Non-operadic Operations on Loop Cohomology

Joint work with Samson Saneblidze

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Millersville Univ of Pennsylvania

Modern Algebra and its Applications

22 September 2011

Samson Saneblidze

Goal of the Talk

- To identify a space X such that $H = H^* (\Omega X; \mathbb{Z}_2)$ comes equipped with an induced operation

$$\omega_2^2 : H \otimes H \longrightarrow H \otimes H$$

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- We assume existence of biassociahedra $KK = \{ KK_{n,m} \}$ and bimultiplihedra $JJ = \{ JJ_{n,m} \}$
- The operation ω_2^2 will be defined in terms of $JJ_{2,1} = JJ_{1,2}$ and $JJ_{2,2}$

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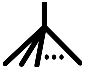

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- Vertices of $K_n \leftrightarrow \{\text{binary planar rooted trees with } n \text{ leaves}\}$

$$\leftrightarrow \{\text{associations of } n \text{ indeterminants}\}$$

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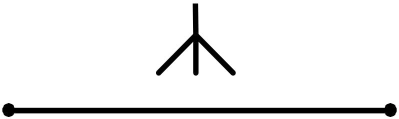

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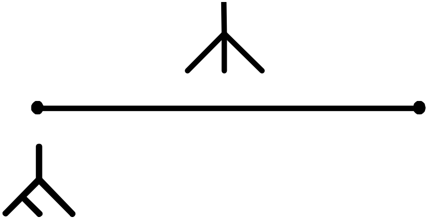

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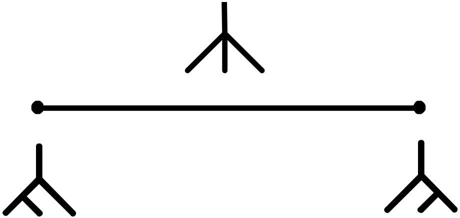

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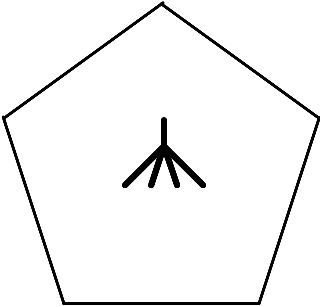

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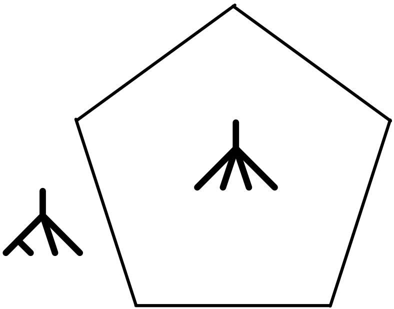

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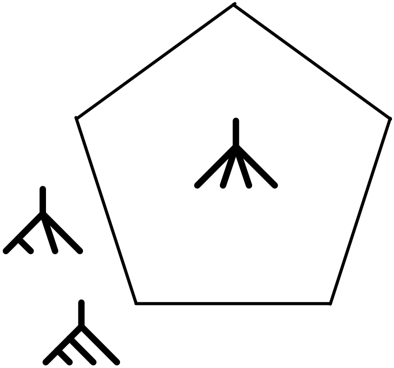

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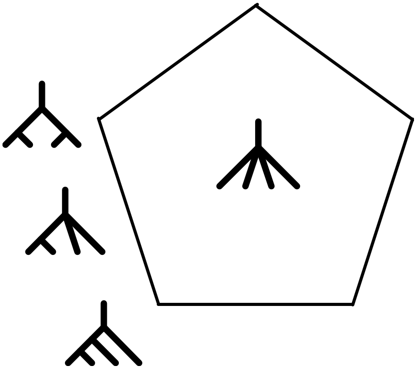

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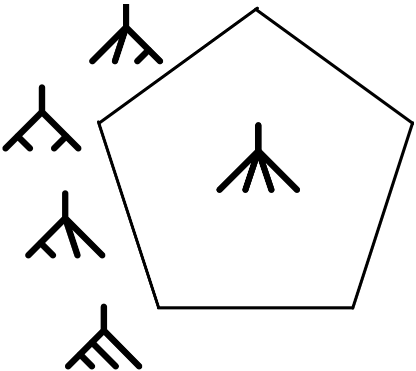

