The Angle Sum of a Triangle
Math 353

In a nutshell, this course is about the angle sum of a triangle. The fundamental result in this course is that the angle sum of a triangle depends on the choice of Parallel Postulate and, given such a choice, is *always less than, equal to, or greater than* $180°$.

On the other hand, if we only assume Euclid's first four postulates, along with the axioms of incidence, congruence, continuity and betweenness, the angle sum of a triangle is always *less than or equal to* $180°$. This geometry is called *neutral (or absolute) geometry.* (For a complete list of incidence, congruence, continuity and betweenness axioms, see the text by Marvin Greenberg, "Euclidean and Non-Euclidean Geometries: Development and History," 3rd Ed., W.H. Freeman and Co., New York, 1993.)

These notes begin with results from neutral geometry and follow with results from Euclidean and non-Euclidean geometries. Many of the proofs follow those in Greenberg's text, and some are my own. The material has been organized so that results on angle sums are obtained as efficiently as possible. Let us begin.

The Angle Sum of a Triangle in Neutral Geometry.

In this section we will prove the Saccheri-Legendre Theorem: *In neutral geometry, the angle sum of a triangle is less than or equal to* $180°$. We will also consider some important consequences of this theorem. Throughout this and the next two sections we assume Euclid's first four postulates and the axioms of incidence, congruence, continuity and betweenness.

Theorem 1 *(Exterior Angle Inequality) The measure of an exterior angle of a triangle is greater than the measure of either remote interior angle.*

Proof. Given $\triangle ABC$, extend side $\overline{BC}$ to ray $\overrightarrow{BC}$ and choose a point D on this ray so that $B-C-D$. I claim that $m\angle ACD > m\angle A$ and $m\angle ACD > m\angle B$. Let M be the midpoint of $\overline{AC}$ and extend the median $\overline{BM}$ so that M is the midpoint of $\overline{BE}$.

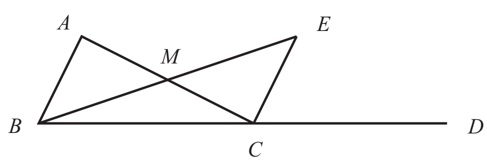

Figure 1.

Then $\angle AMB$ and $\angle CME$ are congruent vertical angles and $\triangle AMB \cong \triangle CME$ by SAS. Consequently, $m\angle ACE = m\angle CAB$ by CPCTC. Now, E lies in the half-plane of A and $\overleftrightarrow{CD}$ since A and E are on the same side of $\overleftrightarrow{CD}$. Also, E lies in the half-plane of D and $\overleftrightarrow{AC}$ since D and E are on the same side of $\overleftrightarrow{AC}$. Therefore E lies in the interior of $\angle ACD$, which is the intersection of these two half-planes. Finally, $m\angle ACD = m\angle ACE + m\angle ECD > m\angle ACE = m\angle CAB = m\angle A$. The proof that $m\angle ACD > m\angle B$ is similar and left to the reader. ■

Corollary 2 *The sum of the measures of any two interior angles of a triangle is less than* $180°$.

Proof. *Given* $\triangle ABC$*, extend side* $\overline{BC}$ *to* $\overleftrightarrow{BC}$ *and choose points* E *and* D *on* $\overleftrightarrow{BC}$ *so that* $E-B-C-D$ *(see Figure 2).*

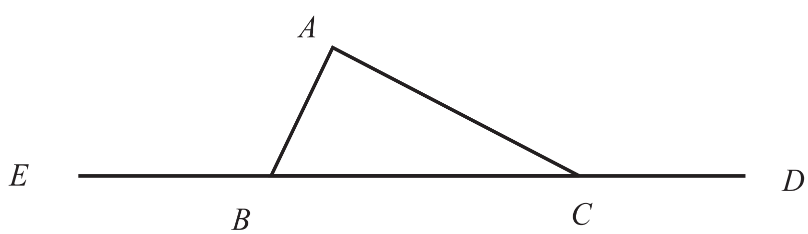

Figure 2.

