Mathematical Research Experiences for Undergraduates at Millersville University

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Millersville University of PA

MAA EPADEL Section Meeting

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- My friend Tom Banchoff calls such problems "found problems"
- Here are three examples:

Found Problem #1:

Tom Banchoff has two daughters Ann and Mary Lynn. Tom was 60 when ML turned 30. A few years later, Ann remarked that Tom's age was twice ML's again! What were their ages the <u>second</u> time?

Ann, Tom, and Mary Lynn

Found Problem #2:

A few years ago, my friend Greg Lapp traveled to Kenya, East Africa, to attend a wedding. While there, he stayed at Tree Tops Nature Preserve very near the equator. Later, when I asked about his trip, Greg remarked that although it was July, there was exactly 12 hour of daylight and 12 hours of darkness every day. And it's the same year 'round. So here's the question ...

Found Problem #2:

Why do people living on the equator have 12 hours of daylight and 12 hours of darkness year round?

Found Problem #3:

About a year ago, my wife and I remodeled our kitchen. At one point I needed some plumbing supplies so I headed for Lowe's. A mile from the store there's a traffic light controlling a conjested intersection, and I arrived just as the light turned red. After what seemed like a very long wait, the light turned green and I went on my way. But I had better luck when returning home. I arrived at the intersection when the light was green, and I continued on my way without stopping. Shortly thereafter, the following question occurred to me...

Found Problem #3:

A traffic light is red for 48 seconds and green for 16. What is the average wait time?

Linear algebra problem solved by Trina Bishop-Armstrong

- Consider a vector space V with a direct sum decomposition

$$V = V_0 \oplus V_1 \oplus V_2 \oplus \cdots$$

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- A *differential* on V is a linear map $d = d_1 + d_2 + \cdots$ where

$$V_0 \xleftarrow{d_1} V_1 \xleftarrow{d_2} V_2 \xleftarrow{d_3} \cdots \quad \text{and} \quad d_{i-1} \circ d_i = 0$$

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- A *deformation* of d is a differential of form $d + p_1 + p_2 + \cdots$

- **Problem:** *Find and classify the deformations of a given d*

Trina's theorem:

There is a vector space (V, d) with an infinite family of non-equivalent deformations $\{d + p_{1,n} + \cdots + p_{n,n} \mid n \geq 1\}$

Trina Bishop-Armstrong

Trina's results appeared in the Proc. AMS (2001)

"Obstructions to Deformations of DG Modules"

Project proposed by and codirected with Frank Morgan

Problem: *Find all minimal paths connecting two points on a tin can*

Tin can team:

Painter, Panofsky, Mohler, Hair-Armstrong, Umble

Tin can team's theorem:

Minimal paths connecting two points A and B on a tin can consist of at most three piece-wise smooth components, each a classical geodesic. If the number of minimal paths connecting A and B is $n < \infty$, then $n \leq 4$.

"Geodesics on a Tin Can" by Robert Painter

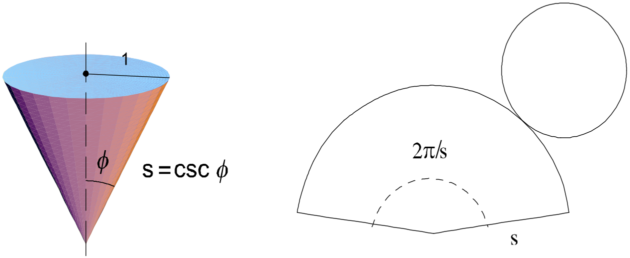
A MathFest 2001 award for her outstanding presentation

Ellen Panofsky

Ellen writes:

"My initial research experience as part of our group of 4 really showed me what it meant to do research in mathematics. As a student, I really did not like writing research papers (and I still don't). I might have said no if you had asked me to help with a research project. Instead you asked if I'd like to work on a problem given to you by Frank Morgan. Before I knew it, we were doing research, and traveling to conferences to present. This experience, followed by research in graph theory for my senior thesis, left me very well prepared for graduate research."

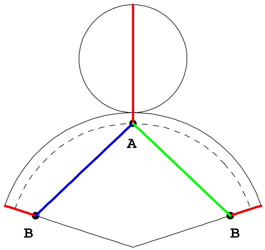
Joel extended the solution to conical cups with lid



Flat model of a conical cup with lid

Joel's conical cup theorem:

Minimal paths connecting two points A and B on a conical cup with lid consist of at most three piece-wise smooth components, each a classical geodesic. If the number of minimal paths connecting A and B is $n < \infty$, then $n \leq 3$.



Three minimal paths from A to B

Joel's poster

Thesis wins EPADEL's 2003 student paper competition

Crannell, Mohler, Umble

Joel's thesis appeared in the PME Journal (2004)

"Minimal Paths on Some Simple Surfaces with Singularities"

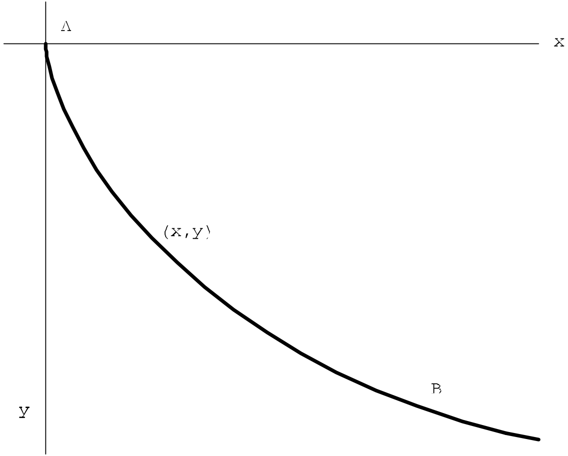
Joel writes:

"Doing mathematical research greatly increased my awareness of the global scope of knowledge. I've greatly enjoyed working with a diverse group of people and building on the work of people through-out the world. This heightened awareness, which was launched during my undergraduate research project, raised my career aspirations and helps to guide the career choices I'm making even now."

Project proposed by and codirected with Mike Nolan

Generalized Brachistochrone Problem: *Consider two points A and B on a smooth frictionless surface S in a (not-necessarily uniform) gravitational field. Find a path from A to B along which a particle released from A reaches B in minimal time.*

Newton's planar solution in a uniform gravitational field

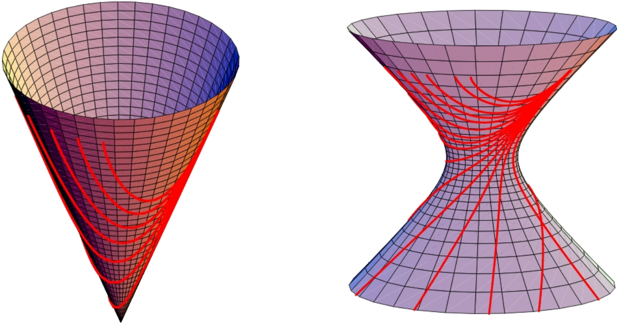


A cycloid

"Animated cycloid" by Robert Painter

John's Theorem:

Solution curves on a large family of surfaces in various gravitational fields have explicit formulations.



