On the Transfer of Multiplicative Structure from Chains to Homology

Joint work with Samson Saneblidze

Ron Umble, speaker
ron.umble@millersville.edu

Lehigh Geometry/Topology Conference

June 6, 2008

Abstract

A chain map $f : B \to A$ of DG modules induces a cochain map $\bar{f} : \left(Hom\left(B^{\otimes n}, B\right); \nabla_B \right) \to \left(Hom\left(B^{\otimes n}, A\right); \nabla \right)$ via $\bar{f}(u) = fu$.

Theorem 1. *Let $(A, d, \mu^n)_{n \geq 2}$ be an A_∞-algebra and let $f : B \to A$ be a chain map. If $\bar{f}$ is a quasi-isomorphism, then*
(i) f transfers the A_∞-algebra structure from A to B; the induced structure on B is unique up to automorphism.
(ii) There is a map $F : B \Rightarrow A$ of A_∞-algebras *extending f.*

This generalizes the work of Kadeishvili, Markl, and others, who assume either that B is free or that f has a left-homotopy inverse. We give an example of an A_∞-algebra with torsion such that every cycle-selection homomorphism $f : H(A) \to A$ fails to have a right-homotopy inverse. Theorem 1 extends immediately to A_∞-bialgebras.

Theorem 3. *Let $\mathbf{k}$ be a field. Then $H_*(\Omega X; \mathbf{k})$ is an A_∞-bialgebra.*

A–infinity Maps and Multiplihedra

- Given A_∞-algebras A and B, and a chain map $f : A \to B$, a morphism

$$F = f + f_2 + f_3 + \cdots : B \Rightarrow A$$

is defined in terms of parameter spaces $\{J_n\}_{n\geq 1}$, called *multiplihedra*

A–infinity Maps and Multiplihedra

- Given A_∞-algebras A and B, and a chain map $f : A \to B$, a morphism

$$F = f + f_2 + f_3 + \cdots : B \Rightarrow A$$

  is defined in terms of parameter spaces $\{J_n\}_{n\geq 1}$, called *multiplihedra*

- Identify f with $J_1 = *$

A–infinity Maps and Multiplihedra

- Given A_∞-algebras A and B, and a chain map $f : A \to B$, a morphism

$$F = f + f_2 + f_3 + \cdots : B \Rightarrow A$$

  is defined in terms of parameter spaces $\{J_n\}_{n\geq 1}$, called *multiplihedra*

- Identify f with $J_1 = *$

- Identify $f_2 \in Hom^1(B \otimes B, A)$ with $J_2 = I$

A–infinity Maps and Multiplihedra

- Given A_∞-algebras A and B, and a chain map $f : A \to B$, a morphism

$$F = f + f_2 + f_3 + \cdots : B \Rightarrow A$$

  is defined in terms of parameter spaces $\{J_n\}_{n \geq 1}$, called *multiplihedra*

- Identify f with $J_1 = *$

- Identify $f_2 \in Hom^1 (B \otimes B, A)$ with $J_2 = I$

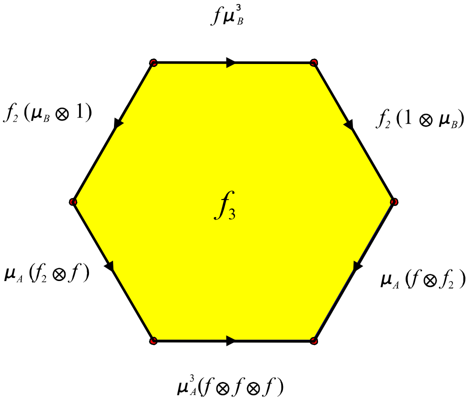
- Identify components of the coboundary

$$\nabla f_2 = \mu_A (f \otimes f) - f \mu_B$$

  with the endpoints of J_2

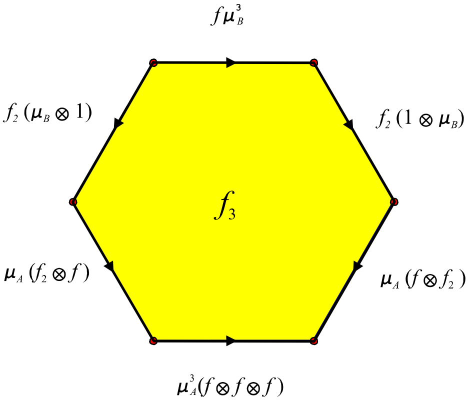
The Multiplihedra

- Identify $f_3 \in Hom^2\left(B^{\otimes 3}, A\right)$ with the hexagonal plane region J_3