By Theorem 1, $m\angle A < m\angle ACD$, $m\angle B < m\angle ACD$, and $m\angle A < m\angle ABE$. By adding $m\angle C = m\angle ACB$ to both sides of the first two inequalities, and by adding $m\angle B = m\angle ABC$ to both sides of the third we obtain

$$\begin{aligned} m\angle A + m\angle C &< m\angle ACD + m\angle ACB = 180^\circ \\ m\angle B + m\angle C &< m\angle ACD + m\angle ACB = 180^\circ \\ m\angle A + m\angle B &< m\angle ABE + m\angle ABC = 180^\circ. \end{aligned}$$

■

Theorem 3 *If two lines are cut by a transversal and a pair of alternate interior angles are congruent, the lines are parallel.*

Proof. We prove the contrapositive. Assume that lines l and m intersect at the point R, and suppose that a transversal t cuts line l at the point A and cuts line m at the point B. Let $\angle 1$ and $\angle 2$ be a pair of alternate interior angles. Then either $\angle 1$ is an exterior angle of $\triangle ABR$ and $\angle 2$ is a remote interior angle, or vise versa.

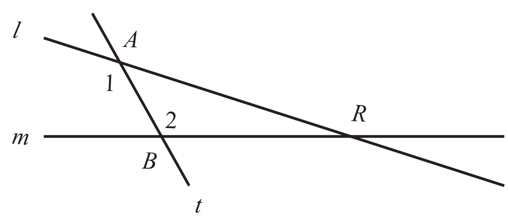

Figure 3.

In either case $m\angle 1 \neq m\angle 2$ by the Exterior Angle Inequality (Theorem 1). ■

Theorem 4 *(Saccheri-Legendre Theorem) The angle sum of a triangle is less than or equal to* 180°.

Proof. Assume, on the contrary, that the angle sum of $\triangle ABC = 180^\circ + p$, for some $p > 0$. Construct the midpoint M of side $\overline{AC}$ then extend $\overline{BM}$ its own length to point E such that $B - M - E$. Note that $\triangle ABM \cong \triangle CEM$ by SAS. Therefore the angle sum of $\triangle ABC$ = angle sum of $\triangle ABM+$ angle sum of $\triangle BMC$ = angle sum of $\triangle CEM+$ angle sum of $\triangle BMC$ = angle sum of $\triangle BEC$. Furthermore, $m\angle BEC = m\angle ABE$ by CPCTC. Therefore, either $m\angle BEC \leq \frac{1}{2}m\angle ABC$ or $m\angle EBC \leq \frac{1}{2}m\angle ABC$. Thus we may replace $\triangle ABC$ with $\triangle BEC$ having the same angle sum as $\triangle ABC$ and one angle whose measure is $\leq \frac{1}{2}m\angle ABC$.

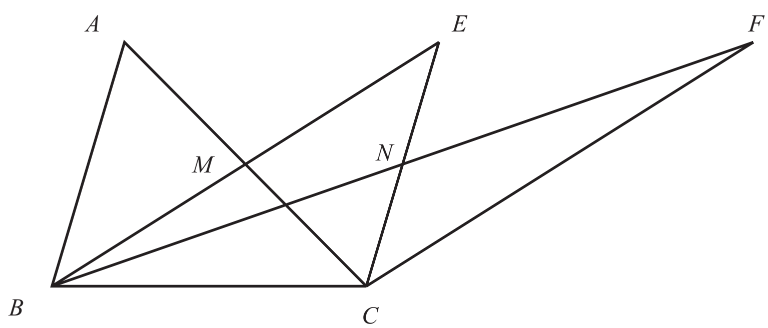

Figure 4.

Now repeat this construction in $\triangle BEC$: If $m\angle EBC \leq \frac{1}{2}m\angle ABC$, construct the midpoint N of $\overline{CE}$ and extend $\overline{BN}$ its own length to point F such that $B - N - F$. Then $\triangle BEC$ and $\triangle BFC$ have the same angle sum and either $m\angle BFC \leq \frac{1}{2}m\angle EBC$ or $m\angle FBC \leq \frac{1}{2}m\angle EBC$. Replace $\triangle EBC$ with $\triangle FBC$ having the same angle sum as $\triangle ABC$ and one angle whose measure is $\leq \frac{1}{4}m\angle ABC$. On the other hand, if $m\angle BEC \leq \frac{1}{2}m\angle ABC$, do same construction with N as the midpoint of $\overline{BC}$ and replace $\triangle EBC$ with $\triangle FEC$. Continue this process indefinitely; the Archimedian property of real numbers guarantees that for sufficiently large n, the triangle obtained after the n^{th} iteration has the same angle sum as $\triangle ABC$ and one angle whose measure is $\leq \frac{1}{2^n}m\angle ABC < p$, in which case the sum of its other two angles is greater than 180°, contradicting Corollary 2. ■

Definition 5 *The* <u>*defect*</u> *of* $\triangle ABC$ *is* $\delta ABC = 180° - m\angle A - m\angle B - m\angle C$.

