

Polygons that Tessellate the Plane When Reflected in Their Edges

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EMU Math Chat

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Tessellations

- A *tessellation* (or *tiling*) of the plane is a collection of plane figures that fills the plane with no overlaps and no gaps.

Tessellations

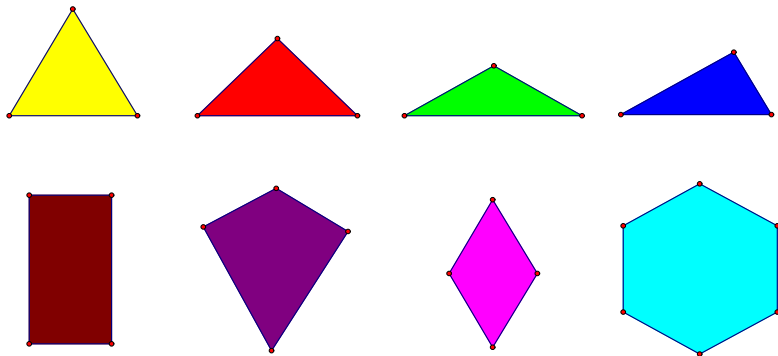
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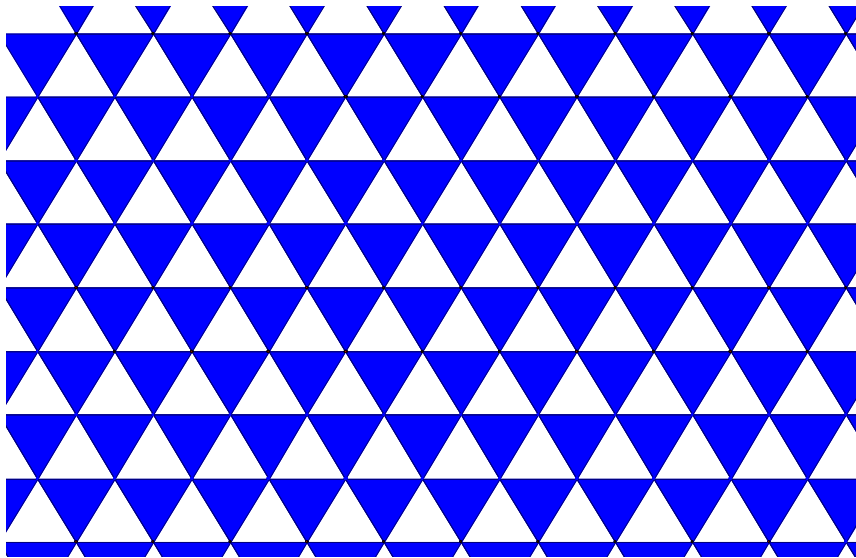
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- Tessellations of the plane generated by reflecting a polygon in its edges are regular.

