The Coherent Framed Join and Biassociahedra

Joint work with Samson Saneblidze

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Abstract

We construct the coherent framed join of finite ordered sets, form the reduced coherent framed join as an appropriate quotient, and use the combinatorics of the reduced framed join to construct the biassociahedra $KK_{n,m}$. In the ranges $1 \leq m \leq 3$ and $1 \leq n \leq 3$, which are sufficient for our applications, the cellular chain complex $C_* \left(KK\right)$ is canonically isomorphic to the free matrad $\mathcal{H}_\infty$.

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- In these ranges, the reduced coherent framed join provides an efficient way to construct a basis, to define the differential, and to determine $KK_{n,m}$
- However, outside these ranges we are unable to define an operator that simultaneously preserves coherency and satisfies $d^2 = 0$. In fact...
- When $m = n = 4$, Saneblidze constructed an example with the following property: *If we use all available components of the face operator to extend the differential, coherency is lost; if we use only those available components that preserve coherency,* $d^2 \neq 0$

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- Given $\alpha, \beta \in \Omega S$, define the *product* $\alpha \cdot \beta \in \Omega S$ by

$$(\alpha \cdot \beta)(t) = \begin{cases} \alpha(2t), & t \in [0, \frac{1}{2}] \\ \beta(2t - 1), & t \in [\frac{1}{2}, 1] \end{cases}$$

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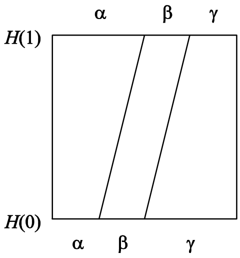
- The loops $(\alpha\beta)\gamma$ and $\alpha(\beta\gamma)$ are homotopic via linear change of parameter



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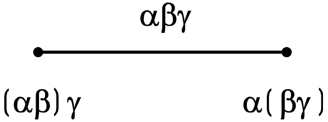

The associahedron K_3

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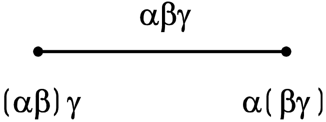

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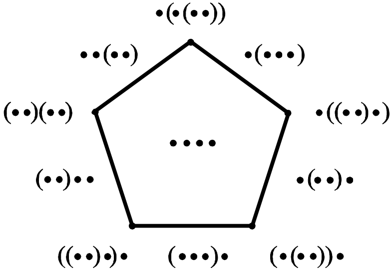
- K_3 controls homotopy associativity in three variables

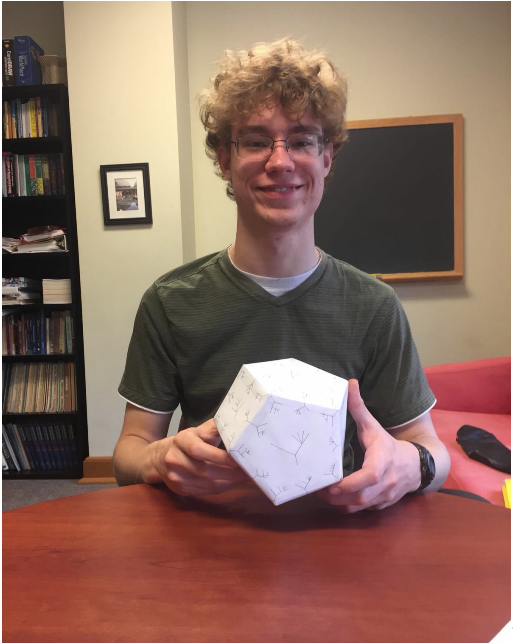
The Associahedron K(4)

- K_4 controls homotopy associativity in four variables



The associahedron K_4

Quinn Minnich's Model of K(5)



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- Let (V, ∂_V) and (W, ∂_W) be d.g. R-modules. A linear map $f : V \to W$ has **degree** p if $f : V_i \to W_{i+p}$