Corollary 6 *Every triangle has non-negative defect.*

Proof. If $\delta ABC = 180° - m\angle A - m\angle B - m\angle C < 0°$, then the angle sum of $\triangle ABC > 180°$, contradicting Theorem 4. ∎

Theorem 7 *(Additivity of defect) Given any triangle* $\triangle ABC$ *and any point* D *between* A *and* B,

$$\delta ABC = \delta ACD + \delta BCD.$$

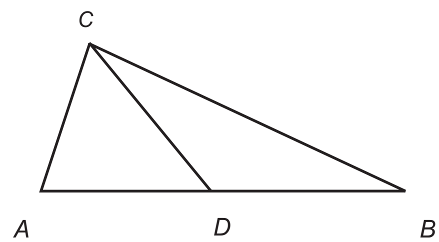

Figure 5.

Proof. Since $\angle ADC$ and $\angle BDC$ are supplementary, $m\angle CDA + m\angle CDB = 180°$. Since $\overrightarrow{CD}$ is in the interior of $\angle ACB$, $m\angle ACB = m\angle ACD + m\angle BCD$. Therefore

$$\begin{aligned} \delta ACD + \delta BCD &= 180° - m\angle ACD - m\angle CDA - m\angle DAC \\ &\quad +180° - m\angle BCD - m\angle CDB - m\angle DBC \\ &= 360° - (m\angle CDA + m\angle CDB) - m\angle DAC \\ &\quad - (m\angle ACD + m\angle BCD) - m\angle DBC \\ &= 180° - m\angle ABC - m\angle BAC - m\angle CDA \\ &= \delta ABC. \end{aligned}$$

∎

Corollary 8 *Given any triangle* $\triangle ABC$ *and any point* D *between* A *and* B, *the angle sum of* $\triangle ABC = 180°$ *if and only if the angle sums of* $\triangle ACD$ *and* $\triangle BCD$ *both equal* $180°$.

Proof. If the angle sums of both $\triangle ACD$ and $\triangle BCD$ equal $180°$, then $\delta ACD = \delta BCD = 0°$. By Theorem 7, $\delta ABC = 0°$ so that the angle sum of $\triangle ABC = 180°$. Conversely, if the angle sum of $\triangle ABC = 180°$, then $\delta ACD + \delta BCD = 0°$. But by Corollary 6, $\delta ACD \geq 0°$ and $\delta BCD \geq 0°$. Therefore $\delta ACD = \delta BCD = 0°$ and both angle sums equal $180°$. ∎

Theorem 9 *If there is a triangle with angle sum* $180°$, *then a rectangle exists.*

Proof. Consider a triangle $\triangle ABC$ with angle sum $180°$. By Corollary 2, the sum of the measures of any two interior angles is less than $180°$, so at most one angle is obtuse. Suppose $\angle A$ and $\angle B$ are acute and construct the altitude $\overline{CD}$. I claim that $A - D - B$. But if not, then either $D - A - B$ or $A - B - D$. Suppose $D - A - B$ and consider $\triangle DAC$.

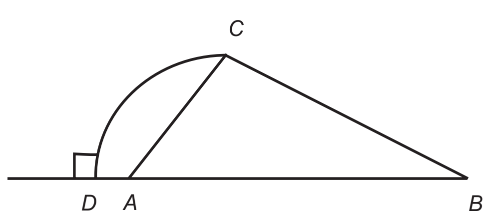

Figure 6.

Then the remote interior angle $\angle CDA$ has measure $90°$, which is greater than the measure of exterior angle $\angle CAB$, contradicting Theorem 1. Assuming that $A - B - D$ leads to a similar contradiction, proving the claim. Then by Corollary 8, $\delta ADC = \delta BDC = 0°$. Let us construct a rectangle from right triangle $\triangle BCD$. By the congruence axioms, there is a unique ray $\overrightarrow{CX}$ with X on the opposite side of $\overleftrightarrow{BC}$ from D such that $\angle CBD \cong \angle BCX$, and there is a unique point E on $\overrightarrow{CX}$ such that $\overline{CE} \cong \overline{BD}$.