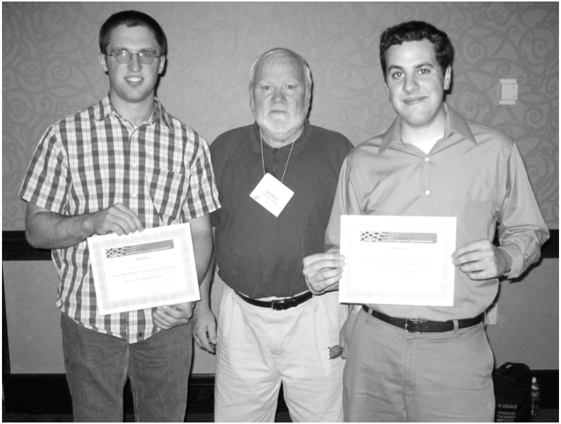
Some solution curves on a cone and hyperboloid

John's poster

John presenting at MathFest 2005

Co-winner SIAM's 2005 prize for best student paper in applied math

John Gemmer and Donald Miller



Andrew Harrell, Donald Miller and John Gemmer

John's thesis appeared in the PME Journal (2011)

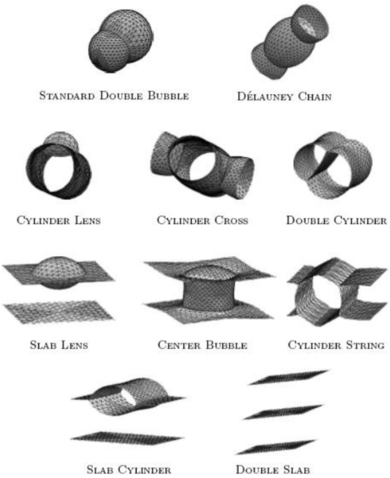
"Generalizations of the Brachistochrone Problem"

John writes:

"Students who want to succeed in graduate school absolutely should participate in an undergraduate research project. My thesis project drastically improved my ability to write mathematics clearly and precisely, taught me how to give clear and concise presentations, introduced me to some tools of the mathematical trade such as Mathematica, Matlab and LaTeX, and taught me how to work independently–a skill that's expected of all entering graduate students."

Project proposed by and codirected with Frank Morgan

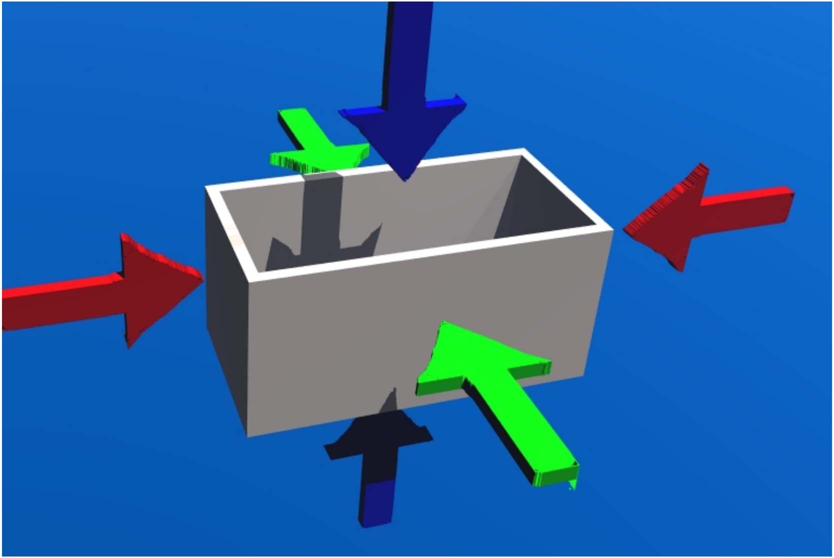
Problem: *Establish physical existence of the following conjectured surface-area minimizing double bubble configurations in a 3-torus*



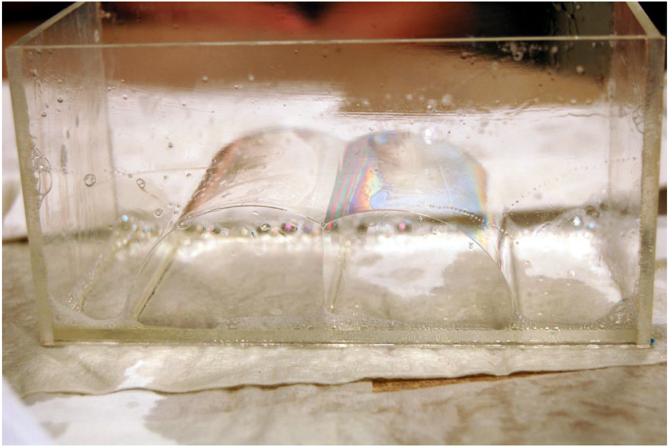
Bubble team:

Evans, Brubaker, Carter,Linn, Peurifoy, Kravatz

A model of the 3-dimensional torus



The double cylinder



Dan Kravatz presenting at MathFest 2007

Bubble team's theorem

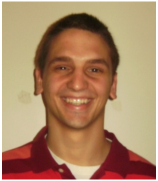
There are at least ten surface-area minimizing double bubble configurations in a 3-torus

Double Bubble Poster

Double bubble results appeared in Math Horizons (2008)

"Double Bubble Experiments in the 3-Torus"

Nick writes:



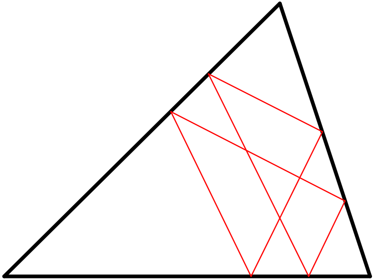
"A lot of graduate students, who haven't had a research experience as an undergraduate, don't know if they can or even want to do research mathematics. My research experiences at Millersville convinced me that I can and want to do research. This helped me focus on research early in my graduate career, and was immensely helpful when applying for my NSF fellowship."

Project proposed by Dennis DeTurck

Codirected with Dennis DeTurck and Zhoude Shao

Problem: *Find and describe the periodic trajectories (orbits) of a billiard ball in motion on a triangular billiard table*

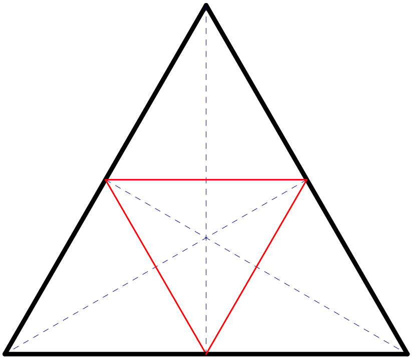
Billiards team I: Baxter, Gemmer, Gerhart, Laverty, Weaver



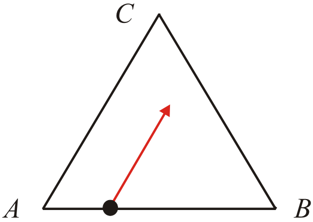
- On certain scalene obtuse triangles, existence is unknown

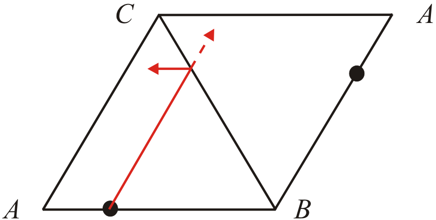
Billiards team's theorem:

Every acute, right, or isosceles triangle admits a periodic orbit. Equilateral triangles admit infinitely many distinct periodic orbits of even period; orbits of odd period are odd multiples of the orthic orbit.