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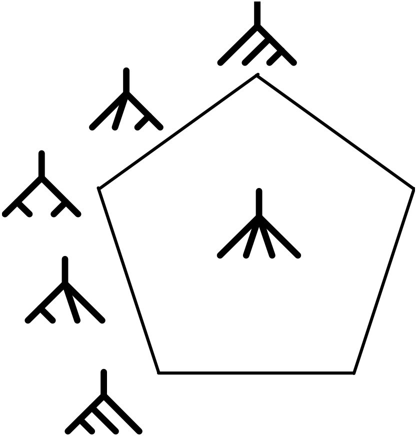

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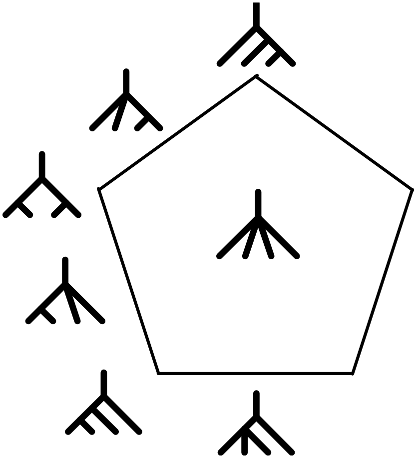

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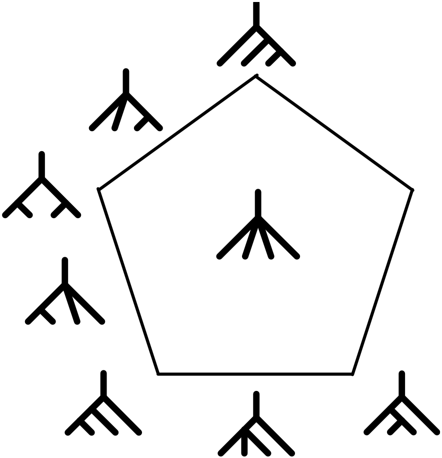

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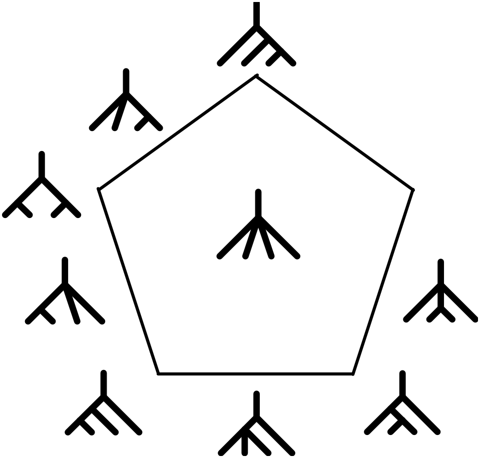

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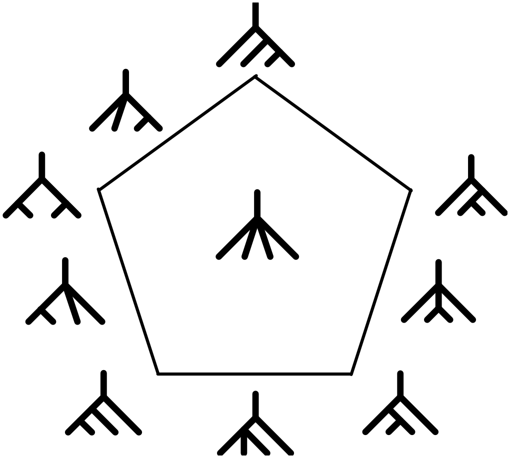

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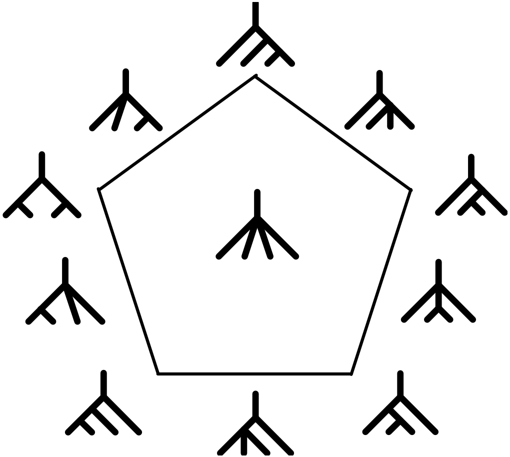