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- $\delta(f) = 0$ iff f is a chain map

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- $(V, \partial, \Delta_2, \Delta_3, \ldots)$ is an A_∞-**coalgebra** if for each $n \geq 2$,

$$\delta(\Delta_n) = \alpha_n \partial(\theta^n)$$

Structure Relations

- Expanding $\alpha_n \partial \left(\theta^n\right)$ expresses the structure relations in the more familiar form

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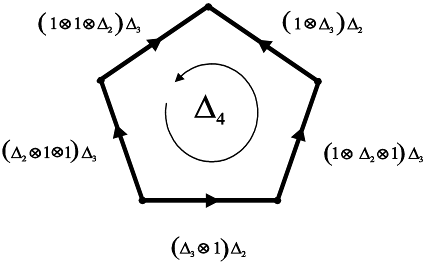
- Δ_n is a chain homotopy among the quadratic compositions encoded by the codim 1 cells of K_n



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- Let us construct $KK_{m,n}$ when $1 \leq m \leq 3$ or $1 \leq n \leq 3$

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- **Example** $\dfrac{7|0|6}{0|1|0} \in P'_3(\{1\}) \times P'_3(\{6,7\})$

Bipartition Matrices

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- **Example** $\begin{pmatrix} \frac{4|5}{1|0} & \frac{5|4}{3|2} \\ \frac{7|0|6}{0|1|0} & \frac{67}{23} \end{pmatrix}$ is a bipartition matrix

over $\mathbf{a}_1 = \{1\}$, $\mathbf{a}_2 = \{2,3\}$, $\mathbf{b}_1 = \{4,5\}$, $\mathbf{b}_2 = \{6,7\}$

with respect to $\begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix}$

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- **Example:** $\mu_{\{2,5\}} 2|1|0|5|4|3 = 12|45|3$

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- Let $\bar{A}_i = A_{\lambda^{i-1}+1} \cup \cdots \cup A_{\lambda^i}$
- The λ-**merging map** $\mu_\lambda : P'_{n+1}(A) \to P'_{k+1}(A)$ is defined

$$\mu_\lambda(A_1|\cdots|A_{n+1}) := \bar{A}_1|\cdots|\bar{A}_{k+1}$$

- **Example:** $\mu_{\{2,5\}}2|1|0|5|4|3 = 12|45|3$
- **Extreme cases**: $\mu_\varnothing(A_1|\cdots|A_{n+1}) = A$

$$\mu_{\{1,2,\ldots,n\}}(A_1|\cdots|A_{n+1}) = A_1|\cdots|A_{n+1}$$

Proposition 1

Given a $q \times p$ bipartition matrix $\left(\frac{\beta_{ij}}{\alpha_{ij}}\right)$ *over* $\{\mathbf{a}_j, \mathbf{b}_i\}$ *w.r.t.* (r_{ij}), *there is a unique $q \times p$ matrix of ordered sets* (λ_{ij}) *such that*

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Example: $$\begin{pmatrix} \frac{45|0}{1|0} & \frac{5|4|0}{0|2|3} \\ \frac{7|0|0|6}{0|1|0|0} & \frac{0|7|6}{2|0|3} \end{pmatrix} \overset{\mu_\lambda}{\rightsquigarrow} \begin{pmatrix} \frac{45|0}{1|0} & \frac{45|0}{2|3} \\ \frac{7|6}{1|0} & \frac{7|6}{2|3} \end{pmatrix},$$

$$\text{where } \lambda = \begin{pmatrix} \{1\} & \{2\} \\ \{2\} & \{2\} \end{pmatrix}$$

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- **Example**

$$\left(\frac{56|7|8}{1|23|4}\right) = \begin{pmatrix} \frac{56}{1} \\ \frac{0}{1} \\ \frac{0}{1} \end{pmatrix} \begin{pmatrix} \frac{7}{0} & \frac{7}{23} \\ \frac{0}{0} & \frac{0}{23} \end{pmatrix} \begin{pmatrix} \frac{8}{0} & \frac{8}{0} & \frac{8}{0} & \frac{8}{4} \end{pmatrix}$$

