aligned} \tag{1.9}$$

A similar calculation gives

$$(\varphi_C \circ \varphi_B)\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x + 2(c_1 - b_1)\\y + 2(c_2 - b_2)\end{bmatrix} = \tau_{\mathbf{AC}}\left(\begin{bmatrix}x\\y\end{bmatrix}\right),$$

completing the proof (see Figure 1.10). ■

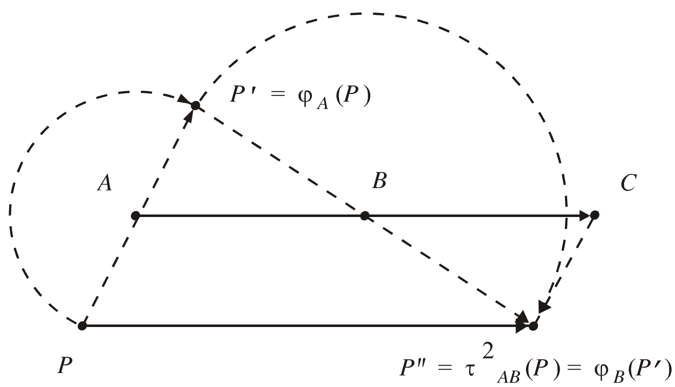


Figure 1.10: A composition of two halfturns

Remark 57 *Since $AC= 2AB$, the calculations in (1.9) tell us that the composition of two halfturns $\varphi_B \circ \varphi_A$ with $A \neq B$ is a translation through twice the directed distance from A to B.*

Theorem 58 *The composition of three halfturns with distinct centers A, B and C is a halfturn. Furthermore, if the point D satisfies $\mathbf{AB} = \mathbf{DC}$, then*

$$\varphi_C \circ \varphi_B \circ \varphi_A = \varphi_D. \tag{1.10}$$

In particular, if A, B and C are non-collinear, then $\square ABCD$ is a parallelogram.

Proof. Given $A = \begin{bmatrix}a_1\\a_2\end{bmatrix}$, $B = \begin{bmatrix}b_1\\b_2\end{bmatrix}$ and $C = \begin{bmatrix}c_1\\c_2\end{bmatrix}$, note that

$$\mathbf{AB} = \begin{bmatrix}b_1 - a_1\\b_2 - a_2\end{bmatrix} = \begin{bmatrix}c_1 - (a_1 - b_1 + c_1)\\c_2 - (a_2 - b_2 + c_2)\end{bmatrix}$$

and the point $D = \left[\begin{smallmatrix} a_1-b_1+c_1 \\ a_2-b_2+c_2 \end{smallmatrix}\right]$ satisfies $\mathbf{AB} = \mathbf{DC}$. Thus if A, B and C are non-collinear, then $\square ABCD$ is a parallelogram. We must show that equation (1.10) holds for $D = \left[\begin{smallmatrix} a_1-b_1+c_1 \\ a_2-b_2+c_2 \end{smallmatrix}\right]$. From the calculation in (1.9) above we have

$$(\varphi_B \circ \varphi_A)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2(b_1 - a_1) \\ y + 2(b_2 - a_2) \end{bmatrix}.$$

Applying φ_C to both sides gives

$$\begin{aligned} (\varphi_C \circ \varphi_B \circ \varphi_A)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \varphi_C\left(\begin{bmatrix} x + 2(b_1 - a_1) \\ y + 2(b_2 - a_2) \end{bmatrix}\right) = \begin{bmatrix} 2c_1 - (x + 2(b_1 - a_1)) \\ 2c_2 - (y + 2(b_2 - a_2)) \end{bmatrix} \\ &= \begin{bmatrix} 2(a_1 - b_1 + c_1) - x \\ 2(a_2 - b_2 + c_2) - y \end{bmatrix}, \end{aligned}$$

which is a halfturn about the point $D = \left[\begin{smallmatrix} a_1-b_1+c_1 \\ a_2-b_2+c_2 \end{smallmatrix}\right]$see Figure 1.11). ■

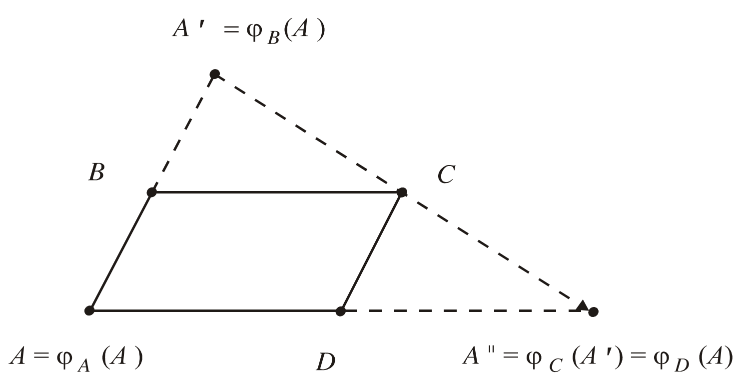


Figure 1.11.