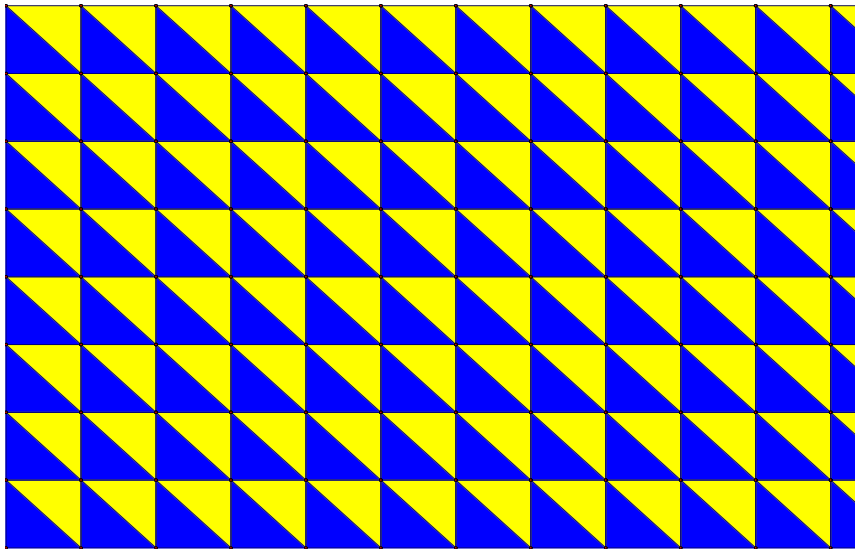
Main Result

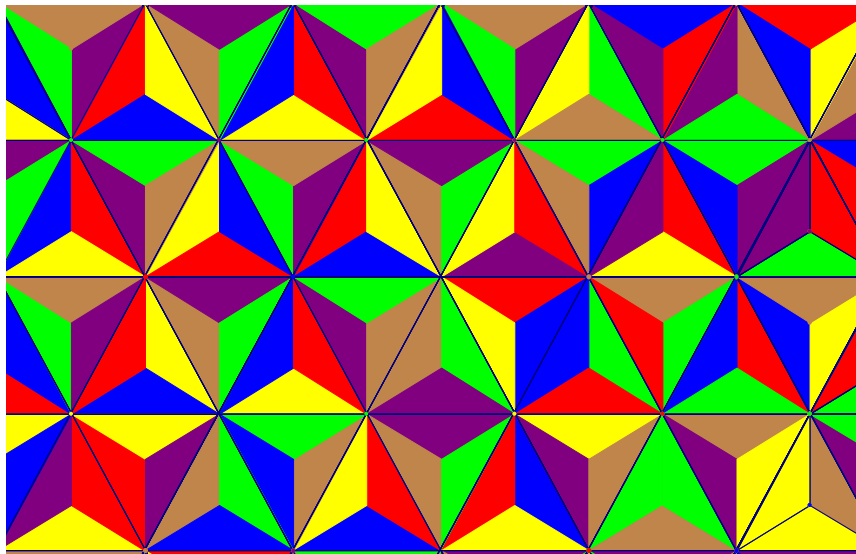
Theorem 1. *Exactly eight polygons tessellate the plane under reflections:*

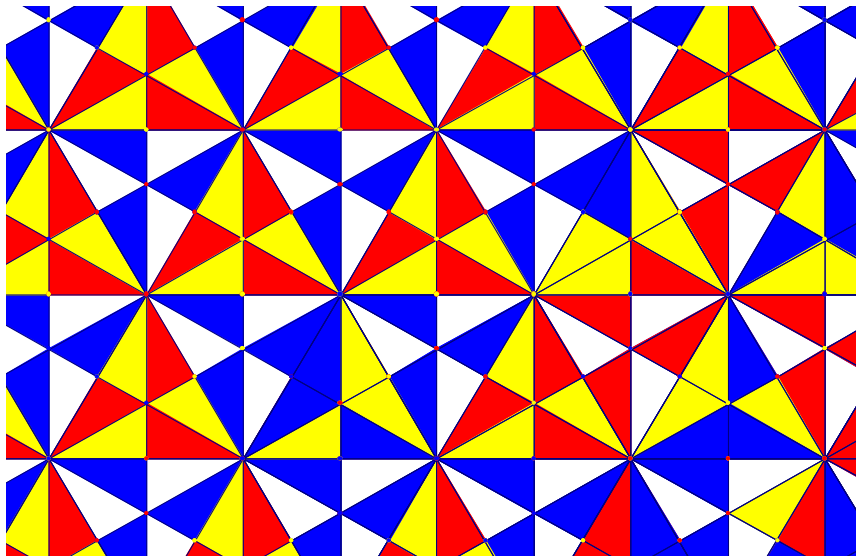


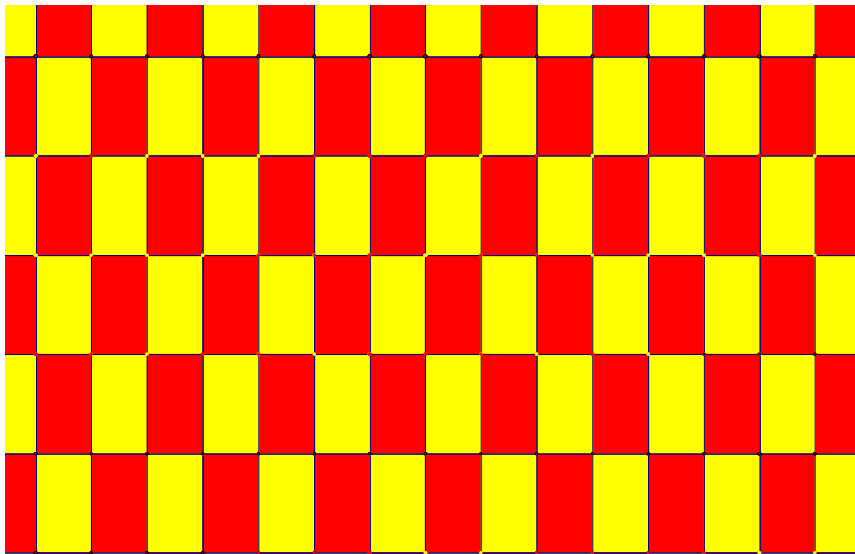
- Most symmetric examples of Laves tilings (Grunbaum & Shephard, Tiling and Patterns, p. 96)