Reflecting an equilateral triangle in its edges unfolds an orbit...

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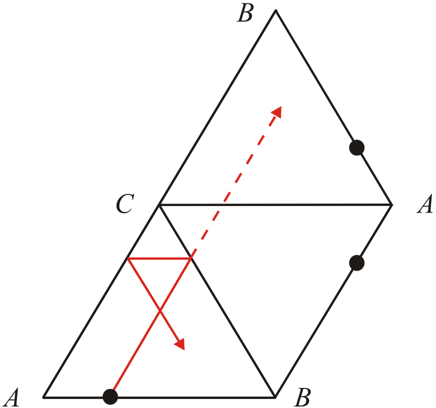
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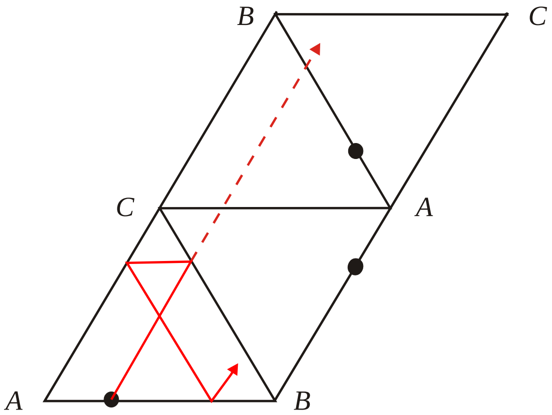
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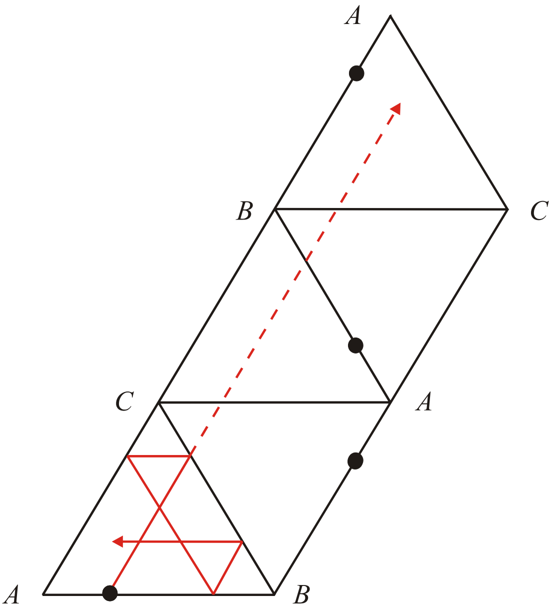
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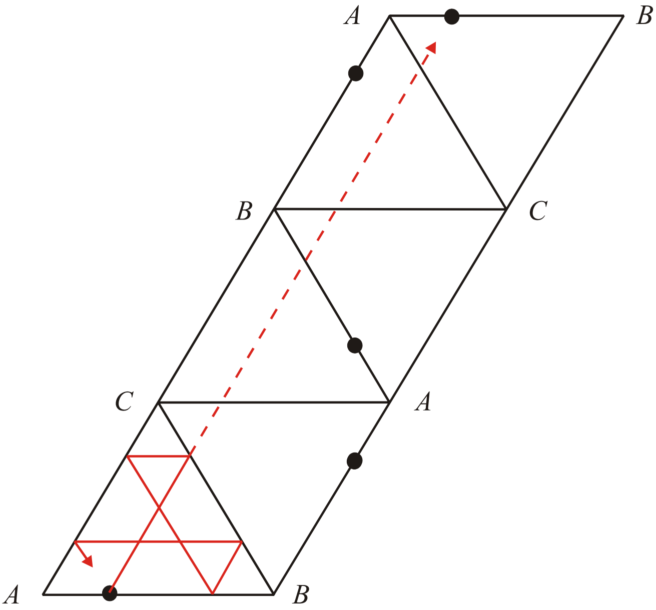
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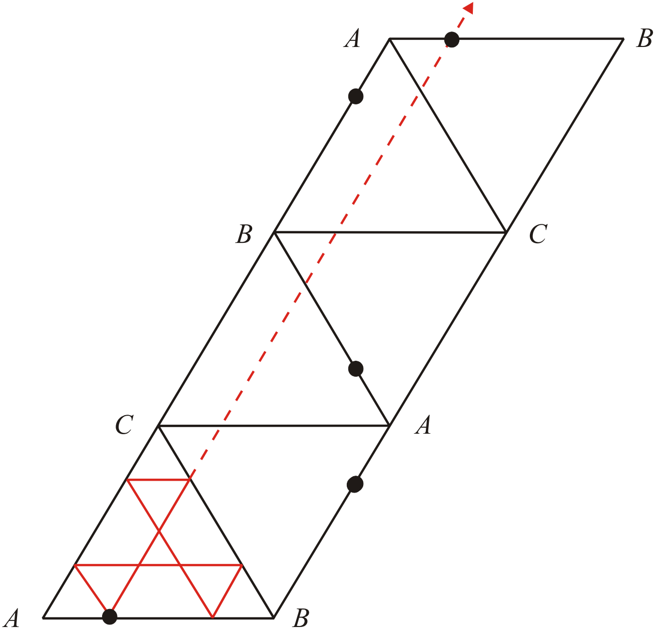
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Software and poster

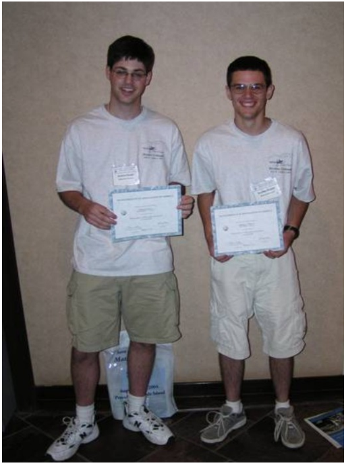
"Unfolding Orbits in a Tessellation" by Steve Weaver

Billiards team's poster

Sean Laverty writes:

"Thanks to the support and encouragement of Millersville faculty, I participated in two NSF REUs, one of which led to my senior thesis project in mathematical biology. Completing a senior thesis laid the groundwork for independent research, and gave me experience in scientific computing, writing, and presenting. Since graduate level research involves a great deal of communication with researchers within and outside my particular field, the experience I had presenting my work in local and regional meetings was invaluable."