Note that Figure 1.11 shows us how to construct the center D of the halfturn $\varphi_D = \varphi_C \circ \varphi_B \circ \varphi_A$.

Theorem 59 *Given any three of the (not necessarily distinct) points A, B, C and D, the fourth point is uniquely determined by the equation*

$$\tau_{\mathbf{AB}} = \varphi_D \circ \varphi_C. \tag{1.11}$$

Proof. First, given equation (1.11) we have

$$B = (\varphi_D \circ \varphi_C)(A). \tag{1.12}$$

Thus B is determined by A, C and D. Apply $\varphi_C \circ \varphi_D$ to both sides of (1.12) and obtain

$$(\varphi_C \circ \varphi_D)(B) = (\varphi_C \circ \varphi_D \circ \varphi_D \circ \varphi_C)(A) = A.$$

Thus A is determined by B, C and D. Next, compose both sides of equation (1.11) with φ_C and obtain

$$\tau_{\mathbf{AB}} \circ \varphi_C = \varphi_D \circ \varphi_C \circ \varphi_C = \varphi_D.$$

Thus given A, B and C, the fourth point D is the midpoint of C and C', where

$$C' = \varphi_D(C) = (\tau_{\mathbf{AB}} \circ \varphi_C)(C) = \tau_{\mathbf{AB}}(C).$$

Finally apply φ_D to both sides of equation (1.11) and obtain

$$\varphi_D \circ \tau_{\mathbf{AB}} = \varphi_D \circ \varphi_D \circ \varphi_C = \varphi_C.$$

Thus given A, B and D, the fourth point C is the midpoint of A and A', where

$$A' = \varphi_C(A) = (\varphi_D \circ \tau_{\mathbf{AB}})(A) = \varphi_D(B).$$

■

Theorem 60 *For any three points P, Q and R,*

$$\varphi_R \circ \varphi_Q \circ \varphi_P = \varphi_P \circ \varphi_Q \circ \varphi_R.$$

Proof. By Theorem 59, given P, Q, R, there is a point S such that

$$\varphi_R \circ \varphi_Q \circ \varphi_P = \varphi_S.$$

Sequentially composing φ_R, φ_Q, and φ_P with both sides on the left gives

$$\varphi_Q \circ \varphi_P = \varphi_R \circ \varphi_S$$

$$\varphi_P = \varphi_Q \circ \varphi_R \circ \varphi_S$$

$$\iota = \varphi_P \circ \varphi_Q \circ \varphi_R \circ \varphi_S.$$

Composing both sides with φ_S on the right gives

$$\varphi_S = \varphi_P \circ \varphi_Q \circ \varphi_R.$$

Therefore $\varphi_R \circ \varphi_Q \circ \varphi_P = \varphi_S = \varphi_P \circ \varphi_Q \circ \varphi_R$. ■

Remark 61 *Halfturns do not commute in general. In fact, if $\varphi_P \circ \varphi_Q = \varphi_Q \circ \varphi_P$ then $\tau_{\mathbf{QP}} = \tau_{\mathbf{PQ}}$, which holds if and only if $P = Q$. Thus the only halfturn that commutes with φ_P is itself.*

Exercises

1. Find all values for a and b such that $\alpha\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} ay \\ x/b \end{bmatrix}$ is an involution.

2. Complete the proof of Theorem 51: Let P and Q be distinct points; let A be a point on $\overleftrightarrow{PQ}$. Then $A' = \tau_{\mathbf{PQ}}(A)$ is a point on $\overleftrightarrow{PQ}$.