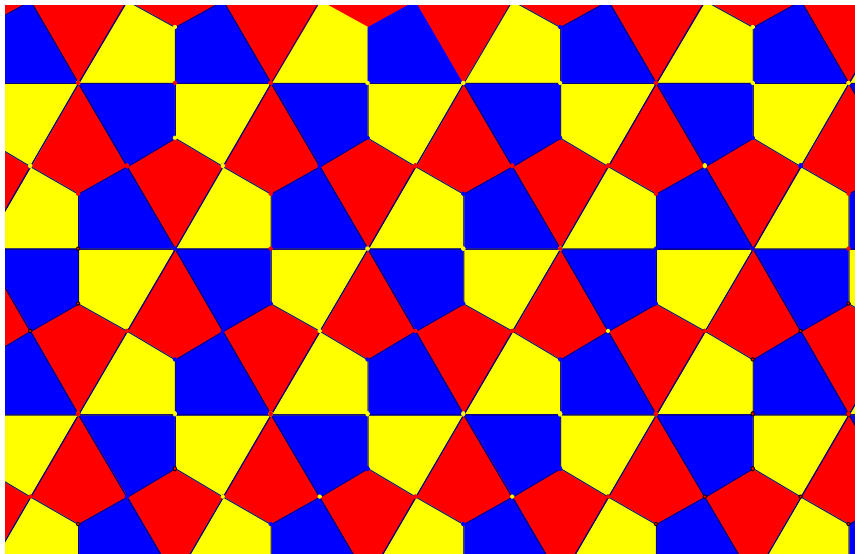


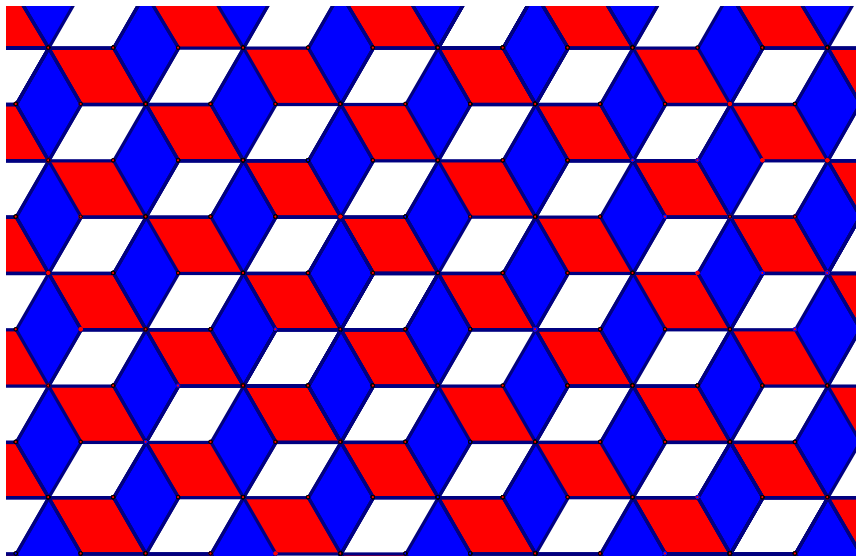


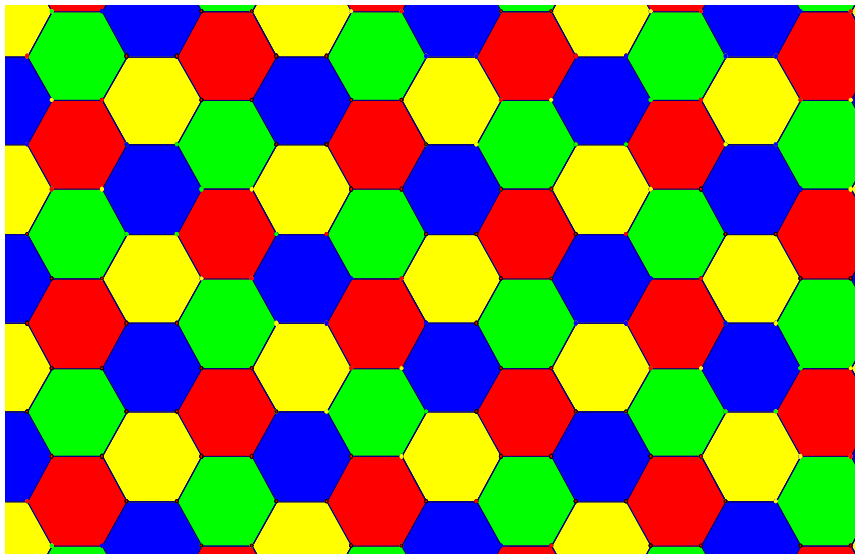












Admissible Interior Angles

- An n -center of a tessellation is the center of a group of n rotational symmetries.

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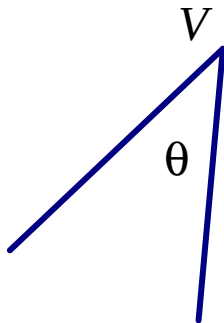
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- **Crystallographic Restriction:** If P is an n -center of a regular tessellation, then $n = 2, 3, 4, 6$.
- If V is a vertex of a polygon G that generates a tessellation under reflections, then $\theta = m\angle V < 180^\circ$.

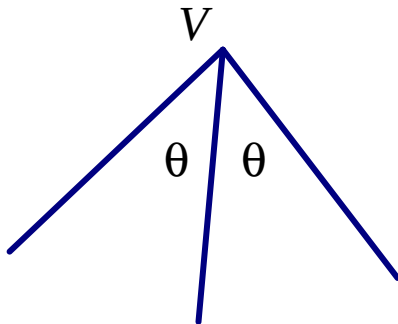