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$$C_k = \begin{pmatrix} \frac{\mathbf{b}_{k,1}}{\mathbf{a}_{k,1}} & \cdots & \frac{\mathbf{b}_{k,1}}{\mathbf{a}_{k,s_k}} \\ \vdots & & \vdots \\ \frac{\mathbf{b}_{k,t_k}}{\mathbf{a}_{k,1}} & \cdots & \frac{\mathbf{b}_{k,t_k}}{\mathbf{a}_{k,s_k}} \end{pmatrix}$$

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$$
\begin{array}{cc}
1 = \mathcal{ACP}_{1}1 & 56|0|0 = \mathcal{ACP}_{5678}56 \\
0|23 = \mathcal{ACP}_{123}23 & 7|0 = \mathcal{ACP}_{78}7 \\
0|0|0|4 = \mathcal{ACP}_{1234}4 & 8 = \mathcal{ACP}_{8}8
\end{array}
$$

Factoring a Bipartition

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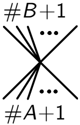
Graphical Representation

- $\dfrac{B}{A} \leftarrow$ 

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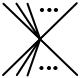
- $\dfrac{B}{A} \leftarrow$ 

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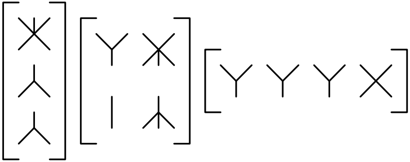
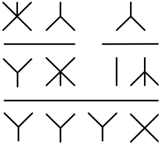
$=$  $=$ 

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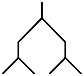
- **Motivation:** $\left(\frac{0|0}{2|13}\right) = \left(\frac{0}{2}\right)\left(\frac{0}{1}\ \frac{0}{3}\right) \leftrightarrow$ 

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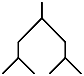
- **Motivation:** $\left(\frac{0|0}{2|13}\right) = \left(\frac{0}{2}\right)\left(\frac{0}{1}\ \frac{0}{3}\right) \leftrightarrow$ 
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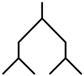
- **Motivation:** $\left(\frac{0|0}{2|13}\right) = \left(\frac{0}{2}\right)\left(\frac{0}{1}\;\frac{0}{3}\right) \leftrightarrow$ 
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- *Dimension is not necessarily the sum of the dim's of its entries*

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- Let $(\lambda_1^i \ \cdots \ \lambda_p^i)$ and $(\lambda_1^j \ \cdots \ \lambda_q^j)^T$ be the associated matrices given by Prop 1 and define

$$\overset{\wedge}{\alpha}_i := \mu_{\lambda_1^i}(\alpha_{i1}) \Cup \cdots \Cup \mu_{\lambda_p^i}(\alpha_{ip}) \in P'(\mathbf{a}_1 \cup \cdots \cup \mathbf{a}_p)$$

$$\overset{\vee}{\beta}_j := \mu_{\lambda_1^j}(\beta_{1j}) \Cup \cdots \Cup \mu_{\lambda_q^j}(\beta_{qj}) \in P'(\mathbf{b}_1 \cup \cdots \cup \mathbf{b}_q)$$

Dimension of a Bipartition Matrix

1. If $c_{ij} = \frac{0|\cdots|0}{\alpha_j}$ for all i, define $|C| := \sum_{1 \leq i \leq q} |\overset{\wedge}{\alpha}_i|$

Example $\left| \begin{pmatrix} \frac{0}{12} & \frac{0}{34} \\ \frac{0|0}{1|2} & \frac{0|0}{3|4} \end{pmatrix} \right| = |1234| + |13|24| = 3 + 2 = 5$