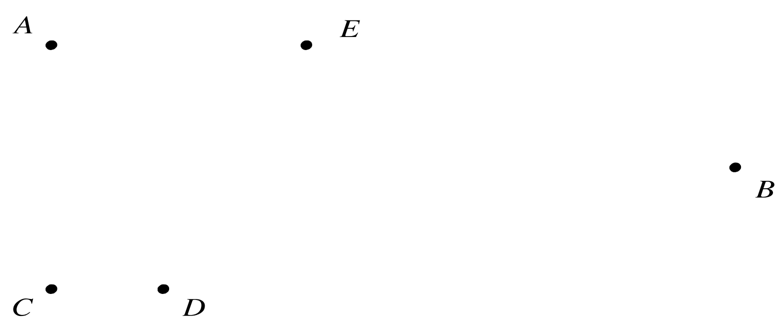
3. Let A, B and P be distinct points. Prove that $\tau_{\mathbf{AB}} \circ \varphi_P \circ \tau_{\mathbf{BA}} = \varphi_{\tau_{\mathbf{AB}}(P)}$.

4. In the figure below, sketch points X, Y, Z such that

   a. $\varphi_A \circ \varphi_E \circ \varphi_D = \varphi_X$

   b. $\varphi_D \circ \tau_{\mathbf{AC}} = \varphi_Y$

   c. $\tau_{\mathbf{BC}} \circ \tau_{\mathbf{AB}} \circ \tau_{\mathbf{EA}}(A) = Z$.



5. Let τ be a translation; let C be a point. Let D be the midpoint of C and $\tau(C)$.

   a. Show that $\tau \circ \varphi_C$ is the halfturn with center D.

   b. Show that $\varphi_C \circ \tau$ is the halfturn with center $E = \tau^{-1}(D)$.

6. Prove Theorem 50: A halfturn is a dilatation.

7. Prove Proposition 54: A reflection is an involution.

8. If coin A in the figure below is rolled around coin B until coin A is directly under coin B, will the head on coin A right-side up or up-side down?

1.7 General Rotations

Halfturns are 180° rotations. In this section we define general rotations, derive their equations, and prove some fundamental properties. We begin with a discussion of angle measure. Assume that the sine and cosine functions are defined with respect to degree measure.

Definition 62 *Given $\Phi, \Theta \in \mathbb{R}$, let $A = \begin{bmatrix} \cos\Phi \\ \sin\Phi \end{bmatrix}$ and $B = \begin{bmatrix} \cos\Theta \\ \sin\Theta \end{bmatrix}$. The directed angle from $\overrightarrow{OA}$ to $\overrightarrow{OB}$ is the angle with initial side $\overrightarrow{OA}$, terminal side $\overrightarrow{OB}$ and measure $\Theta - \Phi$. A directed angle is positively directed if its measure is positive; it is negatively directed if its measure is negative.*

Positively directed angles are measured counterclockwise; negatively directed angles are measured clockwise.

Definition 63 *Two real numbers Φ and Θ are congruent $\bmod 360$ if and only if $\Phi - \Theta = 360k$ for some $k \in \mathbb{Z}$. The set $\Theta^\circ = \{\Theta + 360k \mid k \in \mathbb{Z}\}$ is called the congruence class of Θ and we define $-(\Theta^\circ) = (-\Theta)^\circ$ and $\Theta^\circ + \Phi^\circ = (\Theta + \Phi)^\circ$.*

Each congruence class contains exactly one real number Θ in the range $0 \leq \Theta < 360$. For example, $10 \in 370^\circ$ and $60 \in -300^\circ$. This allows us to compare congruence classes.

Definition 64 *Given congruence classes Θ° and Φ°, say that $\Theta^\circ < \Phi^\circ$ if and only if there exist representatives $\Theta \in \Theta^\circ$ and $\Phi \in \Phi^\circ$ such that $0 \leq \Theta < \Phi < 360$.*

Consequently, $370^\circ < -300^\circ$ since 10 and 60 are the unique class representatives in the range $0 \leq \Theta < 360$.