Admissible Interior Angles

- If G' is the reflection of G in an edge containing V , the interior angle of G' at V has measure θ ; inductively, every interior angle at V has measure θ .



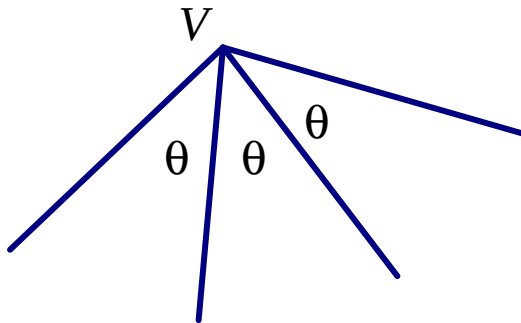
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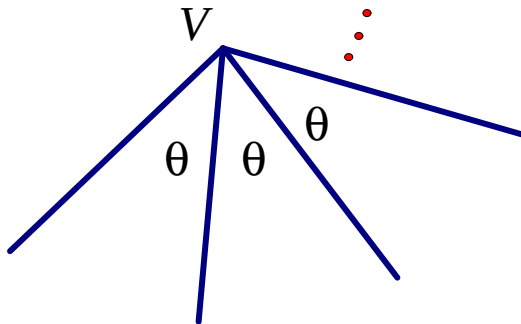
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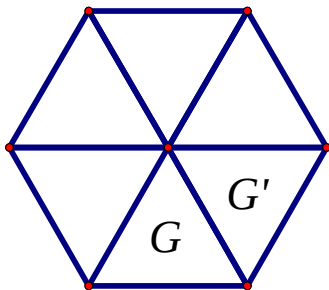
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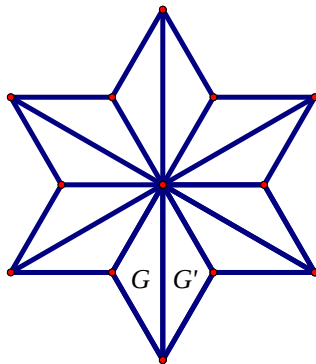
Since successively reflecting in the edges of G that share vertex V is a rotational symmetry, V is an n -center for some $n = 2, 3, 4, 6$.