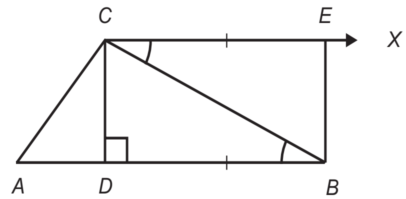

Figure 7.

Then $\triangle CBD \cong \triangle BCE$ by SAS; therefore $\triangle BCE$ is a right triangle with $\delta BCE = 0°$ and right angle at E. Also, since $m\angle DBC + m\angle BCD = 90°$, substituting corresponding parts gives $m\angle ECB + m\angle BCD = 90°$ and $m\angle DBC + m\angle EBC = 90°$. Furthermore, since alternate interior angles $\angle ECB$ and $\angle DBC$ are congruent, $\overleftrightarrow{CE} \| \overleftrightarrow{DB}$ by Theorem 3. Therefore B is an interior point of $\angle ECD$. By the same argument, $\overleftrightarrow{CD} \| \overleftrightarrow{EB}$ and C is an interior point of $\angle EBD$. Therefore $m\angle ECD = m\angle EBD = 90°$ and $\square CDEB$ is a rectangle. ■

Theorem 10 *If a rectangle exists, then the angle sum of every triangle is* $180°$.

Proof. We first prove that every right triangle has angle sum $180°$. Given a rectangle, we can use the Archimedian property to lengthen or shorten the sides and obtain a rectangle $\square AFBC$ with sides AC and BC of any prescribed length. Now given a right triangle $\triangle E'C'D'$, construct a rectangle $\square AFBC$ such that $AC > D'C'$ and $BC > E'C'$. There is a unique point D on $\overline{AC}$ and a unique point E on $\overline{BC}$ such that $\triangle ECD \cong \triangle E'C'D'$ as shown in Figure 9.

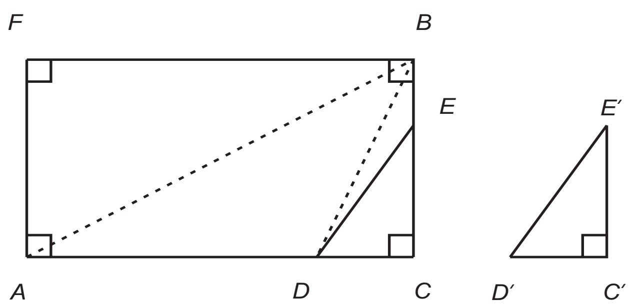

Figure 8.

I claim $\delta ABC = 0°$. If not, then $\delta ABC > 0°$ by Corollary 6 and consequently $m\angle ABC + m\angle BAC < 90°$. But $m\angle CBF = m\angle ABC + m\angle ABF = 90°$ and $m\angle CAF = m\angle BAC + m\angle BAF = 90°$. Therefore $m\angle ABF = 90° - m\angle ABC$ and $m\angle BAF = 90° - m\angle BAC$ so that

$$\begin{aligned} \delta ABF &= 180° - 90° - m\angle ABF - m\angle BAF \\ &= 90° - (90° - m\angle ABC) - (90° - m\angle BAC) \\ &= m\angle ABC + m\angle BAC - 90° < 0°, \end{aligned}$$

contradicting Corollary 6 and proving the claim. Now by repeated applications of Corollary 8 we have $\delta BCD = 0°$ and $\delta ECD = 0°$. But $\triangle ECD \cong \triangle E'C'D'$ implies $\delta E'C'D' = 0°$. Thus every right triangle has zero defect. Now by the construction in Theorem 9, an arbitrary triangle $\triangle ABC$ can be appropriately labeled so that its altitude $\overline{CD}$ lies in the interior of $\triangle ABC$ and subdivides the triangle into two right triangles (see Figure 7), each having zero defect. Thus $\delta ABC = 0°$ by Corollay 8. ■

Corollary 11 *A rectangle exists if and only if every triangle has angle sum* $180°$.

The Angle Sum of a Triangle in Parabolic Geometry.

In this section we assume Playfair's form of Euclid's Parallel Postulate and prove that the angle sum of a triangle is exactly 180°. Geometry that assumes Eucild's Parallel Postulate is called *parabolic geometry.*

Postulate 12 *(Playfair) Given a line l and a point P off l, there is exactly one line through P parallel to l.*

We first prove that the converse of Theorem 3 holds in parabolic geometry.