Definition 65 *Let A, B and C be distinct points. The (undirected) angle from $\overrightarrow{CA}$ to $\overrightarrow{CB}$, denoted by $\angle ACB$, is the angle with initial side $\overrightarrow{CA}$ and terminal side $\overrightarrow{CB}$. The measure of $\angle ACB$, denoted by $m\angle ACB$, is the congruence class* *defined as follows: Let* $A' = \tau_{\mathbf{CO}}(A)$ *and* $B' = \tau_{\mathbf{CO}}(B)$; *choose* Θ *and* Φ *so that* $\left[\begin{smallmatrix}\cos\Theta\\ \sin\Theta\end{smallmatrix}\right]$ *lies on* $\overrightarrow{OA'}$ *and* $\left[\begin{smallmatrix}\cos\Phi\\ \sin\Phi\end{smallmatrix}\right]$ *lies on* $\overrightarrow{OB'}$. *Define* $m\angle ACB = (\Phi - \Theta)^\circ$. *(see Figure 1.12).*

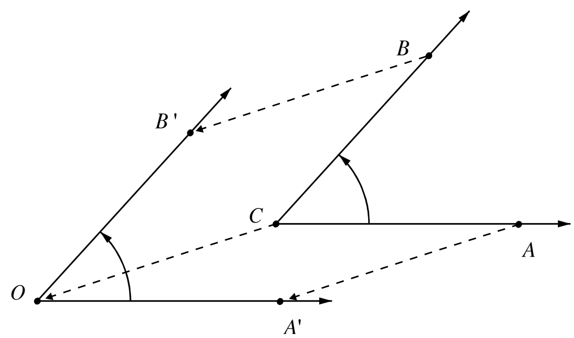


Figure 1.12: $m\angle ACB = m\angle A'OB'$

Given distinct points A, B and C, there is a unique real number $\Theta \in [0, 360)$ such that $m\angle ACB = \Theta^\circ$. Note that a congruence class of angles has undefined "direction" since it contains both positive and negative directed angle measures. Nevertheless, one must be pay careful attention to the order of A, B and C since $m\angle BCA = -m\angle ACB$.

Definition 66 *Let* C *be a point and let* $\Theta \in \mathbb{R}$. *The rotation about* C *through angle* Θ *is the transformation* $\rho_{C,\Theta} : \mathbb{R}^2 \to \mathbb{R}^2$ *with the following properties:*

1. $\rho_{C,\Theta}(C) = C$.
2. *If* $P \neq C$ *and* $P' = \rho_{C,\Theta}(P)$, *then* $CP' = CP$ *and* $m\angle PCP' = \Theta^\circ$ *(see Figure 1.13).*

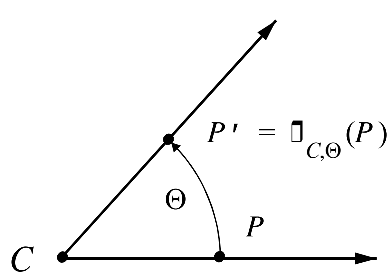


Figure 1.13.

Theorem 67 *A rotation is an isometry.*

Proof. Let $\rho_{C,\Theta}$ be a rotation. Let C, P and Q be points with P and Q distinct and let $P' = \rho_{C,\Theta}(P)$ and $Q' = \rho_{C,\Theta}(Q)$. If $P = C$, then by definition, $PQ = CQ = CQ' = P'Q'$. So assume that C, P and Q are all distinct. If C, P, and Q are non-collinear, then $\triangle PCQ \cong \triangle P'CQ'$ by SAS, and $PQ = P'Q'$ since $CPCTC$. If C, P, and Q are collinear with P between C and Q, then $PQ = CQ - CP = CQ' - CP' = P'Q'$, since $CP = CP'$ and $CQ = CQ'$ by definition, and similarly for Q between C and P. But if C is between P and Q, then C is between P' and Q'since $m\angle PCP' = m\angle QCQ' = \Theta°$. Therefore $PQ = CP + CQ = CP' + CQ' = P'Q'$. ■

We begin our derivation of the equations of a general rotation by considering the special case of rotations about the origin O. From linear algebra we know that a rotation about the origin is a linear transformation, i.e., given vectors $\left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$,

$$\rho_{O,\Theta}\left(a\begin{bmatrix} s \\ t \end{bmatrix} + b\begin{bmatrix} u \\ v \end{bmatrix}\right) = a\rho_{O,\Theta}\left(\begin{bmatrix} s \\ t \end{bmatrix}\right) + b\rho_{O,\Theta}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right)$$

for all $a, b \in \mathbb{R}$. Let $E_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $E_2 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$; let $E_1' = \rho_{O,\Theta}(E_1)$ and $E_2' = \rho_{O,\Theta}(E_2)$. Then

$$\begin{aligned} \rho_{O,\Theta}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \rho_{O,\Theta}\left(x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ &= x\rho_{O,\Theta}(E_1) + y\rho_{O,\Theta}(E_2) \\ &= xE_1' + yE_2' \end{aligned}$$