If G' is the image of G under a rotational symmetry at V , the number k of copies of G sharing vertex V is the order of the rotational subgroup at V . Thus $k = 3, 4, 6$ and $\theta = 60^\circ, 90^\circ, 120^\circ$.



Admissible Interior Angles

Otherwise, the number k of copies of G sharing vertex V is twice the order of the rotational subgroup at V . Thus $k = 4, 6, 8, 12$ and $\theta = 30^\circ, 45^\circ, 60^\circ, 90^\circ$.



Finding Generating Polygons

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- For triangles:

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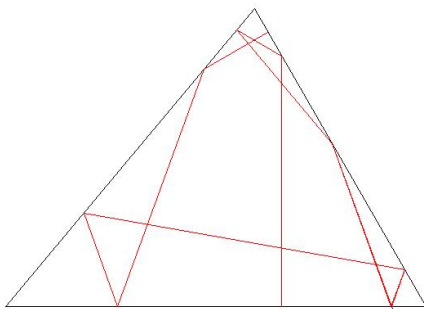
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- Using the combinatorial data, construct all possible n -gons and check which ones tessellate.

Periodic Trajectories and Orbits

- A *periodic orbit* of a billiard ball in motion on a frictionless table Ω bounded by a polygon G is a piecewise-linear constant speed curve $\alpha : \mathbb{R} \rightarrow \Omega$ such that $\alpha(a) = \alpha(b)$ for some $a < b$ and $\alpha'(a+t) = \alpha'(b+t)$ for almost all $t \in \mathbb{R}$.



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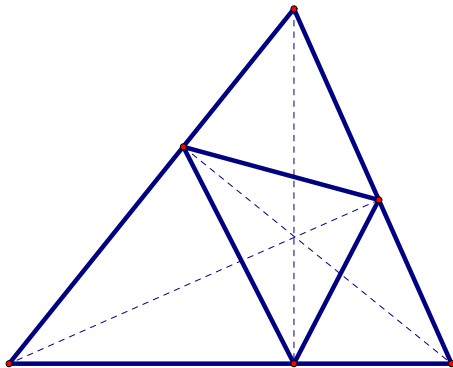
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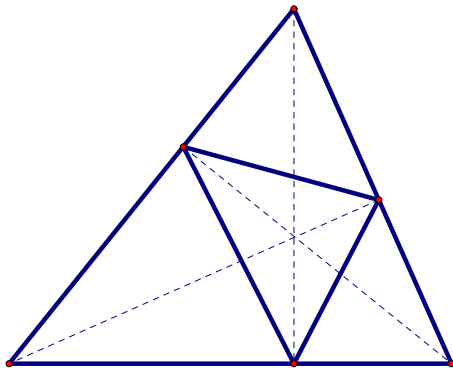
Fagnano's Period 3 Orbit on Acute Triangles

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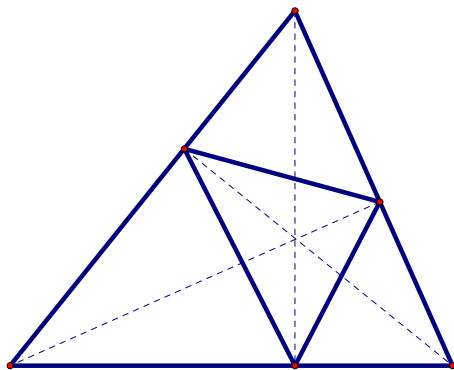
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Fagnano's Period 3 Orbit on Acute Triangles

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- Vertices are feet of the altitudes
- Altitudes bisect the interior angles

