Polygons that Tessellate the Plane Under Reflections

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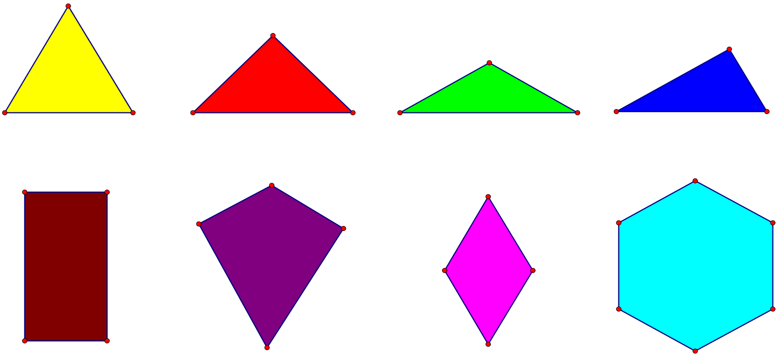
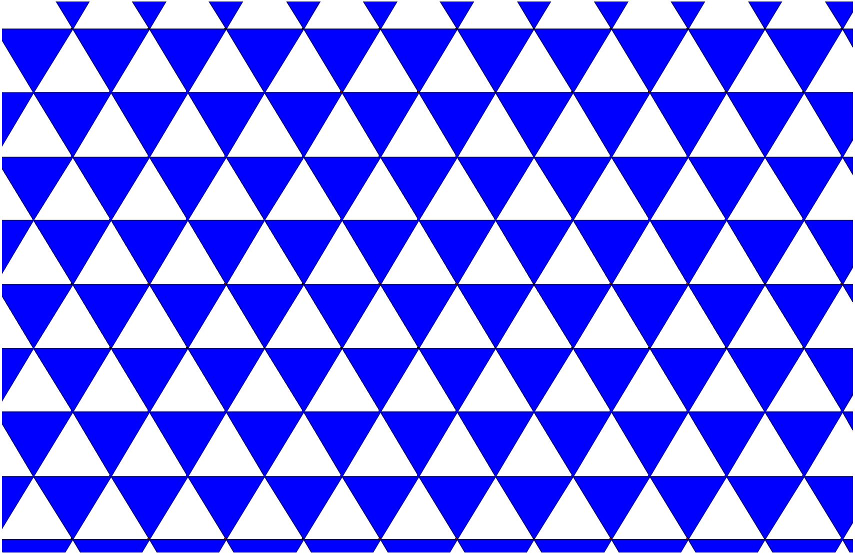
Dickinson College Math Chat