Theorem 13 *If parallel lines are cut by a transversal, then alternate interior angles are congruent.*

Proof. Suppose parallel lines l and m are cut by a transversal t intersecting l at A and m at B. Consider a pair of alternate interior angles $\angle 1$ and $\angle 2$ and suppose that $\angle 1 \ncong \angle 2$; then either $m\angle 1 < m\angle 2$ or vice versa. If $m\angle 1 < m\angle 2$, there is a line $l' \neq l$ through A such that $m\angle 1' = m\angle 2$ by continuity.

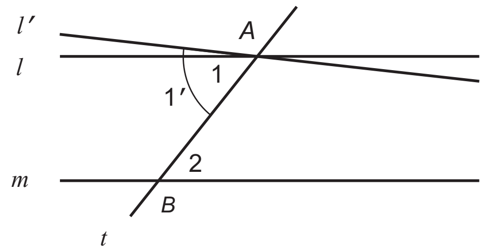

Figure 9.

Then $\angle 1'$ and $\angle 2$ are congruent alternate interior angles and $l' \parallel \overleftrightarrow{BC}$ by Theorem 3. But this contradicts Playfair's form of Eucild's Parallel Postulate. ■

Theorem 14 *The angle sum of a triangle is exactly* 180°.

Proof. Consider a (non-degenerate) triangle $\triangle ABC$ and extend side $\overline{BC}$ to $\overleftrightarrow{BC}$. Since A is off $\overleftrightarrow{BC}$, there is a unique line l through A parallel to $\overleftrightarrow{BC}$. Label the angles as shown in Figure 10.

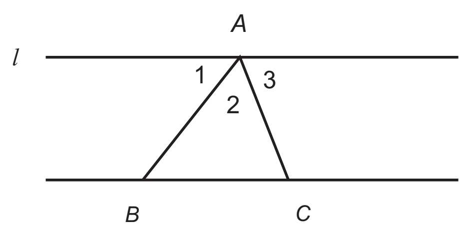

Figure 10.

Since $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$ and alternate interior angles are congruent by Theorem 13, the conclusion follows. ■

The converse of Theorem 14 is also true:

Theorem 15 *If the angle sum of every triangle is* 180°, *then Eucild's Parallel Postulate holds.*

Proof. Let l be a line and let A be a point off l. Let B be the foot of the perpendicular from A to l and choose a point C on l distinct from B. Then $\triangle ABC$ has defect zero and a right angle at B. Construct the line m through A perpendicular to $\overleftrightarrow{AB}$ and choose a point D distinct from A on m and on the opposite side of $\overleftrightarrow{AC}$ from B. Then $m \parallel l$ by Theorem 3, $\overleftrightarrow{AC}$ is a transversal, and alternate interior angles $\angle ACB$ and $\angle DAC$ are congruent by Theorem 13.

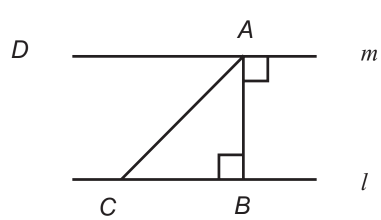

Figure 11.

Let m' be any line parallel to l that passes through A and choose a point D' on m' on the side of $\overleftrightarrow{AC}$ opposite from B. Then $\overleftrightarrow{AC}$ is a transversal and $\angle D'AC \cong \angle ACB$ by Theorem 13. But $\angle ACB \cong \angle DAC$ so that $\angle D'AC \cong \angle DAC$ and $m' = m$. Thus m is the unique line through A parallel to l. ■

Corollary 16 *A rectangle exists if and only if Eucild's Parallel Postulate holds.*

Proof. Combine Corollary 11 with Theorems 14 and 15. ■

Now consider the negation of Playfair's form of Euclid's Parallel Postulate: *There is a line l and a point P off l such that either no line through P is parallel to l or more than one line through P is parallel to l.* This suggests two alternatives to Eucild's Parallel Postulate:

1. (Hyperbolic Parallel Postulate) *There is a line l and a point P off l such that more than one line through P is parallel to l.*

2. (Elliptic Parallel Postulate) *There is a line l and a point P off l such that no line through P is parallel to l.*

We consider these alternatives in the next two sections.