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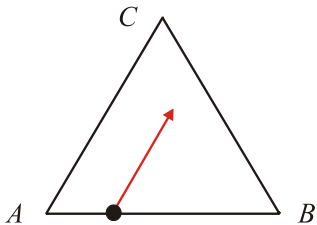
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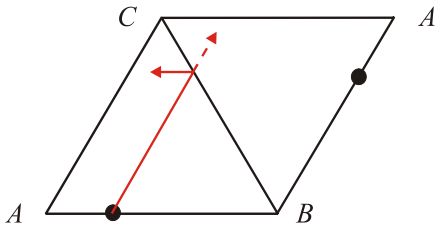
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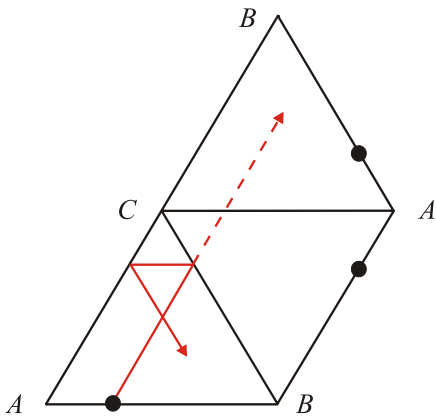
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- Repeat until the ball arrives at some image Q of P with velocity $\vec{v}$

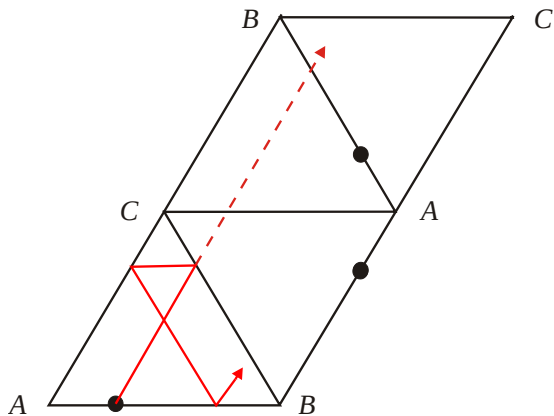
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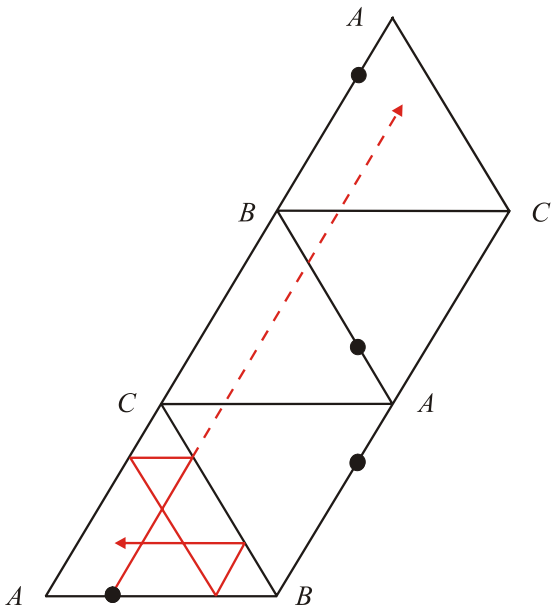
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- $\overline{PQ}$ is an *unfolding* of the orbit