The Multiplihedra

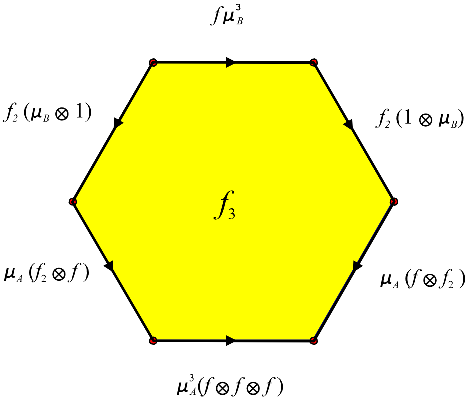
- Identify $f_3 \in Hom^2\left(B^{\otimes 3}, A\right)$ with the hexagonal plane region J_3



- $\nabla f_3 = \mu_A^3 f^{\otimes 3} + \mu_A\left(f_2 \otimes f - f \otimes f_2\right) + f_2\left(\mu_B \otimes 1 - 1 \otimes \mu_B\right) - f\mu_B^3$

The Multiplihedra

- Identify $f_3 \in Hom^2\left(B^{\otimes 3}, A\right)$ with the hexagonal plane region J_3



- $\nabla f_3 = \mu_A^3 f^{\otimes 3} + \mu_A\left(f_2 \otimes f - f \otimes f_2\right) + f_2\left(\mu_B \otimes 1 - 1 \otimes \mu_B\right) - f\mu_B^3$
- Identify f_n with the $(n-1)$-dimensional multiplihedron J_n

The Transfer Problem

- $Q^{n-2} = \partial J_n - \{\text{interior of the cell identified with } f\mu_B^n\}$

The Transfer Problem

- $Q^{n-2} = \partial J_n - \{$interior of the cell identified with $f\mu_B^n\}$
- $\ominus_n \in Hom^{n-2}\left(B^{\otimes n}, A\right)$ is the cochain identified with Q^{n-2}

The Transfer Problem

- $Q^{n-2} = \partial J_n - \{\text{interior of the cell identified with } f\mu_B^n\}$
- $\ominus_n \in Hom^{n-2}\left(B^{\otimes n}, A\right)$ is the cochain identified with Q^{n-2}
- **Problem:** *Given* $\left\{\mu_B^i,\ f_i\right\}_{1\leq i<n}$ *define* f_n *and* μ_B^n *so that*

$$\nabla f_n = \ominus_n - f\mu_B^n$$

The Transfer Problem

- $Q^{n-2} = \partial J_n - \{$interior of the cell identified with $f\mu_B^n\}$
- $\ominus_n \in Hom^{n-2}\left(B^{\otimes n}, A\right)$ is the cochain identified with Q^{n-2}
- **Problem:** *Given* $\left\{\mu_B^i,\ f_i\right\}_{1\leq i<n}$ *define* f_n *and* μ_B^n *so that*

$$\nabla f_n = \ominus_n - f\mu_B^n$$

- Has a solution whenever $\bar{f}$ is a quasi-isomorphism

First Transfer Theorem

Let A and B be DGMs over a commutative ring with unity

- A chain map $f : B \to A$ induces a cochain map

$$\bar{f} : \left(Hom\left(B^{\otimes n}, B\right); \nabla_B\right) \to \left(Hom\left(B^{\otimes n}, A\right); \nabla\right)$$

given by $\bar{f}\,(u) = fu$

First Transfer Theorem

Let A and B be DGMs over a commutative ring with unity

- A chain map $f : B \to A$ induces a cochain map

$$\bar{f} : \left(Hom\left(B^{\otimes n}, B\right); \nabla_B\right) \to \left(Hom\left(B^{\otimes n}, A\right); \nabla\right)$$

  given by $\bar{f}(u) = fu$

- **Theorem 1**. *Let A be an A_∞-algebra and let $f : B \to A$* be a chain map. *If $\bar{f}$ is a quasi-isomorphism, then*

  (i) f transfers the A_∞-algebra structure from A to B; the induced structure on B is unique up to automorphism