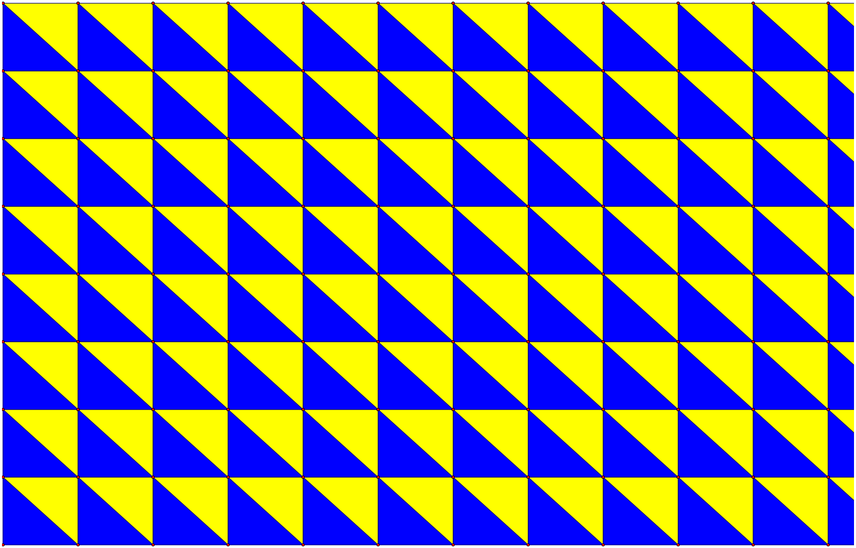
April 14, 2008

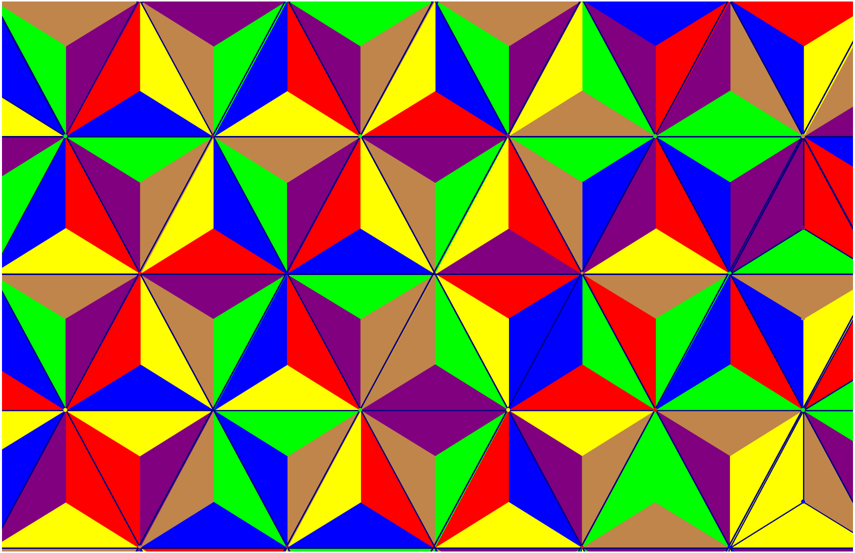
Theorem 1. *Exactly eight polygons tessellate the plane under reflections:*

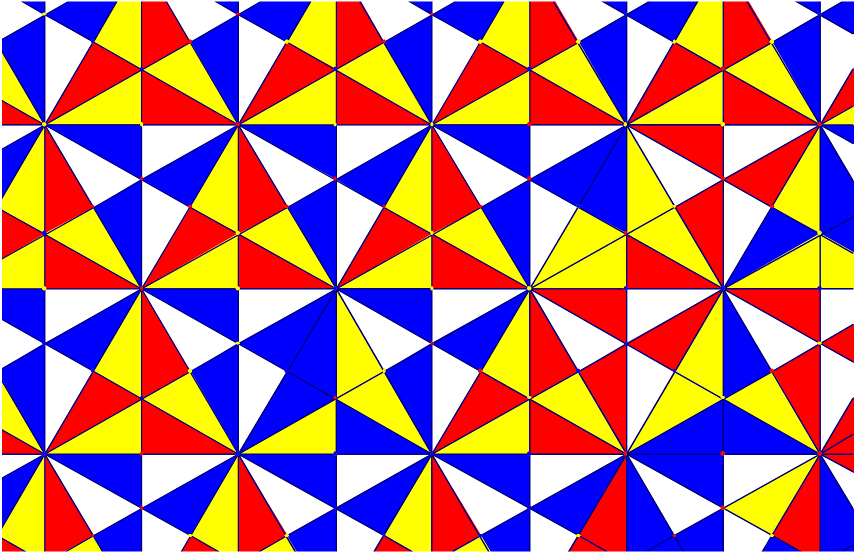


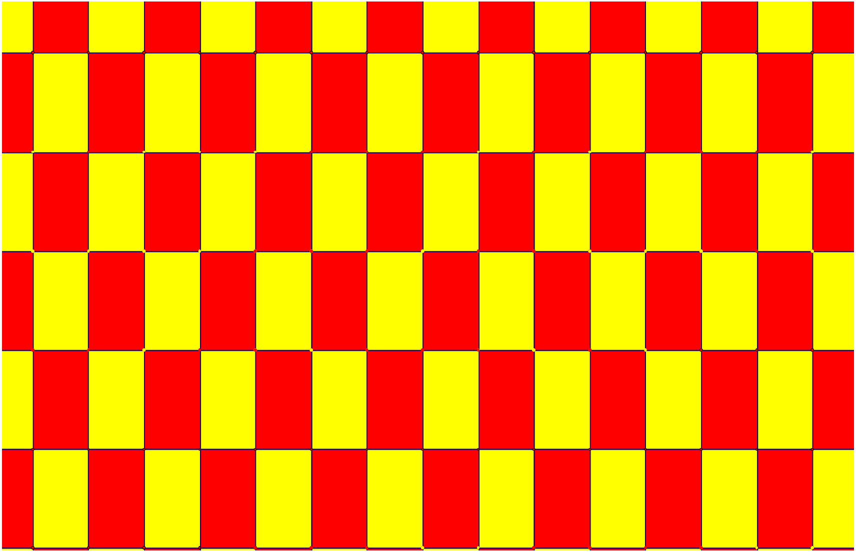
- Most symmetric examples of Laves tilings (Grunbaum & Shephard, Tiling and Patterns, p. 96)