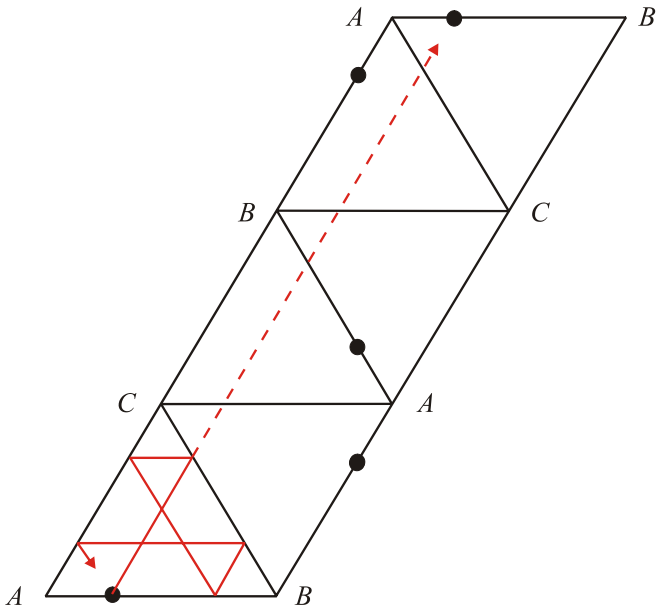


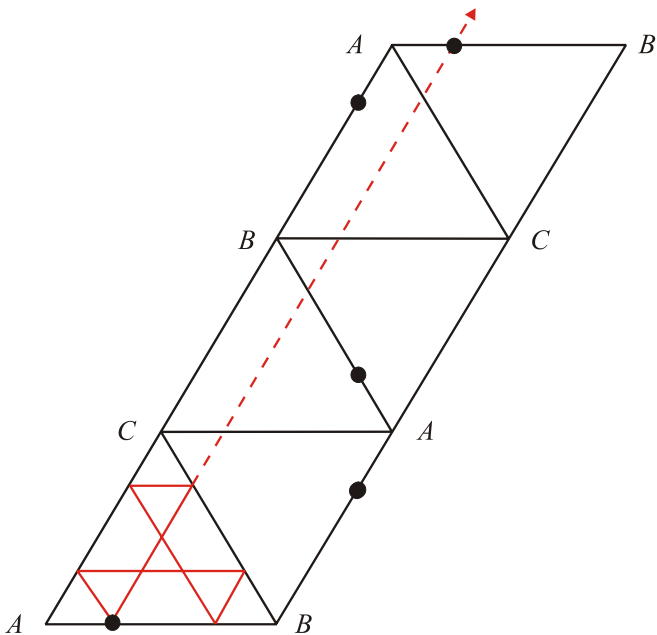






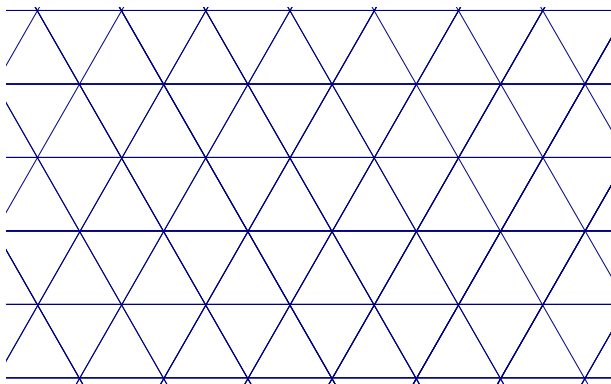






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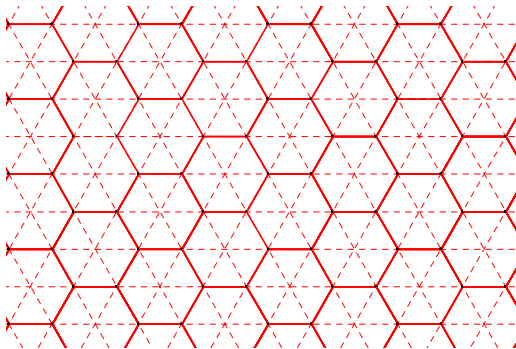
Let $\mathcal{T}$ be the tessellation generated by repeatedly reflecting $\triangle ABC$



Unfolding Orbits on Equilateral Triangles

Let H be the group generated by reflections in the lines of $\mathcal{T}$.

The action of H on an edge of a basic triangle generates a tessellation $\mathcal{H}$ by hexagons.



Unfolding Orbits on Equilateral Triangles

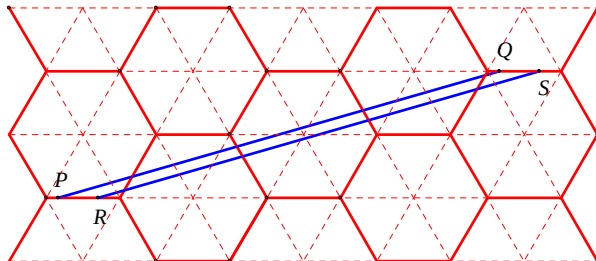
Theorem 2. *Let α be an orbit and let $\overline{PQ}$ be an unfolding. Then*

(i) α has even period iff Q lies on a horizontal edge of $\mathcal{H}$.

(ii) α has odd period iff $\alpha = \gamma^{2k-1}$ for some $k \geq 1$.