A MathFest 2003 award for their outstanding presentation



Andrew Baxter and Steve Weaver

EPADEL's 2005 student paper competition winner

Umble, Sevilla, Baxter, DeTurck

Andrew's thesis appeared in the Monthly (2008)

"Periodic Orbits for Billiards on an Equilateral Triangle"

Andrew writes:

"The opportunity to do undergraduate research helped me realize how different research is from coursework. It's freeing to have the option to modify the problem and make it more tractable. You gave me the freedom to investigate whichever facets of the billiards problem I wanted to."

Project to extend our billiards results to other polygons

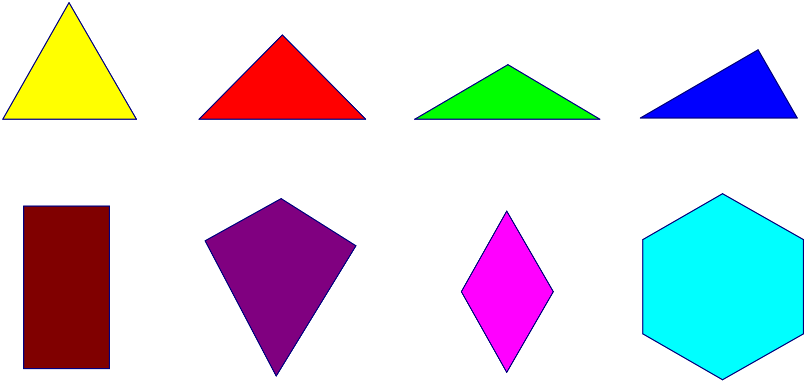
Problem: *Which polygons tessellate the plane when reflected in their edges?*

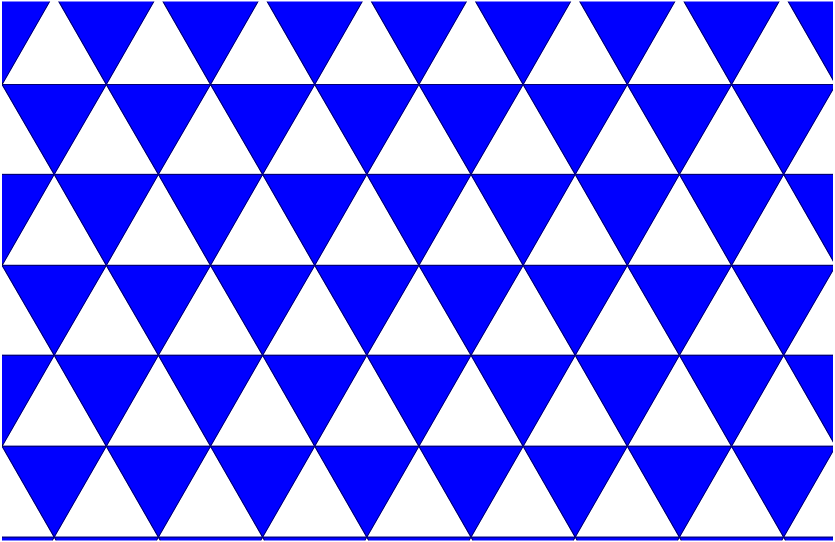
Tessellation team:

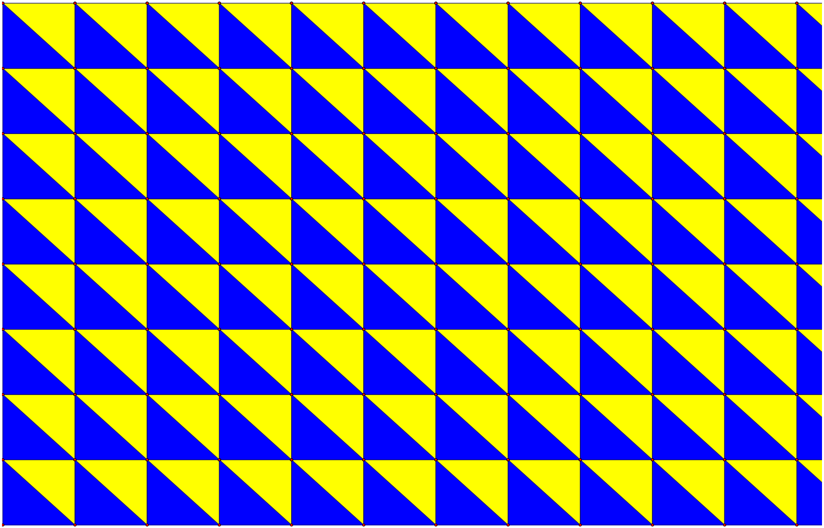
Matthew Kirby, Josh York, Andrew Hall

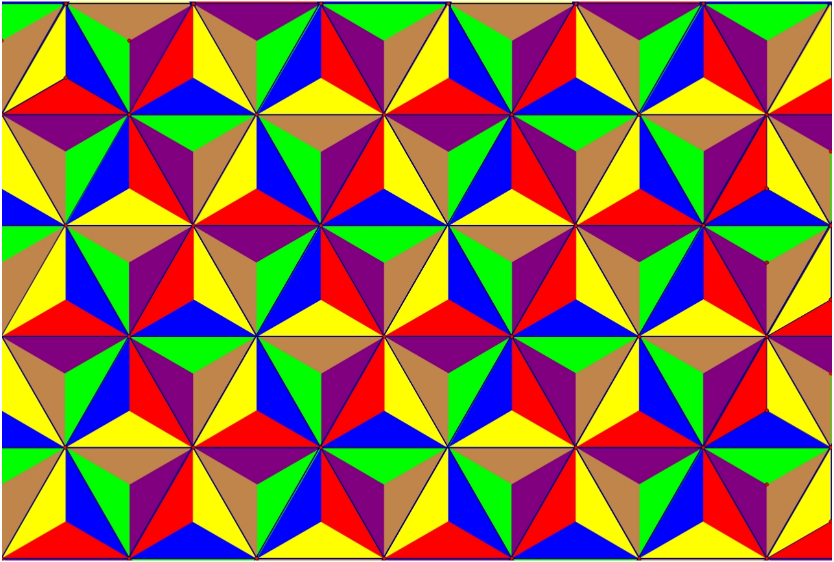
Tessellation team's theorem:

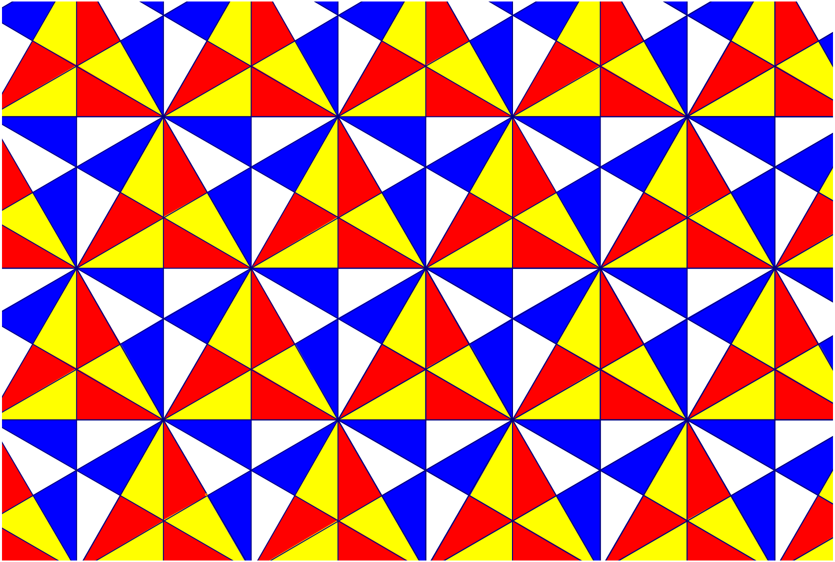
Exactly eight polygons generate edge tessellations:

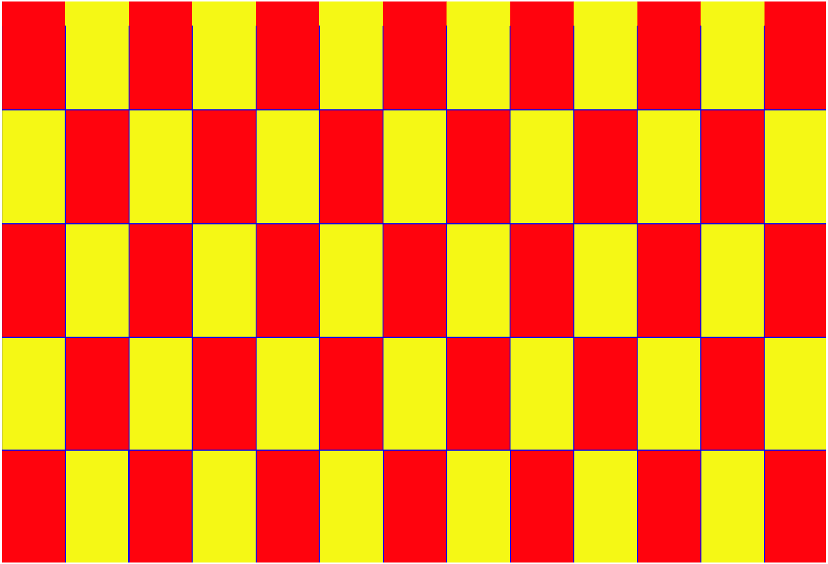


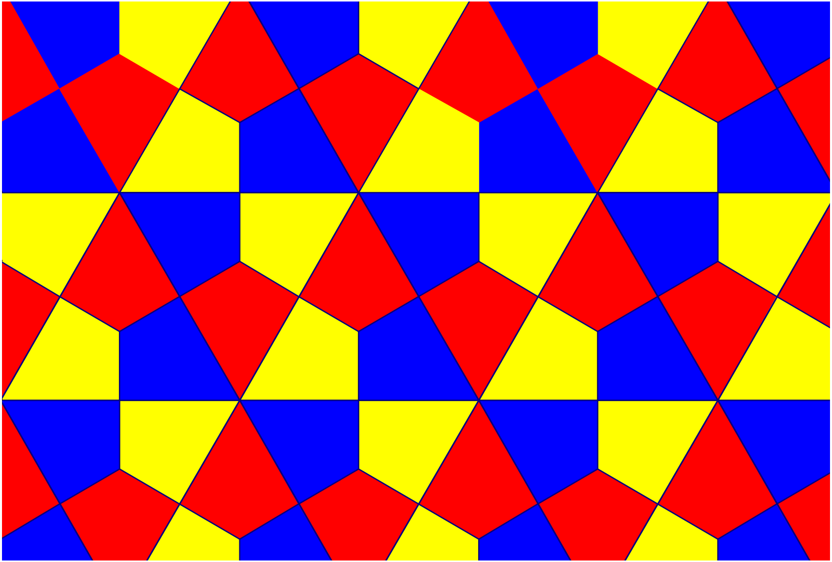


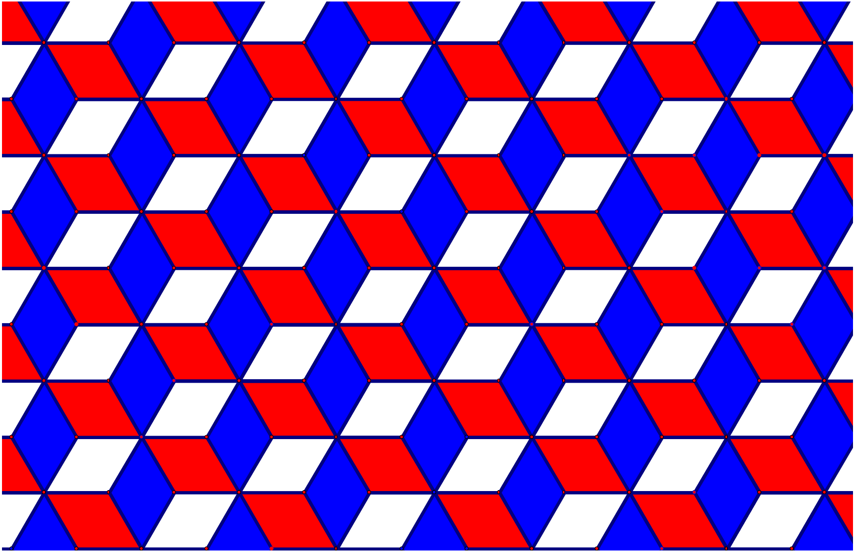


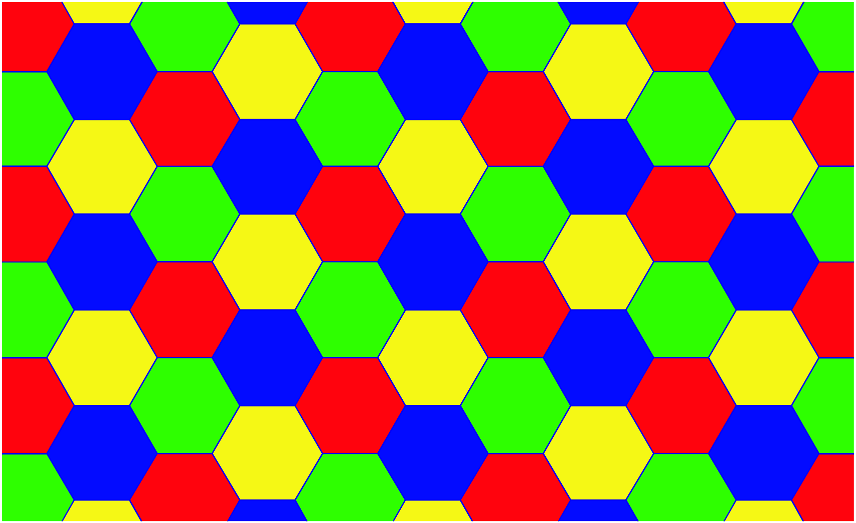






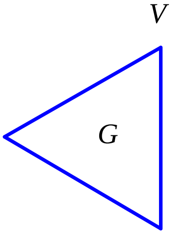






Tessellation team's proof

- Let V be a vertex of a generating polygon G



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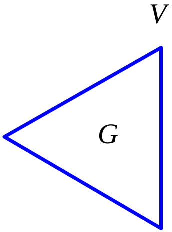