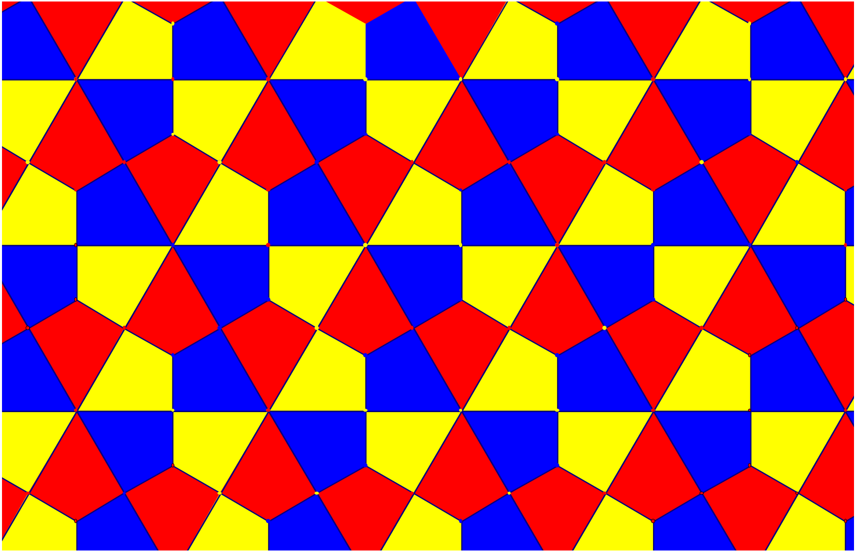


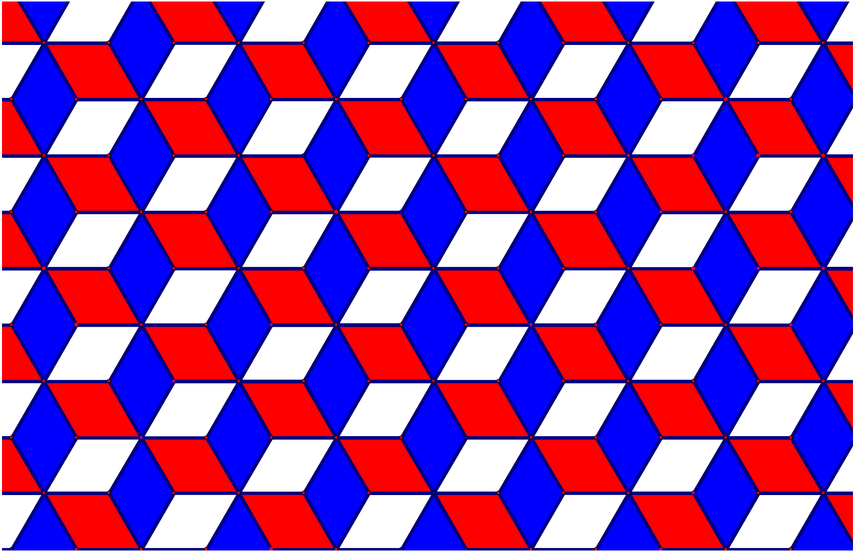


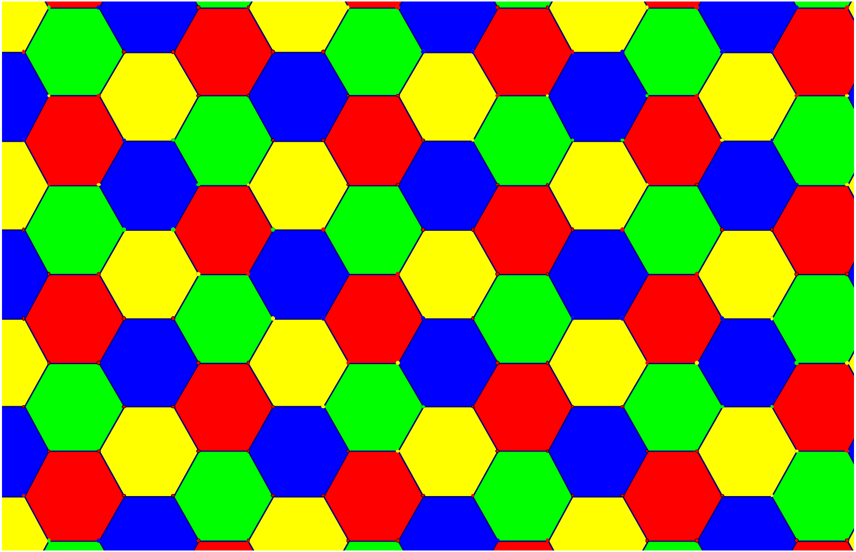












Admissible Interior Angles

- **Crystallographic Restriction:** *If P is an n-center of a wallpaper pattern, i.e., the center of a subgroup of n rotational symmetries, then* $n = 2, 3, 4, 6$.

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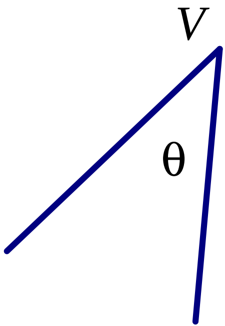
- If V is a vertex of a polygon G that generates a tessellation under reflections, then $\theta = m\angle V < 180^\circ$.

Admissible Interior Angles

- If G' is the reflection of G in an edge containing V, the interior angle of G' at V has measure θ; inductively, every interior angle at V has measure θ.