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3. Otherwise

$$|C| := \sum_{1 \leq i \leq q} |C_{i*}|^{row} + \sum_{1 \leq j \leq p} |C_{*j}|^{col} + \sum_{1 \leq i \leq q, 1 \leq j \leq p} |c_{ij}|^{ss}$$

whose components are defined inductively as follows:

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- Define $|B|^{ss} := \sum_{k \notin I \cup J} |B_k|$ inductively

Conventions for Bipartition Matrices

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- **Example** Discard the 1-dim'l indecomposable matrix

$$C = \left(\frac{0|1}{1|0} \;\; \frac{0|1}{1|0} \;\; \frac{1}{1} \right)$$

Inserting empty blocks in the third entry transforms C into the 3-dim'l decomposable

$$\left(\frac{0|1}{1|0} \;\; \frac{0|1}{1|0} \;\; \frac{0|1}{0|1} \right) = \begin{pmatrix} \frac{0}{1} & \frac{0}{1} & \frac{0}{0} \\ \frac{0}{1} & \frac{0}{1} & \frac{0}{0} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{1}{0} & \frac{1}{0} & \frac{1}{0} & \frac{1}{1} \end{pmatrix}.$$

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- **Example** $\begin{pmatrix} \frac{0}{1} & \frac{0}{3} \\[2ex] \frac{0|0|0}{0|1|0} & \frac{0|0|0}{0|0|3} \end{pmatrix} = \begin{pmatrix} \frac{0}{1} & \frac{0}{3} \\[2ex] \frac{0|0}{1|0} & \frac{0|0}{0|3} \end{pmatrix}$

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- **Example** Preserve all empty blocks in

$$C = \begin{pmatrix} \frac{0}{1} & \frac{0}{3} \\[2ex] \frac{0|0}{1|0} & \frac{0|0}{0|3} \end{pmatrix}$$

Removing empty blocks in the second row increases dimension

Framed Elements

- Given $\mathbf{a}(m)$ and $\mathbf{b}(n)$ of orders m and n, and $r \geq 1$, let

$$\frac{\beta}{\alpha} \in P'_r(\mathbf{a}(m)) \times P'_r(\mathbf{b}(n))$$

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- Inductively, assume the set of framed elements $\alpha' \Cup_f \beta'$ has been defined for all $\frac{\beta'}{\alpha'} \in P'(\mathbf{a}(s)) \times P'(\mathbf{b}(t))$ such that $(s, t) \leq (m, n)$ and $s + t < m + n$

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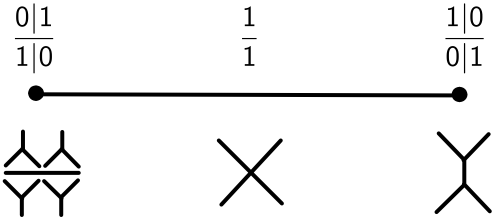

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- Replace entries in all possible ways to obtain coherence

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- **Example**

$$\tilde{\partial}(\mathfrak{2} \Cup_c \mathfrak{1}) = \left\{ \frac{0|1}{1|2}, \frac{0|1}{2|1}, \frac{1|0}{1|2}, \frac{1|0}{2|1}, \frac{1|0}{0|12}, \begin{pmatrix} \frac{0|0}{2|1} \\ \frac{0}{12} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{1}{0} & \frac{1}{0} \end{pmatrix}, \begin{pmatrix} \frac{0}{12} \\ \frac{0|0}{1|2} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{1}{0} & \frac{1}{0} \end{pmatrix} \right\}$$