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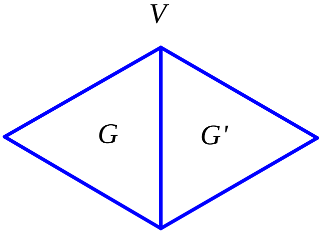

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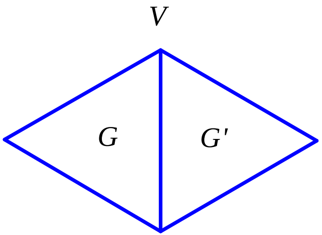
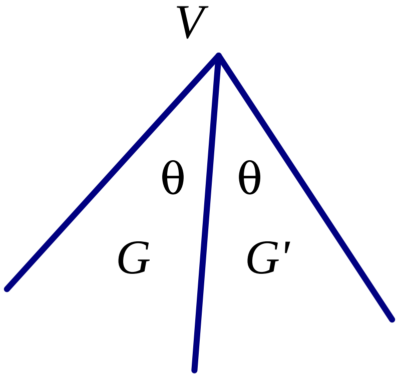


- Let G' be the reflection of G in an edge containing vertex V
- Interior angles at V measure $< 180^\circ$

Determining admissible interior angles

Interior angles at V are congruent



Determining admissible interior angles

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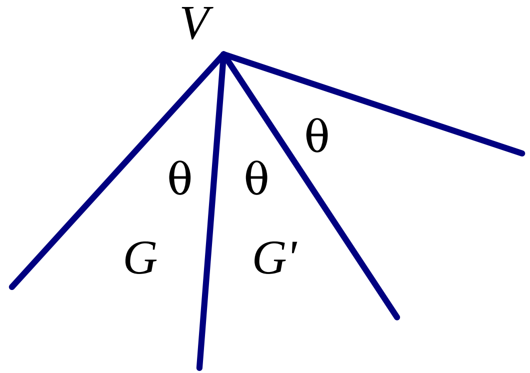

Determining admissible interior angles

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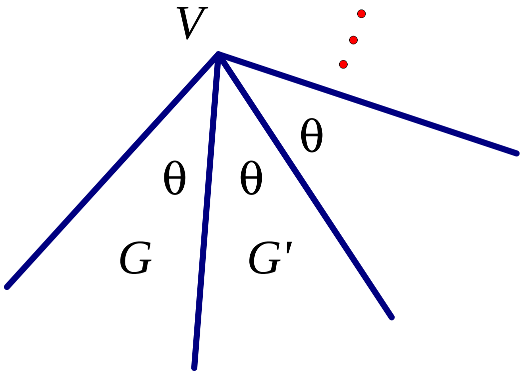
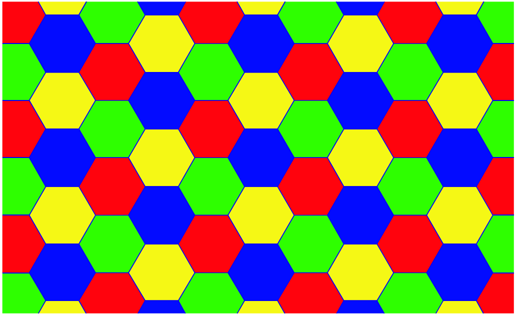


Determining admissible interior angles

- A point P in a tessellation is an *n-center* if exactly n rotational symmetries have center P

Determining admissible interior angles

- A point P in a tessellation is an *n-center* if exactly n rotational symmetries have center P
- The vertices of a honeycomb tessellation are 3-centers



Determining admissible interior angles

- **Crystallographic Restriction:**

  If P is an n-center, $n = 2, 3, 4, 6$

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- Reflecting in adjacent edges sharing V is a *rotational symmetry*

Determining admissible interior angles

- **Crystallographic Restriction:**

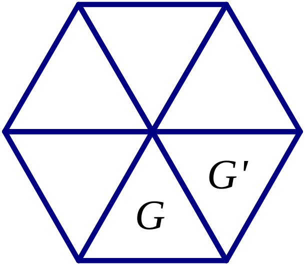
  If P is an n-center, $n = 2, 3, 4, 6$

- Reflecting in adjacent edges sharing V is a *rotational symmetry*

- *V is an n-center!*

Determining admissible interior angles – Case 1:

- G' is the rotational image of G about the n-center V



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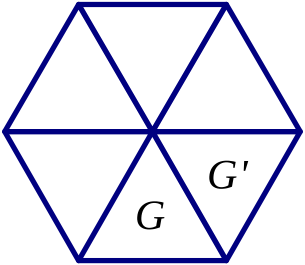

  - 3, 4, or 6 copies of G share vertex V

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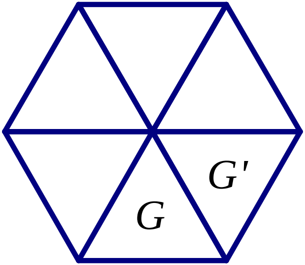
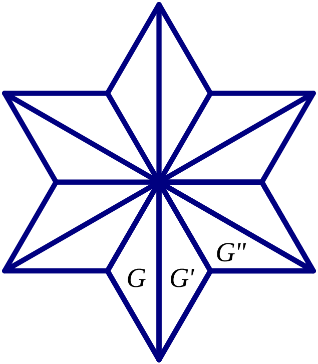


  - 3, 4, or 6 copies of G share vertex V
  - $m\angle V = 120^\circ, 90^\circ, 60^\circ$

Determining admissible interior angles – Case 2:

- G'' is the rotational image of G about the n-center V



Determining admissible interior angles – Case 2:

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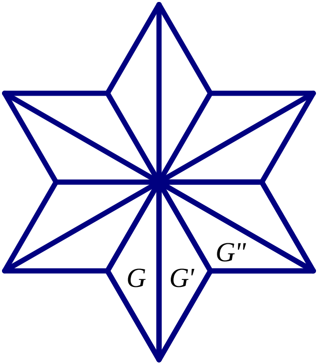
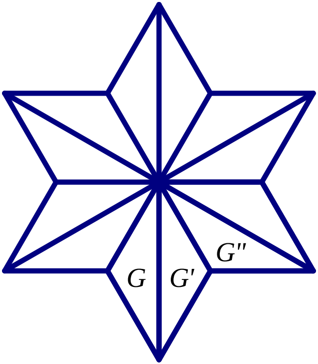


  - 4, 6, 8, or 12 copies of G share vertex V

Determining admissible interior angles – Case 2:

- G'' is the rotational image of G about the n-center V



  - 4, 6, 8, or 12 copies of G share vertex V
  - $m\angle V = 90^\circ, 60^\circ, 45^\circ, 30^\circ$

Conclusion–

The admissible interior angles of an edge tessellating polygon are

$$30^\circ, 45^\circ, 60^\circ, 90^\circ, 120^\circ$$

To identify the edge tessellating polygons–

- Identify all polygons with admissible interior angles

To identify the edge tessellating polygons–

- Identify all polygons with admissible interior angles
- Check which ones generate an edge tessellation

To identify the edge tessellating polygons–

- Identify all polygons with admissible interior angles
- Check which ones generate an edge tessellation
- For example...