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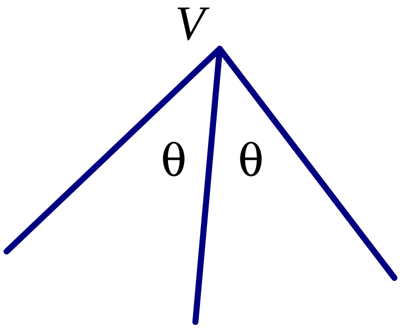

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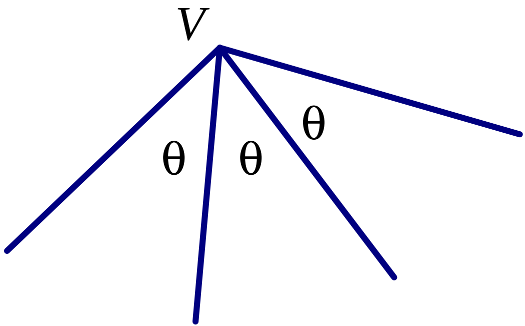

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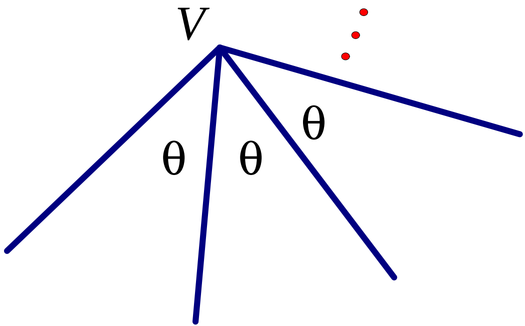
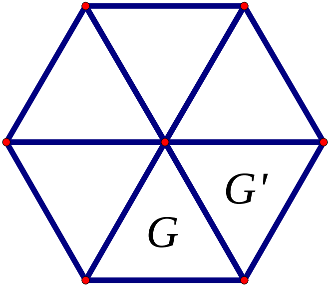


Admissible Interior Angles

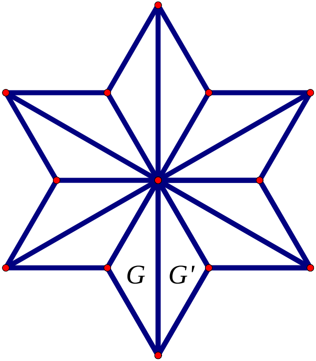
Since successively reflecting in the edges of G that share vertex V is a rotational symmetry, V is an n-center for some $n = 2, 3, 4, 6$.

If G' is the image of G under a rotational symmetry at V, the number k of copies of G sharing vertex V is the order of the rotational subgroup at V. Thus $k = 3, 4, 6$ and $\theta = 60^\circ, 90^\circ, 120^\circ$.