The Angle Sum of a Triangle in Hyperbolic Geometry.

In this section we assume the Hyperbolic Parallel Postulate and prove that the angle sum of a triangle is strictly less than 180°. Geometry that assumes the Hyperbolic Parallel Postulate is called *hyperbolic geometry.*

Our first order of business is to strengthen the Hyperbolic Parallel Postulate.

Theorem 17 *(Hyperbolic Parallel Theorem) Given any line l and any point P off l, there is more than one line through P parallel to l.*

Proof. Consider a line l and a point P off l. Let Q be the foot of the perpendicular from P to l. Construct a line n through P and perpendicular to $\overleftrightarrow{PQ}$. Then l is parallel to n by Theorem 3, since alternate interior angles are right angles. Choose a point R on l and distinct from Q and construct a line t through R perpendicular l. Let S be the foot of the perpendicular from P to t; then $\overleftrightarrow{PS}$ is parallel to l again by Theorem 3.

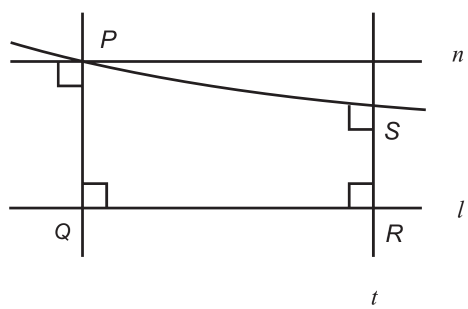

Figure 12.

Consider quadrilateral $PQRS$, in which $\overline{PQ} \perp \overline{QR}$, $\overline{QR} \perp \overline{RS}$, and $\overline{RS} \perp \overline{SP}$. Then S cannot lie on n, for if it did, $PQRS$ would be a rectangle, contradicting Corollary 16. Therefore $\overleftrightarrow{PS}$ is parallel to l and distinct from n. ■

Corollary 18 *Given any line l and any point P off l, there are infinitely many lines through P parallel to l.*

Proof. Vary the point R in the proof of Theorem 17. ■

Theorem 19 *The angle sum of every triangle is less than* 180°.

Proof. Since Euclid's Parallel Postulate fails, Corollary 16 implies that rectangles do not exist. Since rectangles do not exist, Corollary 11 implies that triangles have non-zero defect. But by Corollary 6, every triangle has non-negative defect. Therefore the defect is positive. ■

We conclude our discussion of hyperbolic geometry with the striking observation that two triangles are congruent whenever their corresponding angles are congruent, i.e., AAA implies congruence. The requires the following corollary to Theorem 19:

Corollary 20 *All convex quadrilaterals have angle sum less than* $360°$.

Proof. Consider a convex quadrilateral $\square ABCD$ and construct diagonal $\overline{AC}$.

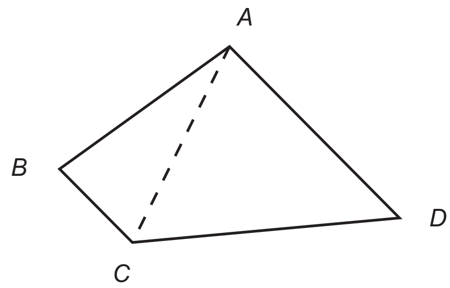

Figure 13.

By Theorem 19, the angle sums of $\triangle ABC$ and $\triangle ACD$ are less than $180°$. By convexity, $\overrightarrow{AB} - \overrightarrow{AC} - \overrightarrow{AD}$ and $\overrightarrow{CB} - \overrightarrow{CA} - \overrightarrow{CD}$. Therefore $m\angle BAC + m\angle CAD = m\angle BAD$ and $m\angle BCA + m\angle ACD = m\angle BCD$. It follows that the angle sum of $\square ABCD$ is the sum of the measures of the six interior angles of $\triangle ABC$ and $\triangle ACD$, which is less than $360°$. ■

Theorem 21 *If two triangles are similar, they are congruent.*

Proof. On the contrary, suppose there exist similar non-congruent triangles $\triangle ABC$ and $\triangle A'B'C'$. Then corresponding angles are congruent but no pair of corresponding sides are congruent, for otherwise the triangles would be congruent by ASA. Consider the triples (AB, AC, BC) and $(A'B', A'C', B'C')$. One of these triples contains at least two lengths that are larger than the corresponding lengths in the other triple. So suppose that $AB > A'B'$ and $AC > A'C'$. Then by definition, there exist points B'' on $\overline{AB}$ and C'' on $\overline{AC}$ such that $AB'' = A'B'$ and AC”” $= A'C'$.