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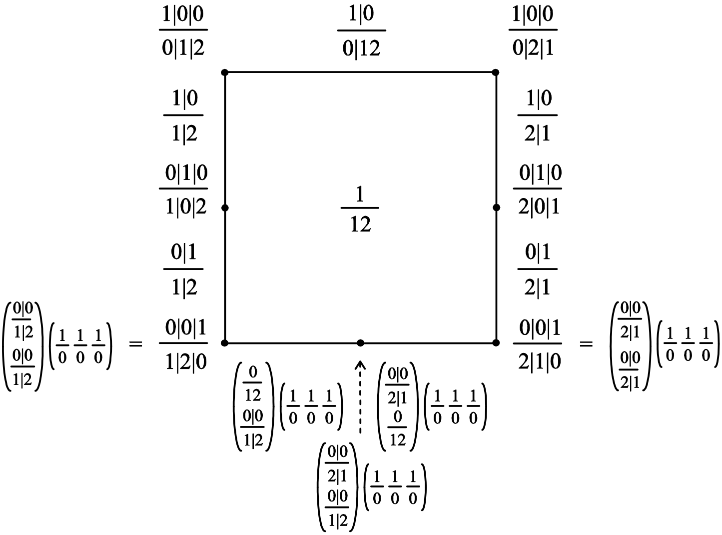
PP(2,3) = KK(2,3)



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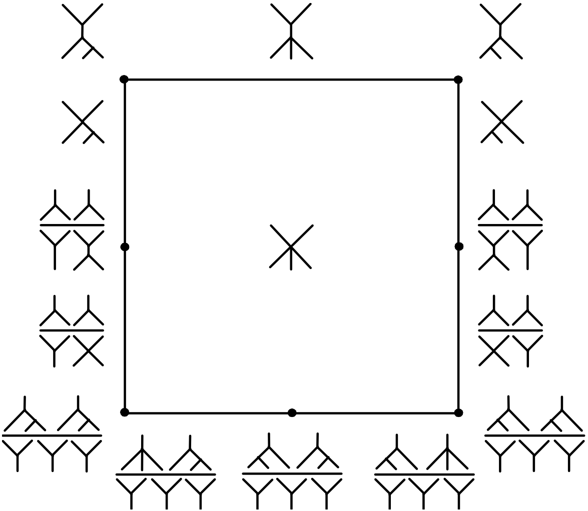

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- $KK_{n+1,m+1} \leftrightarrow \mathfrak{m} \circledast_{kk} \mathfrak{n}$

Stasheff's Associahedron K(4)

- In $KK_{1,4} \leftrightarrow 3 \circledast_{kk} 0$ we have

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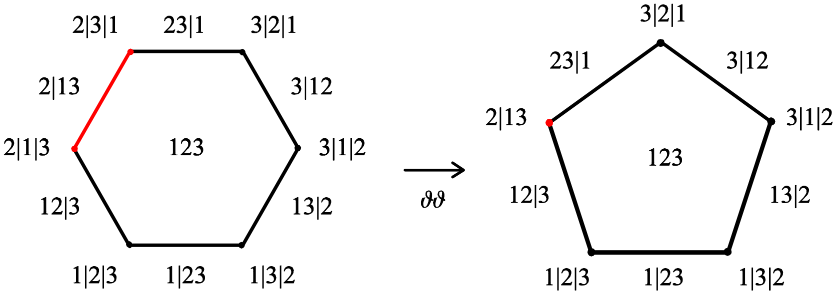
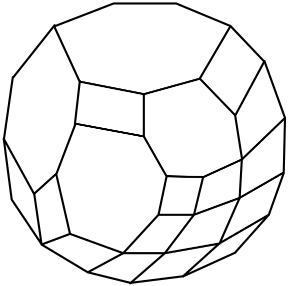
- $\vartheta\vartheta : P_4 = PP_{1,4} \to KK_{1,4} = K_4$

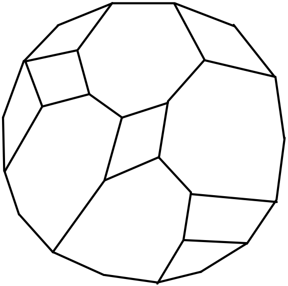
Projection to K(4)



KK(3,3)



Front view



Rear view

- $\partial KK_{3,3}$ consists of 8 heptagons and 22 squares

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- Prior to this work, all known rational homology invariants of $\Omega\Sigma X$ were trivial

The End

Thank you!