Admissible Interior Angles

Otherwise, the number k of copies of G sharing vertex V is twice the order of the rotational subgroup at V. Thus $k = 4, 6, 8, 12$ and $\theta = 30^\circ, 45^\circ, 60^\circ, 90^\circ$.



Finding Generating Polygons

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- For triangles:

$$\begin{array}{ccccccccccc} 30a & + & 45b & + & 60c & + & 90d & + & 120e & = & 180 \\ a & + & b & + & c & + & d & + & e & = & 3 \end{array}$$

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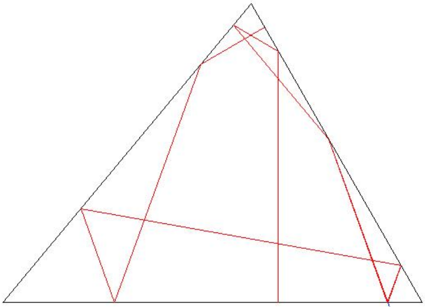
- Using the combinatorial data, construct all possible n-gons and check which ones tessellate.

Periodic Trajectories and Orbits

- A *periodic orbit* of a billiard ball in motion on a frictionless table Ω bounded by a polygon G is a piecewise-linear constant speed curve $\alpha : \mathbb{R} \rightarrow \Omega$ such that $\alpha(a) = \alpha(b)$ for some $a < b$ and $\alpha'(a+t) = \alpha'(b+t)$ for almost all $t \in \mathbb{R}$.



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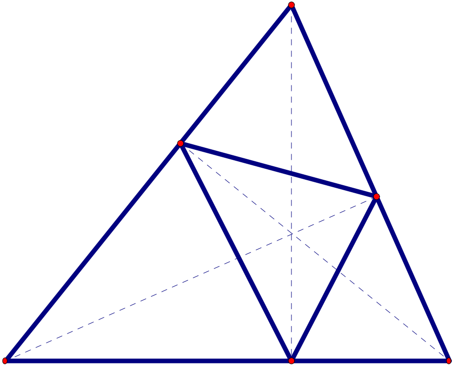
Fagnano's Period 3 Orbit on Acute Triangles

- The *orthic triangle* is the inscribed triangle of least perimeter



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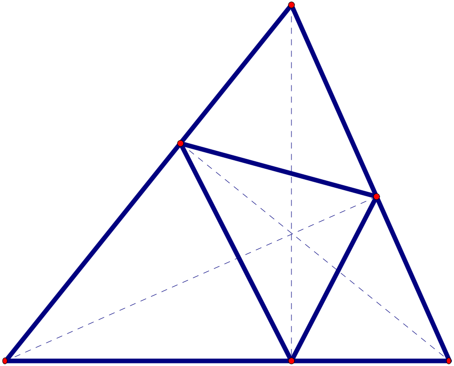

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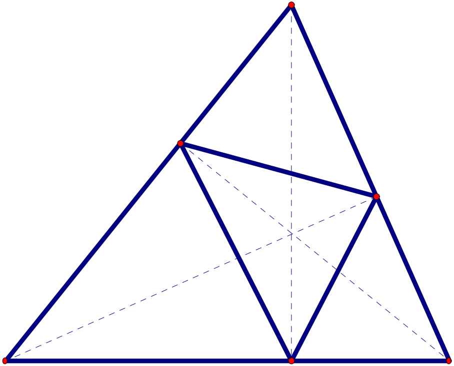

- Vertices are feet of the altitudes
- Altitudes bisect the interior angles

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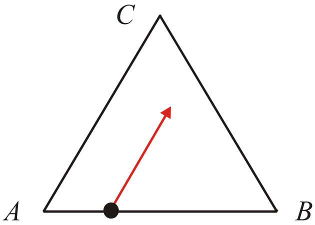
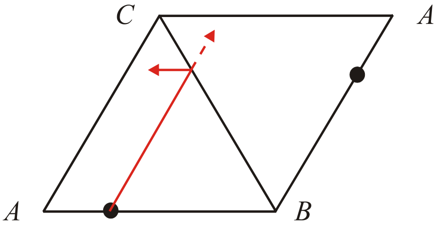
- $\overline{PQ}$ is an *unfolding* of the orbit