Interior angles of edge tessellating triangles

are solutions of

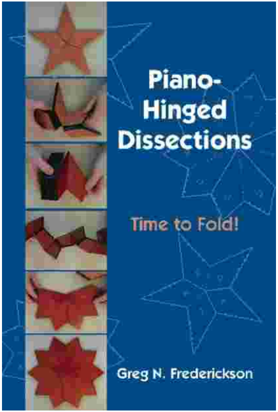
$$\left\{\begin{array}{ccccccccccc} 30a & + & 45b & + & 60c & + & 90d & + & 120e & = & 180 \\ a & + & b & + & c & + & d & + & e & = & 3 \end{array}\right.$$

$$\implies \left\{\begin{array}{l} a = -3 + c + 3d + 5e \\ b = 6 - 2c - 4d - 6e \end{array}\right.$$

(c, d, e) :	$(3, 0, 0)$	$(1, 1, 0)$	$(0, 1, 0)$	$(0, 0, 1)$
30°	0	1	0	2
45°	0	0	2	0
60°	3	1	0	0
90°	0	1	1	0
120°	0	0	0	1

Fredrickson's Conjecture:

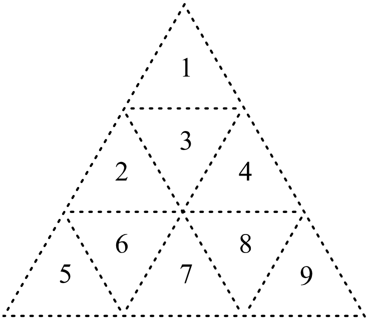
"Although triangular stamps have come in a variety of different triangular shapes, only three shapes seem suitable for [stamp] folding puzzles: equilateral, isosceles right triangles, and 60°-right triangles." See p. 144 of:



Stamp Folding Puzzle #1

Fold this block of equilateral triangular stamps into a packet 9-deep with stamps in the following order:

2 6 7 5 9 3 4 1 8



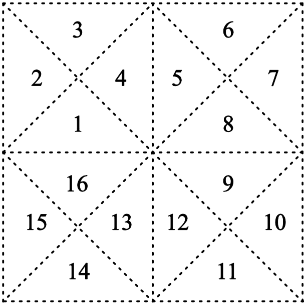
<u>Hint:</u> *Tuck 5 between 7 and 9*

<u>Solution:</u> *Fredrickson p 143*

Stamp Folding Puzzle #2

Fold this block of isosceles right triangular stamps into a packet 16-deep with stamps in the following order:

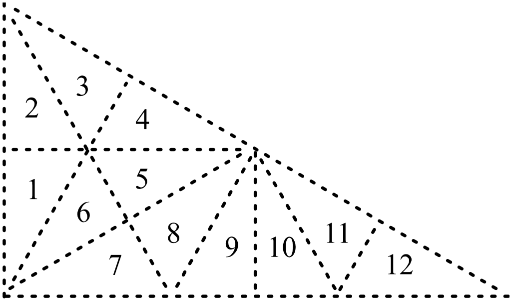
4 1 16 6 5 15 14 8 7 13 11 12 2 3 9 10



Stamp Folding Puzzle #3

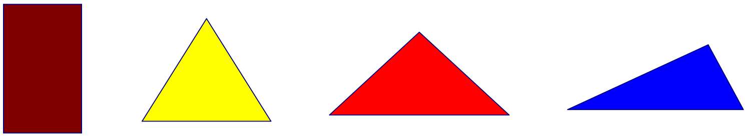
Fold this block of 60°-right triangular stamps into a packet 12-deep with stamps in the following order:

5 2 8 9 7 3 4 11 12 1 6 10



HYKU Corollary

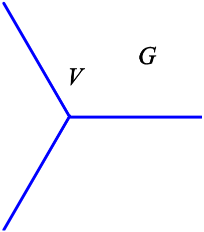
Exactly four shapes are suitable for stamp folding puzzles:



Proof of Fredrickson's Conjecture

If the angle at vertex V of an edge tessellating polygon G is *obtuse*

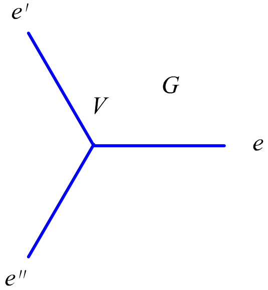
- $m\angle V = 120^\circ$ and three copies of G share vertex V



Proof of Fredrickson's Conjecture

If an edge tessellating polygon G has an obtuse angle at vertex V

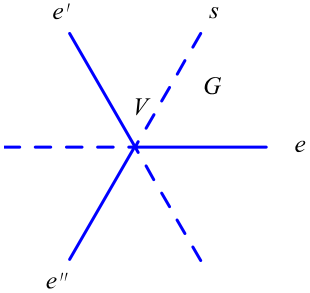
- $m\angle V = 120^\circ$ and three copies of G share vertex V



- Label the edges that meet at V by e, e', and e'' as shown

Proof of Fredrickson's Conjecture

- e'' is collinear with bisector s of $\angle V$



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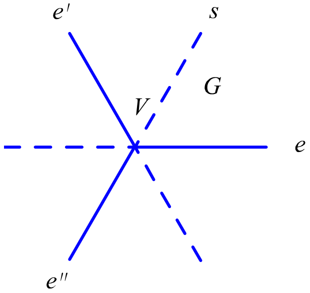
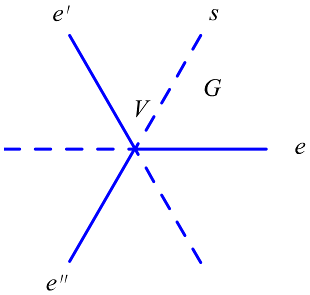