and we see that $\rho_{O,\Theta}$ is completely determined by its action on E_1 and E_2. Since the directed angle measure from $\overrightarrow{OE_1}$ to $\overrightarrow{OE_1'}$ is Θ,

$$E_1' = \begin{bmatrix} \cos\Theta \\ \sin\Theta \end{bmatrix}.$$

Furthermore, the directed angle measure from $\overrightarrow{OE_1}$ to $\overrightarrow{OE_2}$ is 90 and from $\overrightarrow{OE_2}$ to $\overrightarrow{OE_2'}$ is Θ, so the directed angle measure from $\overrightarrow{OE_1}$ to $\overrightarrow{OE_2'}$ is $\Theta + 90$ and

$$E_2' = \rho_{O,\Theta+90}(E_1) = \begin{bmatrix} \cos(\Theta + 90) \\ \sin(\Theta + 90) \end{bmatrix} = \begin{bmatrix} -\sin\Theta \\ \cos\Theta \end{bmatrix}.$$

Consequently,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = xE_1' + yE_2' = \begin{bmatrix} x\cos\Theta \\ x\sin\Theta \end{bmatrix} + \begin{bmatrix} -y\sin\Theta \\ y\cos\Theta \end{bmatrix} = \begin{bmatrix} x\cos\Theta - y\sin\Theta \\ x\sin\Theta + y\cos\Theta \end{bmatrix}.$$

We have proved:

Theorem 68 *Let $\Theta \in \mathbb{R}$. The equations for $\rho_{O,\Theta}$ are*

$$\begin{array}{l} x' = x\cos\Theta - y\sin\Theta \\ y' = x\sin\Theta + y\cos\Theta \end{array}.$$

Now rotate the plane about a general point $C = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ through directed angle Θ by performing the following three operations in sequence:

1. Translate by vector $\mathbf{CO}$;
2. Rotate about the origin O through directed angle Θ;
3. Translate by vector $\mathbf{OC}$.

Then

$$\rho_{c,\Theta} = \tau_{\mathbf{OC}} \circ \rho_{O,\Theta} \circ \tau_{\mathbf{CO}}$$

and we obtain

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \left(\tau_{\mathbf{OC}} \circ \rho_{O,\Theta} \circ \tau_{\mathbf{CO}}\right)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \tau_{\mathbf{OC}}\left(\rho_{O,\Theta}\left(\begin{bmatrix} x-a \\ y-b \end{bmatrix}\right)\right) \\ &= \tau_{\mathbf{OC}}\left(\begin{bmatrix} (x-a)\cos\Theta - (y-b)\sin\Theta \\ (x-a)\sin\Theta + (y-b)\cos\Theta \end{bmatrix}\right) \\ &= \begin{bmatrix} (x-a)\cos\Theta - (y-b)\sin\Theta + a \\ (x-a)\sin\Theta + (y-b)\cos\Theta + b \end{bmatrix}. \end{aligned}$$

In summary, we have proved:

Theorem 69 *Let $C = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ and let $\Theta \in \mathbb{R}$. The equations for the rotation $\rho_{c,\Theta}$ are*

$$\begin{array}{l} x' = (x-a)\cos\Theta - (y-b)\sin\Theta + a \\ y' = (x-a)\sin\Theta + (y-b)\cos\Theta + b \end{array}.$$

Proposition 70 *A non-identity rotation $\rho_{C,\Theta}$ fixes exactly one point, namely its center C, and fixes every circle with center C.*

Proof. That $\rho_{C,\Theta}$ fixes C follows from the definition. From the equations of a rotation it is evident that $\rho_{C,\Theta} = \iota$ (the identity) if and only if $\Theta = 360k$ for some $k \in \mathbb{Z}$. So $\Theta^\circ \neq 0^\circ$ by assumption. Now if P is distinct from C and $P' = \rho_{C,\Theta}(P)$, then $P \neq P'$ since $m\angle PCP' = \Theta^\circ$. Therefore C is the only point fixed by $\rho_{C,\Theta}$. Furthermore, let Q be a point on C_P and let $Q' = \rho_{C,\Theta}(Q)$. Then by definition $CP = CQ = CQ'$ so that Q' lies on C_P. Therefore $\rho_{C,\Theta}(C_P) = C_P$.