C
A
B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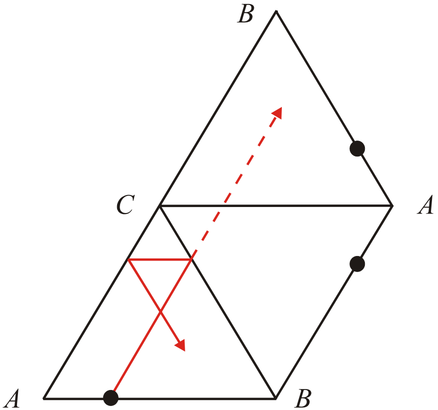
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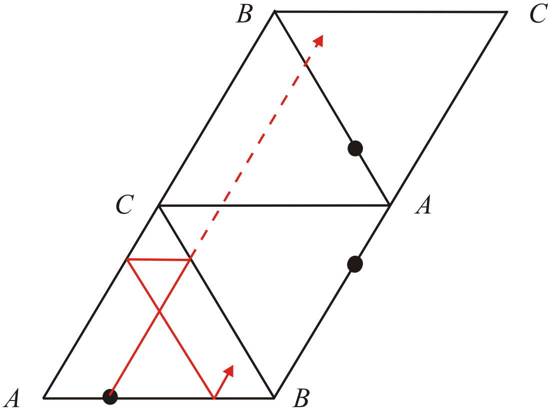
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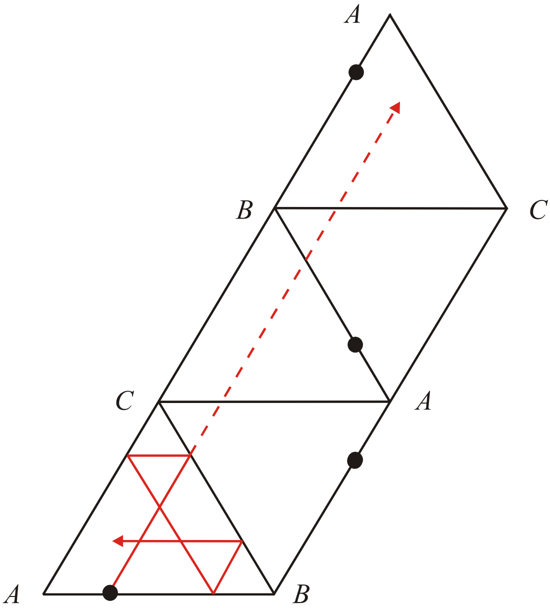
A
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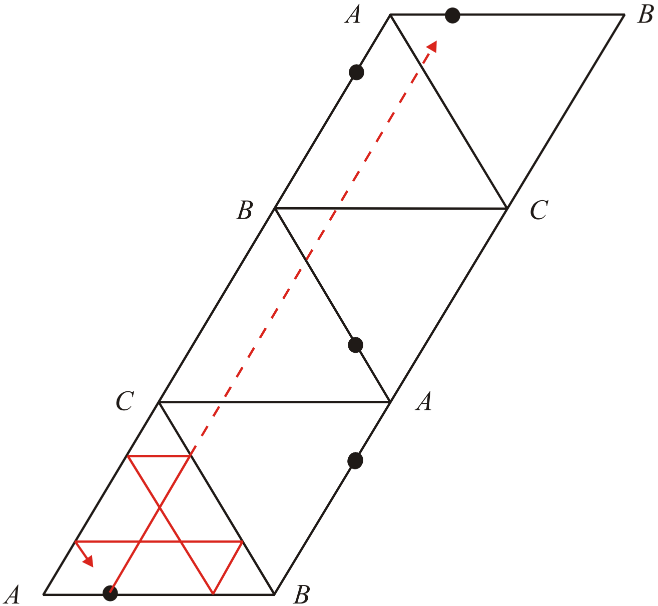
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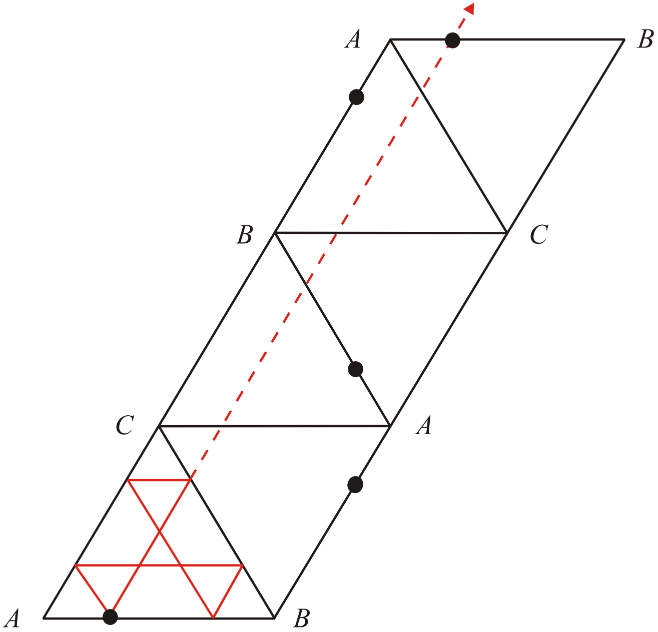
A
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C
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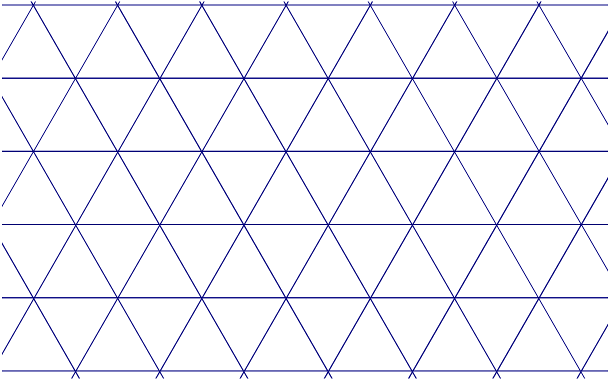
Unfolding Orbits on Equilateral Triangles

