Tensor Products of A_∞-algebras with Homotopy IPs (Joint work with Thomas Tradler)

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AMS Special Session on Physically Inspired Homotopy Algebra

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- The S-U diagonal on cellular chains

$$\Delta_K : C_*K \to C_*K \otimes C_*K$$

induces an A_∞-algebra structure on $A \otimes B$ as follows:

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- Operadic representations of A_∞-algebra structures

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$$\xi_n : C_* K \to Hom\left(B^{\otimes n}, B\right)$$

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- Define the representation

$$\begin{array}{ccc} C_*K & \overset{\varphi_n}{- - \rightarrow} & Hom\left((A \otimes B)^{\otimes n}, A \otimes B\right) \\ \Delta_K \downarrow & & \uparrow \approx \\ C_*K \otimes C_*K & \overset{\zeta_n \otimes \xi_n}{\longrightarrow} & Hom\left(A^{\otimes n}, A\right) \otimes Hom\left(B^{\otimes n}, B\right) \end{array}$$

Cyclic A-infinity Algebras

- A *cyclic* A_∞-algebra (A, μ) is equipped with a cyclically invariant inner product:

$$\langle \mu_n (a_1, \ldots, a_n), a_{n+1} \rangle = \langle \mu_n (a_2, \ldots, a_{n+1}), a_1 \rangle$$

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- Given cyclic A_∞-algebras A and B, consider the A_∞-algebra $(A \otimes B, \varphi)$ and the inner product

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- Is $\langle -,-\rangle_{A\otimes B}$ cyclically invariant?

Tensor Product of Cyclic A-infinity Algebras

- The differential and product are cyclically invariant:

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- There is a chain homotopy $\varrho_{2,0} : (A \otimes B)^{\otimes 4} \to R$ such that

$$\left(\varrho_{2,0} \circ \varphi_1\right) (a|b, c|d, e|f, g|h) =$$

$$\langle \varphi_3 (a|b, c|d, e|f), g|h \rangle - \langle \varphi_3 (c|d, e|f, g|h), a|b \rangle$$

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- We construct a diagonal Δ_C on cellular chains of pairahedra
- Δ_C induces a homotopy inner product (HIP) structure on $A \otimes B$

A-infinity Algebras with HIPs

- An A_∞-*algebra* A *with HIPs* is equipped with "compatible" module maps
$$\lambda_{j,k} : A^{\otimes j} \otimes A \otimes A^{\otimes k} \longrightarrow A$$
and HIPs
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- There is a cubical subdivision Q_*A as in B-V's W-construction

- Serre's diagonal on I^n induces a coassociative diagonal
$$\Delta_Q : Q_*\mathcal{A} \to Q_*\mathcal{A} \otimes Q_*\mathcal{A}$$

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- Δ_Q induces a non-coassociative diagonal

$$\Delta_C : C_*\mathcal{A} \xrightarrow{q} Q_*\mathcal{A} \xrightarrow{\Delta_Q} Q_*\mathcal{A} \otimes Q_*\mathcal{A} \xrightarrow{p \otimes p} C_*\mathcal{A} \otimes C_*\mathcal{A}$$

Tensor Product of HIP Structures

- Operadic representations of HIP structures

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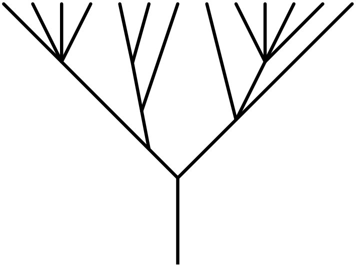
The 3-colored operad CA

$C_*\mathcal{A}$ is genrated by three types of planar diagrams

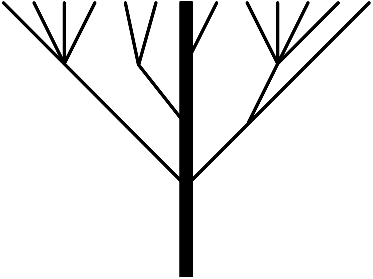
Colors: Empty, thin, thick

1. **Planar trees:** Control A_∞-algebra structure
   - *Thin leaves and root*

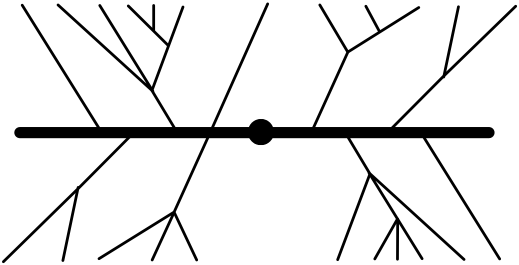


The 3-colored operad CA

2. **Module trees:** Control homotopy bimodule structure
   - *Thick vertical root and leaf*
   - *j thin leaves in left half-plane*
   - *k thin leaves in right half-plane*



3. **Inner product diagrams:** Control HIP structure
   - *Empty root and two thick horizontal leaves*
   - *j thin leaves in upper half-plane*
   - *k thin leaves in lower half-plane*



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- Two inner product diagrams cannot be composed

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- **Boundary:** $\partial_C(D) := \sum_{D'/e=D} D'$

  where e is an edge of D'

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- The **coloring** of a diagram D with n leaves is a pair