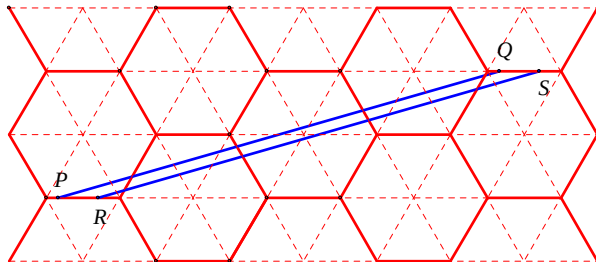
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- The *period of a class* is the period of its elements
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- Since $\mathcal{H}$ has countably many edges, there are countably many even classes

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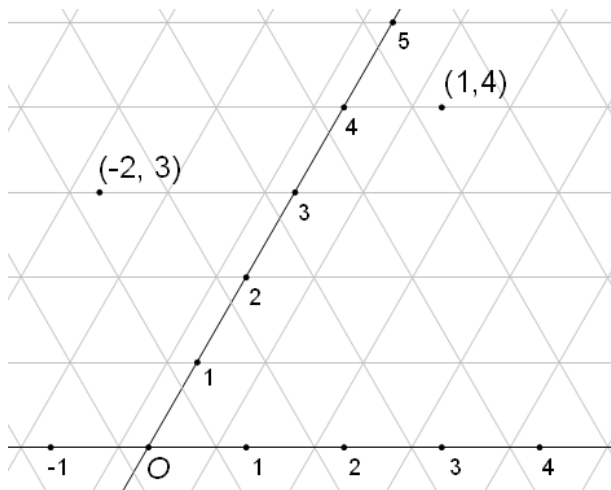
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- The line through O with inclination 60° is the y -axis

Rhombic Coordinates



Periodic Orbits and Rhombic Coordinates

Software for finding periodic orbits:

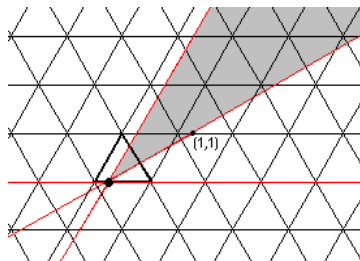
“Orbit Tracer 2” by Stephen Weaver at

http://www.millersville.edu/~rumble/StudentProjects/Weaver/Tessellation_v1.1.exe

The Fundamental Region

Proposition 1. *Every periodic orbit strikes some side of $\triangle ABC$ with an incidence angle in the range $30^\circ \leq \theta \leq 60^\circ$.*

Every periodic orbit has an unfolding in $\Gamma_O = \{(x, y) \mid 0 \leq x \leq y\}$



Classification of Even Orbits

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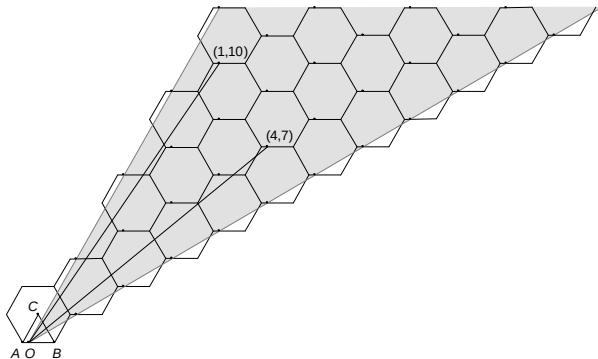
- **Theorem 4. (Classification)**

(i) α has odd period iff $\alpha = \gamma^{2k-1}$ for some $k \geq 1$.

(ii) $\alpha \leftrightarrow (x, y)$ has period $2n$ iff $x \equiv y \pmod{3}$ and $x + y = n$.

Counting Classes of Periodic Orbits

- Distinct classes may have the same period:



Theorem 5. (Counting Formulas)

(i) *The number of classes with period $2n$ is exactly*

$$\mathcal{O}(n) = \left\lfloor \frac{n+2}{2} \right\rfloor - \left\lfloor \frac{n+2}{3} \right\rfloor.$$

(ii) *The number of primitive classes with period $2n$ is exactly*

$$\mathcal{P}(n) = \sum_{d|n} \mu(d) \mathcal{O}(n/d),$$

$$\text{where } \mu(d) = \begin{cases} 1, & d = 1 \\ (-1)^r, & d = p_1 p_2 \cdots p_r \text{ for distinct primes } p_i \\ 0, & \text{otherwise.} \end{cases}$$

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- The only primitive odd orbit is Fagnano's period 3
- There are no primitive orbits of period 8, 12, or 20
- *All* classes of period $2n$ are primitive iff n is prime

Project Proposal

- Given a polygon G that tessellates the plane under reflections, find, classify and count the classes of periodic orbits on G .

Thank you!