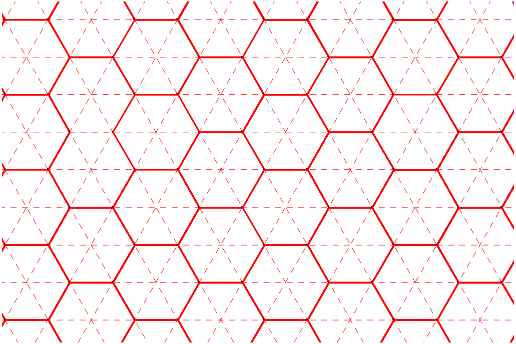
Let $\mathcal{T}$ be the tessellation generated by repeatedly reflecting $\triangle ABC$



Unfolding Orbits on Equilateral Triangles

Let H be the group generated by reflections in the lines of $\mathcal{T}$.

The action of H on an edge of a basic triangle generates a tessellation $\mathcal{H}$ by hexagons.



Unfolding Orbits on Equilateral Triangles

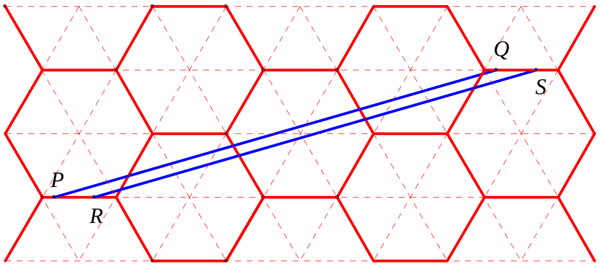
Theorem 2. *Let* α be an orbit and let $\overline{PQ}$ *be an unfolding. Then*

(i) α *has even period iff* Q *lies on a horizontal* edge of $\mathcal{H}$.

(ii) α *has odd period iff* $\alpha = \gamma^{2k-1}$ *for some* $k \geq 1$.

Equivalent Orbits

- Periodic orbits α and β are *equivalent* if there exist respective unfoldings $\overline{PQ}$ and $\overline{RS}$ and a horizontal translation τ such that $\overline{RS} = \tau\left(\overline{PQ}\right)$.



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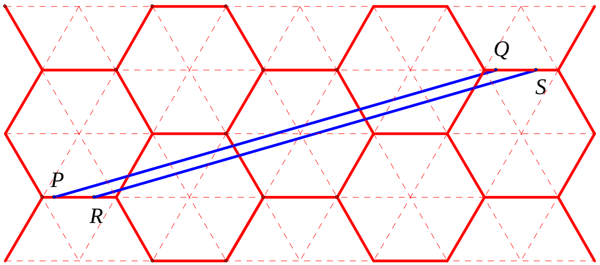

- Two unfoldings that terminate on the same horizontal edge of $\mathcal{H}$ are equivalent.