- e' is the reflection of e in bisector s

Proof of Fredrickson's Conjecture

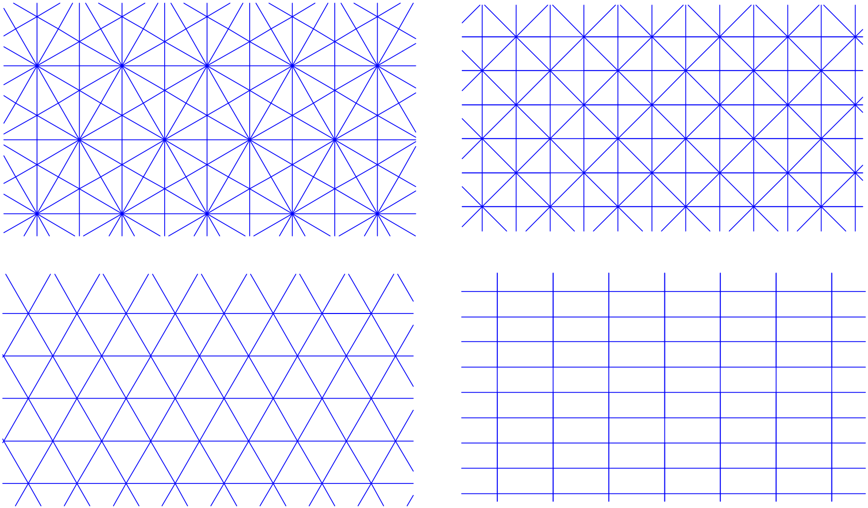
- e'' is collinear with bisector s of $\angle V$



- e' is the reflection of e in bisector s
- Folding along e'' creases a stamp, which is not allowed

Conclusion:

Only **non-obtuse** *polygons are suitable for stamp folding puzzles*

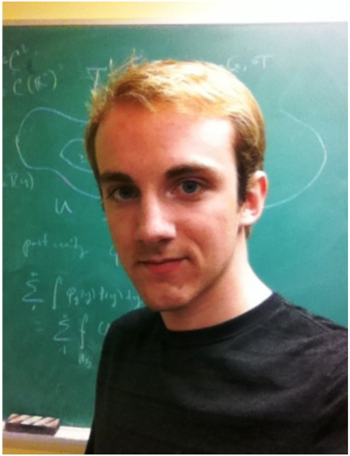


Results appeared in the Math Magazine (2011)

"Edge Tessellations and Stamp Folding Puzzles"

Non-obtuse edge tessellating polygons analyzed

Rutgers REU directed by Dr. Andrew Baxter, summer 2011

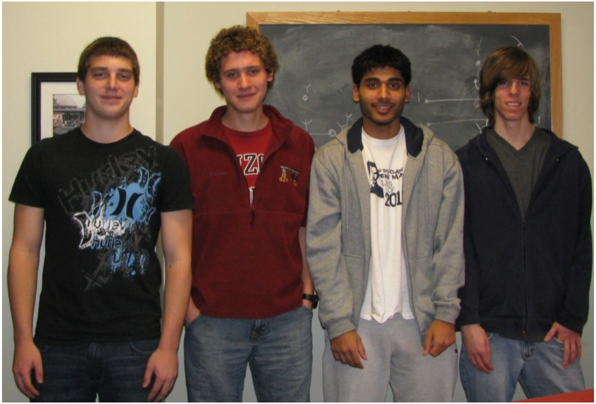


Jonathan Eskreis-Winkler and Ethan McCarthy

Analysis of obtuse edge tessellating polygons in progress

Project codirected with Zhigang Han

Billiards team II:



Pavoncello, Baer, Gilani, and Brown

Open Problems for Undergraduates

- Find the minimal paths connecting two points on an "ice cream cone".

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- Find the minimal paths connecting two points on general piece-wise smooth surfaces.

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- Describe the periodic orbits on obtuse edge tessellating polygons.

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- Find the minimal paths connecting two points on general piece-wise smooth surfaces.
- Solve the Brachistochrone Problem on general piece-wise smooth surfaces.
- Describe the periodic orbits on obtuse edge tessellating polygons.
- Describe the periodic orbits on a general isosceles triangle.

  "Orbits on a General Triangle" by Steve Weaver

Found Problem #1 Solution

- Mary Lynn turned 30 while Tom was 60.

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- Mary Lynn turned 30 while Tom was 60.
- Tom turned 61 while ML was 30.

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- *Tom turned 62 while ML was 31!*

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- Mary Lynn turned 30 while Tom was 60.
- Tom turned 61 while ML was 30.
- ML turned 31 while Tom was 61.
- *Tom turned 62 while ML was 31!*

- **Remark:** If person A is more than a year older than person B, and their birthdays fall on different days of the year, the age of A will be twice the age of B *twice*.

Found Problem #2 Solution:

- The great circle separating day and night intersects the equator at diametrically opposite points.

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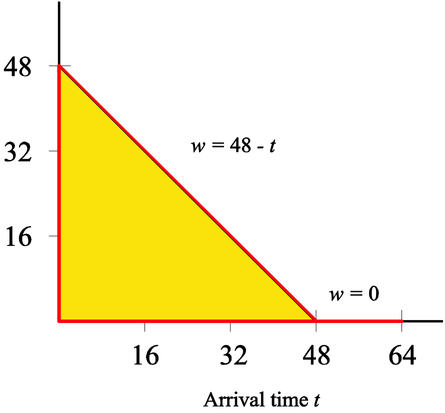
- The great circle separating day and night intersects the equator at diametrically opposite points.
- *Thus at each instant, exactly half of the equator is light and half is dark.*

<u>**Animation of rotating earth**</u>

Found Problem #3 Solution:

- Given an arrival time t, the wait time w is



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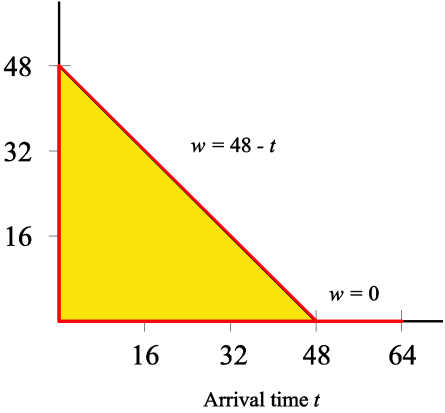

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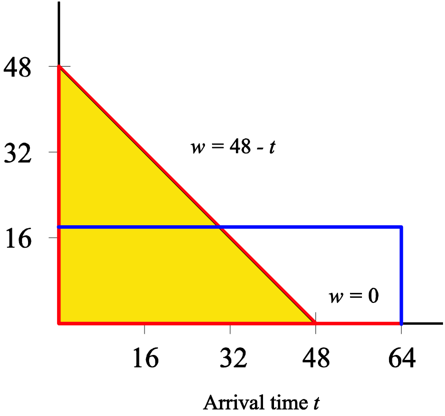

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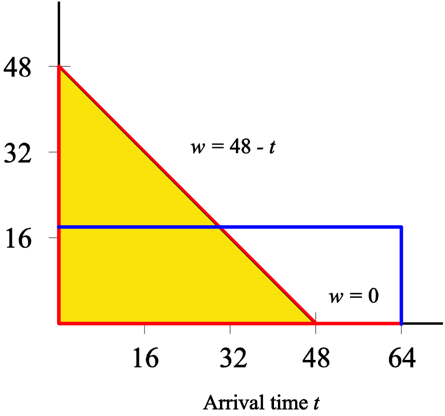

- Average wait time is the height of the rectangle with length 64 and area $48^2/2 = 1152 = 64 \cdot 18$
- *Wow! Only 18 seconds!!*

A lesson for life–

Cheer up!!

Things are often better than they seem!

HAPPY RESEARCHING!