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- μ is *homotopy associative in* 3 *variables* if there is a chain homotopy

$$\mu_3 : A^{\otimes 3} \to A$$

such that

$$\nabla \mu_3 = \mu(1 \otimes \mu) - \mu(\mu \otimes 1),$$

where ∇ is defined on $f \in Hom(V, W)$ by $\nabla f = d_W f - (-1)^{|f|} f d_V$

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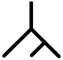
- Then $\mu(1 \otimes \mu) - \mu(\mu \otimes 1) = \alpha\Big($  $-$  $\Big) = \alpha\partial\Big($  $\Big)$

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  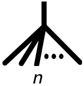
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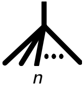

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- $\mathcal{J}_\infty$ is a free $\mathcal{A}_\infty$-bimodule

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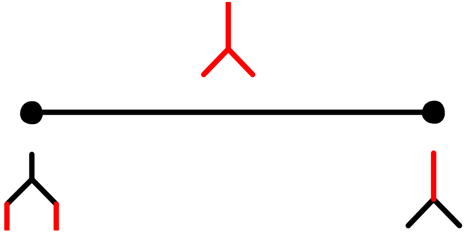

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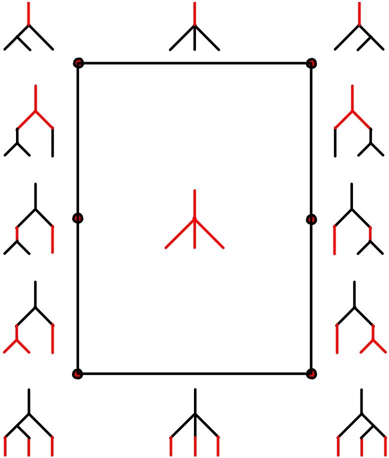
- J_3 :



Morphisms of A-infinity Algebras

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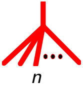

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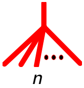

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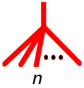

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- Analogously, the free matrad $\mathcal{H}_\infty$ controls A_∞-bialgebra structures
- The free relative matrad $\mathcal{J}\mathcal{J}_\infty$ controls morphisms thereof

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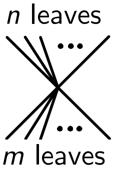
- Top dim'l cell of $KK_{n,m}$ is identified with the matradic generator



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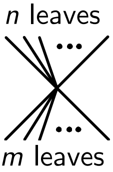
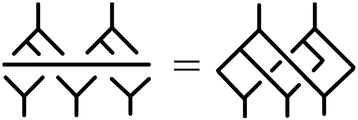
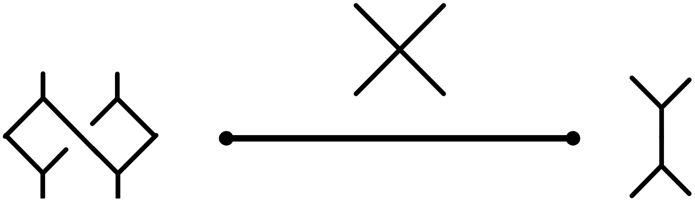


- Cells of KK are identified with certain $\circledcirc$-product monomials



$\leftrightarrow$ a vertex of $KK_{2,3}$

The Biassociahedron $KK(2,2)$:



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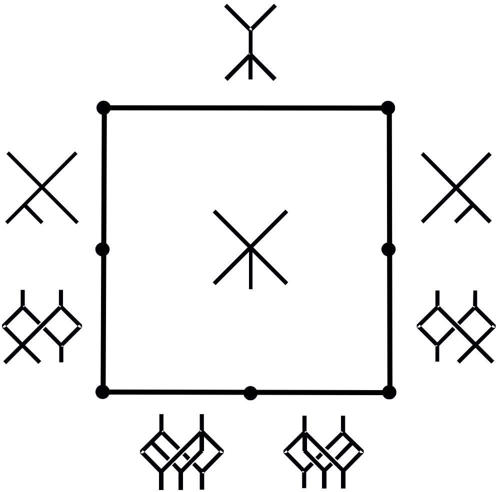

A-infinity Bialgebras

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$$\omega = \left\{\omega_m^n \in Hom\left(A^{\otimes m}, A^{\otimes n}\right) \mid mn > 1\right\}$$