■

Proposition 71 *Let C be a point and let $\Theta, \Phi \in \mathbb{R}$. Then*

$$\rho_{C,\Phi} \circ \rho_{C,\Theta} = \rho_{C,\Theta+\Phi} = \rho_{C,\Theta} \circ \rho_{C,\Phi}.$$

Proof. Let P be any point distinct from C; let $P' = \rho_{C,\Theta}(P)$ and $P'' = \rho_{C,\Phi}(P')$ (see Figure 1.14).

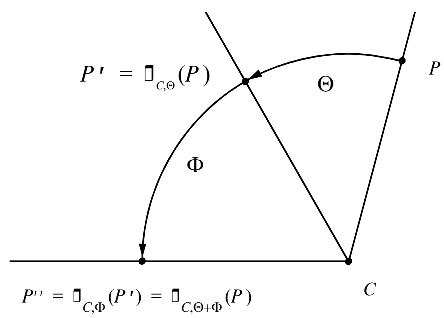


Figure 1.14.

Then by definition, the directed angle measure from $\overrightarrow{CP}$ to $\overrightarrow{CP'}$ is Θ, the directed angle measure from $\overrightarrow{CP'}$ to $\overrightarrow{CP''}$ is Φ, and $CP = CP' = CP''$. But

$$P'' = \rho_{C,\Phi}(P') = \rho_{C,\Phi}(\rho_{C,\Theta}(P)) = (\rho_{C,\Phi} \circ \rho_{C,\Theta})(P)$$

and the directed angle measure from $\overrightarrow{CP}$ to $\overrightarrow{CP''}$ is $\Theta + \Phi$. Therefore $\rho_{C,\Phi} \circ \rho_{C,\Theta} = \rho_{C,\Theta+\Phi}$, by definition. Similarly, $\rho_{C,\Theta} \circ \rho_{C,\Phi} = \rho_{C,\Theta+\Phi}$. ■

Corollary 72 *Involutory rotations are halfturns.*

Proof. An involutory rotation $\rho_{C,\Theta}$ is a non-identity rotation such that $\rho_{C,\Theta} \circ \rho_{C,\Theta} = \iota = \rho_{C,0}$. By Proposition 71, $\rho_{C,\Theta} \circ \rho_{C,\Theta} = \rho_{C,2\Theta}$ so that $\rho_{C,2\Theta} = \rho_{C,0}$ and $(2\Theta)^\circ = 0^\circ$. Hence there exists some $k \in \mathbb{Z}$, such that $2\Theta = 360k$ and consequently $\Theta = 180k$. Now k cannot be even since $\rho_{C,\Theta} \neq \iota$ implies that Θ is not a multiple of 360. Therefore $k = 2n + 1$ is odd and $\Theta = 180 + 360n$. It follows that $\Theta^\circ = 180^\circ$ and $\rho_{C,\Theta} = \varphi_C$ as claimed. ■

Exercises

1. Find the coordinates of the points $\rho_{O,30}\left(\begin{bmatrix}3\\6\end{bmatrix}\right)$.

2. Let $Q = \begin{bmatrix}-3\\5\end{bmatrix}$. Find the coordinates of the point $\rho_{Q,45}\left(\begin{bmatrix}3\\6\end{bmatrix}\right)$.

3. Let ℓ be the line with equation $2X + 3Y + 4 = 0$.

   a. Find the equation of the line $\rho_{O,30}(\ell)$.

   b. Let $Q = \begin{bmatrix}-3\\5\end{bmatrix}$. Find the equation of the line $\rho_{Q,45}(\ell)$.

4. Let C be a point and let $\Theta \in \mathbb{R}$. Prove that $\rho_{C,\Theta}^{-1} = \rho_{C,-\Theta}$.

5. Let ℓ and m be the lines with respective equations $X + Y - 2 = 0$ and $Y = 3$.

   a. Compose the equations of σ_m and σ_ℓ and show that the composition $\sigma_m \circ \sigma_\ell$ is a rotation $\rho_{C,\Theta}$.

   b. Find the center and directed angle of rotation C and Θ; compare Θ with the directed angle from ℓ to m.

6. Let P and R be distinct points and let Q be the midpoint of P and R. Let c be a line such that $\tau_{\mathbf{PR}}(c) = c$. Using the fact that $\tau_{\mathbf{PR}} = \varphi_Q \circ \varphi_P$, prove that $\sigma_c \circ \tau_{\mathbf{PR}} = \tau_{\mathbf{PR}} \circ \sigma_c$.