  (ii) There is a map $F : B \Rightarrow A$ of A_∞-algebras *extending f*

Remarks

- If $\bar{f}$ is a quasi-isomorphism, so is f, but not conversely

Remarks

- If $\bar{f}$ is a quasi-isomorphism, so is f, but not conversely
- However, the converse holds whenever B is free

Remarks

- If $\bar{f}$ is a quasi-isomorphism, so is f, but not conversely
- However, the converse holds whenever B is free
- Theorem 1 generalizes work of T. Kadeishvili, M. Markl, et al

Remarks

- If $\bar{f}$ is a quasi-isomorphism, so is f, but not conversely
- However, the converse holds whenever B is free
- Theorem 1 generalizes work of T. Kadeishvili, M. Markl, et al
- Kadeishvili assumes *A and B are free and f is an inclusion*

Remarks

- If $\bar{f}$ is a quasi-isomorphism, so is f, but not conversely
- However, the converse holds whenever B is free
- Theorem 1 generalizes work of T. Kadeishvili, M. Markl, et al
- Kadeishvili assumes *A and B are free and f is an inclusion*
- Markl assumes that *f has a left-homotopy inverse*

Remarks

- If $\bar{f}$ is a quasi-isomorphism, so is f, but not conversely
- However, the converse holds whenever B is free
- Theorem 1 generalizes work of T. Kadeishvili, M. Markl, et al
- Kadeishvili assumes *A and B are free and f is an inclusion*
- Markl assumes that *f has a left-homotopy inverse*
- We only assume that *$\bar{f}$ is a quasi-isomorphism*

Remarks

- If $\bar{f}$ is a quasi-isomorphism, so is f, but not conversely
- However, the converse holds whenever B is free
- Theorem 1 generalizes work of T. Kadeishvili, M. Markl, et al
- Kadeishvili assumes A *and* B *are free and* f *is an inclusion*
- Markl assumes that f *has a left-homotopy inverse*
- We only assume that $\bar{f}$ *is a quasi-isomorphism*
- $B = H(A)$ is an interesting and important special case

Proof of Theorem 1: Transfer of DGA Structure

Given $(A, d, \mu, \mu^n)_{n \geq 3}$ and $f : B \to A$ such that $\bar{f}$ is a quasi-isomorphism

- Consider $\ominus_2 = \mu(f \otimes f) \in Hom^0(B \otimes B, A)$

Proof of Theorem 1: Transfer of DGA Structure

Given $(A, d, \mu, \mu^n)_{n\geq 3}$ and $f : B \longrightarrow A$ such that $\bar{f}$ is a quasi-isomorphism

- Consider $\ominus_2 = \mu (f \otimes f) \in Hom^0 (B \otimes B, A)$
- $\nabla \ominus_2 = d\mu (f \otimes f) - \mu (f \otimes f) (d_B \otimes 1 + 1 \otimes d_B)$

$$= (d\mu - \mu (d \otimes 1 + 1 \otimes d)) (f \otimes f) = 0$$

Proof of Theorem 1: Transfer of DGA Structure

Given $(A, d, \mu, \mu^n)_{n\geq 3}$ and $f : B \to A$ such that $\bar{f}$ is a quasi-isomorphism

- Consider $\ominus_2 = \mu\,(f \otimes f) \in Hom^0\,(B \otimes B, A)$
- $\nabla\ominus_2 = d\mu\,(f \otimes f) - \mu\,(f \otimes f)\,(d_B \otimes 1 + 1 \otimes d_B)$

  $$= (d\mu - \mu\,(d \otimes 1 + 1 \otimes d))\,(f \otimes f) = 0$$
- Using surjectivity of $\bar{f}_*$, choose μ_B such that $\bar{f}_*\,[\mu_B] = [\ominus_2]$

Proof of Theorem 1: Transfer of DGA Structure

Given $(A, d, \mu, \mu^n)_{n\geq 3}$ and $f : B \to A$ such that $\bar{f}$ is a quasi-isomorphism

- Consider $\ominus_2 = \mu (f \otimes f) \in Hom^0 (B \otimes B, A)$

- $\nabla \ominus_2 = d\mu (f \otimes f) - \mu (f \otimes f) (d_B \otimes 1 + 1 \otimes d_B)$

$$= (d\mu - \mu (d \otimes 1 + 1 \otimes d)) (f \otimes f) = 0$$

- Using surjectivity of $\bar{f}_*$, choose μ_B such that $\bar{f}_* [\mu_B] = [\ominus_2]$