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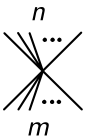

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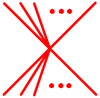

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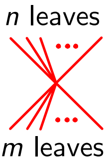
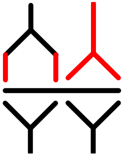


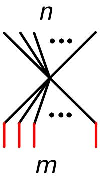
- Cells of JJ are identified with certain fraction product monomials



$\leftrightarrow$ an edge of $JJ_{2,2}$

The Free Relative Matrad

$KK_{n,m}$ embeds in $JJ_{n,m}$ as a codim 1 cell two ways:

- As $KK_{n,m} \times 0 \quad \leftrightarrow$ 

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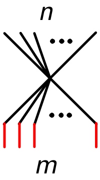
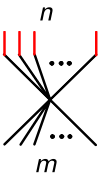
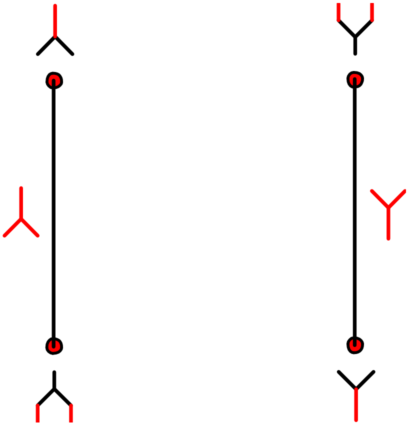
- As $KK_{n,m} \times 0 \quad \leftrightarrow$ 

- As $KK_{n,m} \times 1 \quad \leftrightarrow$ 

The Bimultiplihedra $JJ(1,2)$ and $JJ(2,1)$:



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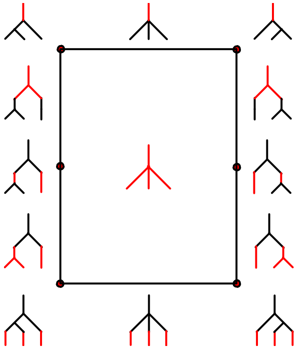

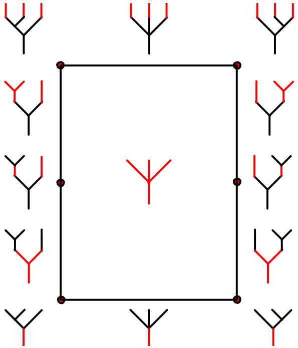

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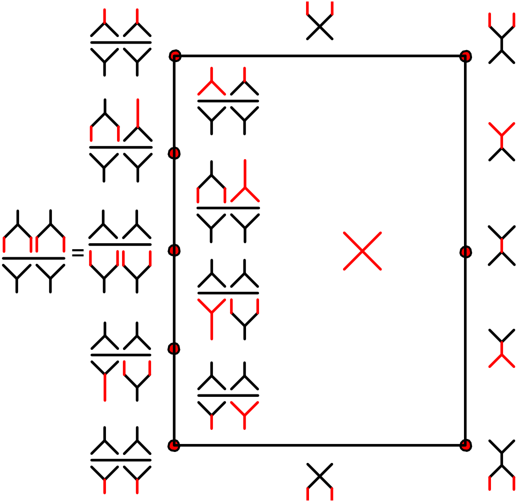

Morphisms of A-infinity Bialgebras

- Let (A, d_A, ω_A) and (B, d_B, ω_B) be A_∞-bialgebras

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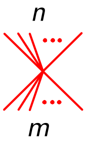
- A *morphism* $A \Rightarrow B$ *of* A_∞*-bialgebras* is a family of maps

$$G = \{g_m^n \in Hom(A^{\otimes m}, B^{\otimes n})\}$$

together with a chain map

$$\beta : C_* \left(JJ\right) \to Hom \left(TA, TB\right)$$

that sends



$$\mapsto g_m^n$$

and preserves relative matradic structure

Induced Cochain Map

- A chain map $g : A \longrightarrow B$ of DGMs induces a cochain map

$$\tilde{g} : Hom\left(TA, TA\right) \to Hom\left(TA, TB\right)$$

  defined on $u \in Hom\left(A^{\otimes m}, A^{\otimes n}\right)$ by $\tilde{g}\left(u\right) = g^{\otimes n} u$