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- The *period of a class* is the period of its elements
- Even (period) classes have cardinality $\mathfrak{c}$
- Odd classes are singletons $\left[\gamma^{2k-1}\right]$
- Since $\mathcal{H}$ has countably many edges, there are countably many even classes

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- The line through O with inclination 60° is the y-axis

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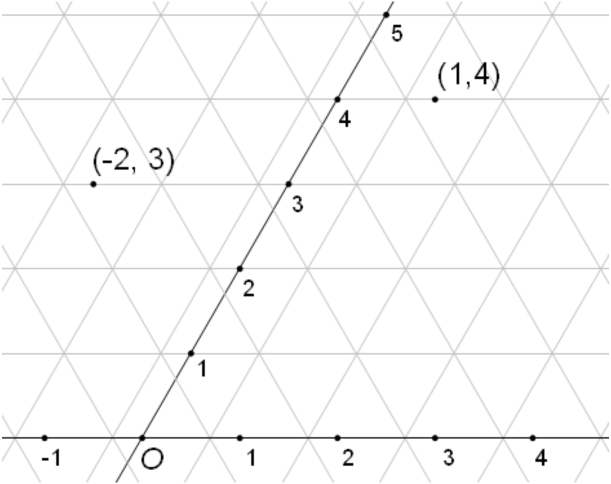


Software for finding periodic orbits:

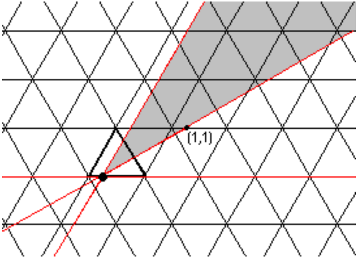
"Orbit Tracer 2" by Stephen Weaver at

http://www.millersville.edu/~rumble/StudentProjects/Weaver/Tessellation_v1.1.exe

The Fundamental Region

Proposition 1. *Every periodic orbit strikes some side of $\triangle ABC$ with an incidence angle in the range $30^\circ \leq \theta \leq 60^\circ$.*

Every periodic orbit has an unfolding in $\Gamma_O = \{(x, y) \mid 0 \leq x \leq y\}$



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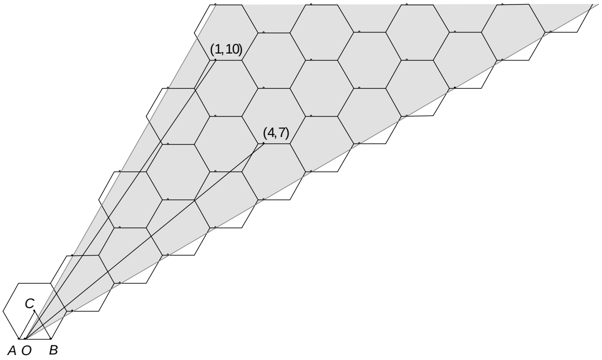
- **Theorem 4. (Classification)**

(i) α *has odd period iff* $\alpha = \gamma^{2k-1}$ *for some* $k \geq 1$.

(ii) $\alpha \leftrightarrow (x, y)$ *has period* $2n$ *iff* $x \equiv y \pmod{3}$ and $x + y = n$.

Counting Classes of Periodic Orbits

- Distinct classes may have the same period:



Theorem 5. (Counting Formulas)

(i) The number of classes with period 2n is exactly

$$\mathcal{O}(n) = \left\lfloor \frac{n+2}{2} \right\rfloor - \left\lfloor \frac{n+2}{3} \right\rfloor.$$

(ii) The number of primitive classes with period 2n is exactly

$$\mathcal{P}(n) = \sum_{d|n} \mu(d)\, \mathcal{O}(n),$$

$$\text{where } \mu(d) = \begin{cases} 1, & d = 1 \\ (-1)^r, & d = p_1 p_2 \cdots p_r \text{ and } p_i \neq p_j \\ 0, & \text{otherwise.} \end{cases}$$

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- There are no primitive orbits of period $2, 8, 12, 20$
- There is at least one class of period $2n$ for each $n \geq 2$
- *All* classes of period $2n$ are primitive iff $n = 1$ or n is prime

Project Proposal

- Given a polygon G that tessellates the plane under reflections, find, classify and count the classes of periodic orbits on G.

Thank you!