- Then $\nabla_B (\mu_B) = d_B \mu_B - \mu_B (d_B \otimes 1 + 1 \otimes d_B) = 0$

Proof of Theorem 1: Transfer of DGA Structure

Given $(A, d, \mu, \mu^n)_{n\geq 3}$ and $f : B \to A$ such that $\bar{f}$ is a quasi-isomorphism

- Consider $\ominus_2 = \mu(f \otimes f) \in Hom^0(B \otimes B, A)$
- $\nabla \ominus_2 = d\mu(f \otimes f) - \mu(f \otimes f)(d_B \otimes 1 + 1 \otimes d_B)$

  $$= (d\mu - \mu(d \otimes 1 + 1 \otimes d))(f \otimes f) = 0$$

- Using surjectivity of $\bar{f}_*$, choose μ_B such that $\bar{f}_*[\mu_B] = [\ominus_2]$
- Then $\nabla_B(\mu_B) = d_B\mu_B - \mu_B(d_B \otimes 1 + 1 \otimes d_B) = 0$
- Cocycles $f\mu_B$ and $\ominus_2$ are cohomologous

Proof of Theorem 1: Transfer of DGA Structure

Given $(A, d, \mu, \mu^n)_{n\geq 3}$ and $f : B \to A$ such that $\bar{f}$ is a quasi-isomorphism

- Consider $\ominus_2 = \mu (f \otimes f) \in Hom^0 (B \otimes B, A)$
- $\nabla \ominus_2 = d\mu (f \otimes f) - \mu (f \otimes f) (d_B \otimes 1 + 1 \otimes d_B)$

  $$= (d\mu - \mu (d \otimes 1 + 1 \otimes d)) (f \otimes f) = 0$$
- Using surjectivity of $\bar{f}_*$, choose μ_B such that $\bar{f}_* [\mu_B] = [\ominus_2]$
- Then $\nabla_B (\mu_B) = d_B \mu_B - \mu_B (d_B \otimes 1 + 1 \otimes d_B) = 0$
- Cocycles $f\mu_B$ and $\ominus_2$ are cohomologous
- Choose f_2 such that $\nabla f_2 = \ominus_2 - f\mu_B$

Transfer of DGA Structure (diagram chase)

$$\begin{array}{ccc}
\mu_B\,(d_B \otimes 1 + 1 \otimes d_B) - d_B\mu_B & \xleftarrow{\nabla_B} & \mu_B \\
\| & & \downarrow \bar{f} \\
0 & \underset{\nabla}{\longleftarrow} & \left\{ \begin{array}{c} f\mu_B \\ \ominus_2 \end{array} \right. \\
 & & \ominus_2 - f\mu_B \quad \underset{\nabla}{\longleftarrow} \quad f_2
\end{array}$$

- (B, d_B, μ_B) is a DGA and $F = f + f_2$ is an A_2-map

Remarks

- If $\phi \in Hom^1(B \otimes B, B)$, then

$$\nabla (f_2 + f\phi) = \ominus_2 - f\mu_B + \nabla f\phi = \ominus_2 - f(\mu_B + \nabla_B \phi)$$

Remarks

- If $\phi \in Hom^1(B \otimes B, B)$, then

$$\nabla(f_2 + f\phi) = \ominus_2 - f\mu_B + \nabla f\phi = \ominus_2 - f(\mu_B + \nabla_B \phi)$$

- Thus $\{\text{multiplications on } H\} \leftrightarrow \mathrm{Im}(\nabla_B) \cap Hom^0(B \otimes B, B)$

Remarks

- If $\phi \in Hom^1(B \otimes B, B)$, then

$$\nabla(f_2 + f\phi) = \ominus_2 - f\mu_B + \nabla f\phi = \ominus_2 - f(\mu_B + \nabla_B \phi)$$

- Thus $\{\text{multiplications on } H\} \leftrightarrow \mathrm{Im}(\nabla_B) \cap Hom^0(B \otimes B, B)$
- There is an induced DGA structure on H whenever $\bar{f}_*$ is surjective

Transfer of A-infinity Structure

- Let $\ominus_3 = \mu^3 f^{\otimes 3} + \mu\left(f_2 \otimes f - f \otimes f_2\right) + f_2\left(\mu_B \otimes 1 - 1 \otimes \mu_B\right)$