The General Transfer Theorem – Generalizing the BPL

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  - *g extends to an A_∞-bialgebra map $G = \{g_m^n \mid g_1^1 = g\} : A \Rightarrow B$*

Remarks

- The Transfer Theorem generalizes the seminal work of T. Kadeishvili in which B is an A_∞-algebra, its homology $B = H(A)$ is free, and g has a right-homotopy inverse $f : B \to A$

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- In the conference proceedings, we apply our Transfer Theorem to an A_∞-algebra whose homology is not free, g has no right-homotopy inverse, and B has no Hodge decomposition

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- The Transfer Thm imposes an A_∞-bialgebra structure on $H(\Omega X; \mathbf{k})$

A Non-operadic Cohomology Operation

- Consider the following pullback of the path fibration:

$$\begin{array}{ccccc} K(\mathbb{Z}_2,4) & \longrightarrow & X & \longrightarrow & \mathcal{L}K(\mathbb{Z}_2,5) \\ & & p\downarrow & & \downarrow \\ & & (S^2\times S^3)\vee\Sigma\mathbb{C}P^2 & \xrightarrow{f} & K(\mathbb{Z}_2,5) \\ & & \bar{a}_2\bar{a}_3+Sq^2\bar{b} & \underset{f^*}{\longleftarrow} & \iota_5 \end{array}$$

where $\bar{a}_i\in H^i\left(S^i;\mathbb{Z}_2\right)$ and $\bar{b}\in H^3\left(\Sigma\mathbb{C}P^2;\mathbb{Z}_2\right)$

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- $\bar{a}_i \mapsto a_i$ and $\bar{b} \mapsto b$ under p^*
- $A = H^*(X;\mathbb{Z}_2) = \{1, a_2, a_3, b, a_2a_3 = Sq^2b, \ldots\}$

Homotopy Gerstenhaber Algebra Structure on A

- The attaching in $\mathbb{C}P^2$ and the twisting in X induce a HGA structure

$$E_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \longrightarrow A$$

with nontrivial action

$$E_{1,0} = E_{0,1} = \mathrm{Id}_A \quad \text{and} \quad E_{1,1}(b; b) = a_2 a_3$$

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- $\phi = E'_{1,0} + E'_{0,1} + E'_{1,1} : BA \otimes BA \to A$ acts nontrivially:

$$\phi([b] \otimes [b]) = a_2 a_3$$

The DG Hopf Algebra BA

- ϕ lifts to a chain map $\mu : BA \otimes BA \to BA$ of DG coalgebras given by

$$\mu = \sum_{k \geq 0} \downarrow^{\otimes k+1} \phi^{\otimes k+1} \bar{\psi}^{(k)}$$

where $\bar{\psi}^{(0)} = \mathrm{Id}_{BA \otimes BA}$ and $\bar{\psi}^{(k)}$ is the left-iterated reduced coproduct

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- Nontrivial product $\mu([b] \otimes [b]) = [a_2 a_3]$

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A Model for Loop Cohomology

- $H = H^* (BA) \approx H^*(\Omega X; \mathbb{Z}_2)$ as graded coalgebras
- Denote: $\alpha_i = \mathrm{cls}\,[a_i] \qquad \beta = \mathrm{cls}\,[b]$
- Choose a cycle-selecting map $g : H \longrightarrow BA$ such that

$$g\,(\mathrm{cls}\,[x_1|\cdots|x_n]) = [x_1|\cdots|x_n]$$

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- *There is a homotopy g_2 satisfying the relation above such that*

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$$=[a_2]\otimes[a_3]$$

- $[a_2]\otimes[a_3]$ represents a non-zero class in $BA\otimes BA$

- Thus no choice of g_2^2 kills the obstruction $[a_2]\otimes[a_3]$

A Non-operadic Cohomology Operation Defined

- *There is an operation* ω_2^2 *and a cochain homotopy* g_2^2 *satisfying the relation on* $JJ_{2,2}$ *such that*

$$\omega_2^2\left(\beta\otimes\beta\right)=\alpha_2\otimes\alpha_3$$

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- Then $(g \otimes g)\,\omega_2^2\,(\beta \otimes \beta) = [a_2] \otimes [a_3]$ kills the obstruction
- ω_2^2 is the first nontrivial non-operadic operation on loop cohomology

The End

Thank you!