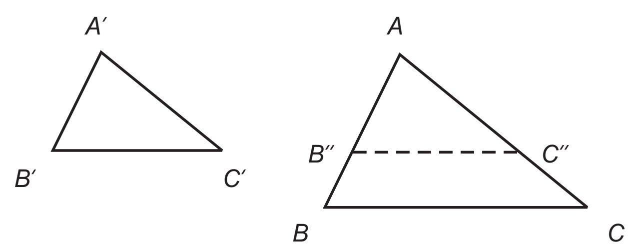

Figure 13.

Now $\triangle AB''C'' \cong \triangle A'B'C'$ by SAS; hence $\angle AB''C'' \cong \angle B'$ and $\angle AC''B'' \cong \angle C'$ by CPCTC. By hypothesis, $\angle B \cong \angle B'$ and $\angle C \cong \angle C'$. Therefore we also have $\angle AB''C'' \cong \angle B$ and $\angle AC''B'' \cong \angle C$ by the congruence axioms. Hence the pair of alternate interior angles, one of which is $\angle ABC$, formed by lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{B''C''}$ cut by transversal $\overleftrightarrow{AB}$ are congruent and $\overleftrightarrow{BC} \| \overleftrightarrow{B''C''}$ by Theorem 3. It follows that quadrilateral $\square BB''C''C$ is convex. Furthermore, $m\angle B + m\angle C''B''B = m\angle AB''C'' + m\angle C''B''B = 180° = m\angle AC''B'' + m\angle B''C''C = m\angle C + m\angle B''C''C$ and it follows that the angle sum of $\square BB''C''C$ is $360°$, contradicting Corollary 20. ■

The Angle Sum of a Triangle in Elliptic Geometry.

In this section we assume the Parabolic Parallel Postulate. In this case, the angle sum of a triangle is strictly greater than $180°$. Geometry that assumes the Parabolic Parallel Postulate is called *parabolic geometry.*

But there's a problem here. We can't simply adjoin the Elliptic Parallel Postulate to neutral geometry as we did in the Parabolic and Hyperbolic cases, for if we did there would be inconsistencies. Indeed, Theorem 3 tells us there are always parallel lines in neutral geometry, but this is contrary to the Parabolic Parallel Postulate. So some adjustments are necessary.

To see which modifications are necessary, consider the geometry of a sphere. Lines on a sphere are great circles, and any two great circles intersect, so there are no parallels. And that's good. But there are still problems here. Note that a pair of antipodal points determine infinitely many great circles (think lines of longitude through the north and south poles). So Eucild's first postulate fails. To remedy this, we identify antipodal points and work in the *(real) projective plane,* which we think of as a hemisphere with antipodal points along its equatorial boundary identified. Unfortunately this is the best we can do, as this surface cannot be embedded in 3-dimensional space.

The projective plane has a surprising property: A line does not divide the plane into two sides, because you can now leap across a great circle by passing from a given point to its (now equal) antipode, which was in the other hemisphere (on the other side of the line) before we made the identifications. If we cut a strip from the projective plane, it would look like a Moebius band, which is a long strip of paper with a half twist and its two ends glued together. A Moebius band is an example of a *one-sided* surface. Surfaces with this property are *nonorientable*.

Some of our axioms must change as well. "Betweenness" no longer makes sense for points on a great circle, so we replace them with *separation axioms* (see the text by Greenberg). Thus the axioms of Elliptic geometry consist of the same incidence, congruence and continuity axioms as neutral geometry, but the betweenness axioms are replaced by separation axioms.

As was the case in hyperbolic geometry, we can strengthen the Elliptic Parallel Postulate.

Theorem 22 *(Elliptic Parallel Theorem) Given any line l and any point P off l, every line through P intersects l.*

Proof. We prove this result in the projective plane. Note that any two distinct great circles intersect at a pair of antipodal points of the sphere. Thus in the projective plane, every pair of distinct lines intersect at exactly one point and there are no parallels. The conclusion now follows. ■

Theorem 23 *The angle sum of a triangle is always greater than* $180°$.

We omit the proof.

November 27, 2007.