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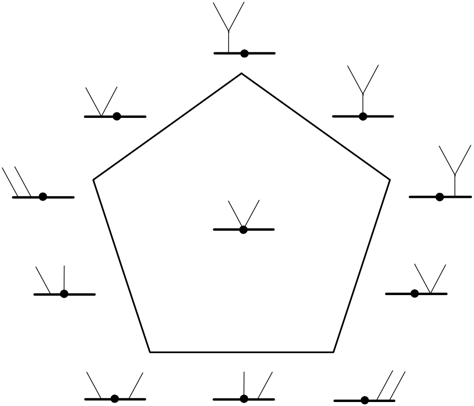
- $C_*\mathrm{A}_y^{\mathbf{x}}$ is generated by diagrams of coloring $\mathbf{x} \times y$
- **Example**: $C_*\mathrm{A}_1^{11\cdots1}$ is generated by planar trees

Example

$C_*\mathrm{A}_0^{1122}$ is generated by faces of the pairahedron $I_{2,0}$:



The 3-Colored Operad QA

- Q_*A is generated by all *metric diagrams* (D, g), where
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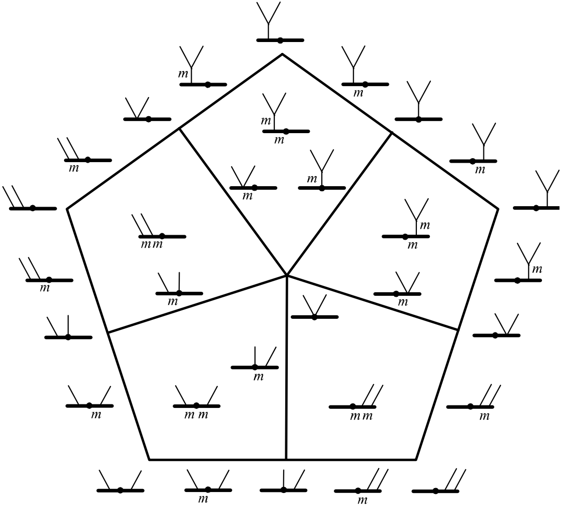
- **Boundary:** $\partial_Q(D) := \sum_{\text{metric } e \subset D} D/e + D_e$

  where D_e is obtained from D by relabeling e non-metric

Example

The cubical subdivision of $I_{2,0}$ (metric labels displayed):



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- q is a chain map

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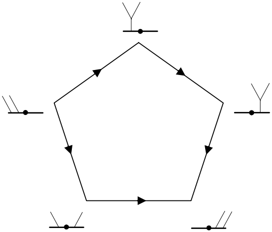
$\mathcal{B}_{I_{2,0}}$:



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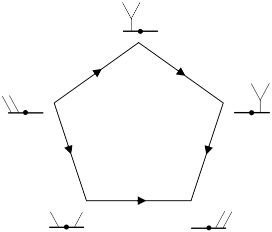

- $\mathcal{B}_D$ has minimal element $D_{\min}$ and maximal element $D_{\max}$

The 2-Sided Homotopy Inverse $p : QA \longrightarrow CA$

- Define p on a *fully metric* diagram $(D, m) \in Q_k\mathcal{A}$ by

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- $pq = \mathrm{Id}$ and $qp \simeq \mathrm{Id}$

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- Define

$$\Delta_Q(D) = \sum_{X \subseteq \{\text{metric edges of } D\}} D/X \otimes D_{\bar{X}}$$

The Diagonal *CA* —> *CA* x *CA*

- $\Delta_C : C_*\mathcal{A} \xrightarrow{q} Q_*\mathcal{A} \xrightarrow{\Delta_Q} Q_*\mathcal{A} \otimes Q_*\mathcal{A} \xrightarrow{p \otimes p} C_*\mathcal{A} \otimes C_*\mathcal{A}$

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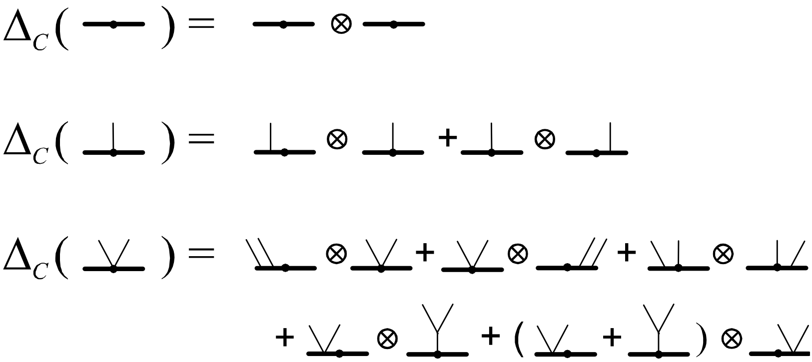
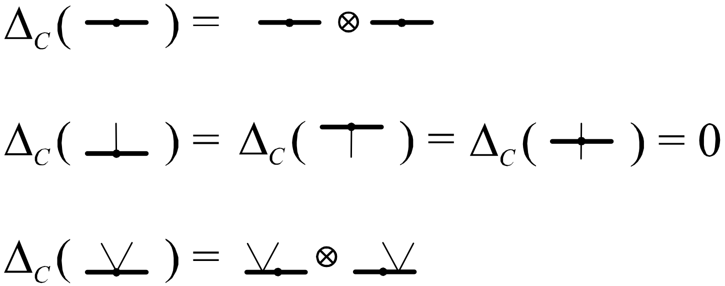
- Let $C_n = C_n\mathcal{A}_y^{\mathbf{x}}$. On a corolla $c \in C_k$ we have

$$\Delta_C(c) = \sum_{\substack{S\otimes T \in C_i \otimes C_j \\ S_{\max} \leq T_{\min} \\ i+j=k}} S \otimes T$$

Example



The Cyclic Case



The End

Thank you!