Transfer of A-infinity Structure

- Let $\ominus_3 = \mu^3 f^{\otimes 3} + \mu\left(f_2 \otimes f - f \otimes f_2\right) + f_2\left(\mu_B \otimes 1 - 1 \otimes \mu_B\right)$
- $\nabla \ominus_3 = f\mu_B\left(1 \otimes \mu_B - \mu_B \otimes 1\right) \in [0]$

Transfer of A-infinity Structure

- Let $\ominus_3 = \mu^3 f^{\otimes 3} + \mu\left(f_2 \otimes f - f \otimes f_2\right) + f_2\left(\mu_B \otimes 1 - 1 \otimes \mu_B\right)$
- $\nabla \ominus_3 = f \mu_B \left(1 \otimes \mu_B - \mu_B \otimes 1\right) \in [0]$
- Hence $\bar{f}_* \left[\mu_B \left(1 \otimes \mu_B - \mu_B \otimes 1\right)\right] = [0]$

Transfer of A-infinity Structure

- Let $\ominus_3 = \mu^3 f^{\otimes 3} + \mu\left(f_2 \otimes f - f \otimes f_2\right) + f_2\left(\mu_B \otimes 1 - 1 \otimes \mu_B\right)$
- $\nabla \ominus_3 = f\mu_B\left(1 \otimes \mu_B - \mu_B \otimes 1\right) \in [0]$
- Hence $\bar{f}_*\left[\mu_B\left(1 \otimes \mu_B - \mu_B \otimes 1\right)\right] = [0]$
- Since $\bar{f}_*$ is injective, $\left[\mu_B\left(1 \otimes \mu_B - \mu_B \otimes 1\right)\right] = [0]$

Transfer of A-infinity Structure

- Let $\ominus_3 = \mu^3 f^{\otimes 3} + \mu \left(f_2 \otimes f - f \otimes f_2\right) + f_2 \left(\mu_B \otimes 1 - 1 \otimes \mu_B\right)$
- $\nabla \ominus_3 = f \mu_B \left(1 \otimes \mu_B - \mu_B \otimes 1\right) \in [0]$
- Hence $\bar{f}_* \left[\mu_B \left(1 \otimes \mu_B - \mu_B \otimes 1\right)\right] = [0]$
- Since $\bar{f}_*$ is injective, $\left[\mu_B \left(1 \otimes \mu_B - \mu_B \otimes 1\right)\right] = [0]$
- Choose ϕ such that $\nabla_B \phi = \mu_B \left(1 \otimes \mu_B - \mu_B \otimes 1\right)$

Transfer of A-infinity Structure

- Let $\ominus_3 = \mu^3 f^{\otimes 3} + \mu \left(f_2 \otimes f - f \otimes f_2\right) + f_2 \left(\mu_B \otimes 1 - 1 \otimes \mu_B\right)$
- $\nabla \ominus_3 = f \mu_B \left(1 \otimes \mu_B - \mu_B \otimes 1\right) \in [0]$
- Hence $\bar{f}_* \left[\mu_B \left(1 \otimes \mu_B - \mu_B \otimes 1\right)\right] = [0]$
- Since $\bar{f}_*$ is injective, $\left[\mu_B \left(1 \otimes \mu_B - \mu_B \otimes 1\right)\right] = [0]$
- Choose ϕ such that $\nabla_B \phi = \mu_B \left(1 \otimes \mu_B - \mu_B \otimes 1\right)$
- Then $\nabla \ominus_3 = f \nabla_B \phi = \nabla f \phi$

Transfer of A-infinity Structure

- Let $\ominus_3 = \mu^3 f^{\otimes 3} + \mu\left(f_2 \otimes f - f \otimes f_2\right) + f_2\left(\mu_B \otimes 1 - 1 \otimes \mu_B\right)$
- $\nabla \ominus_3 = f \mu_B \left(1 \otimes \mu_B - \mu_B \otimes 1\right) \in [0]$
- Hence $\bar{f}_* \left[\mu_B \left(1 \otimes \mu_B - \mu_B \otimes 1\right)\right] = [0]$
- Since $\bar{f}_*$ is injective, $\left[\mu_B \left(1 \otimes \mu_B - \mu_B \otimes 1\right)\right] = [0]$
- Choose ϕ such that $\nabla_B \phi = \mu_B \left(1 \otimes \mu_B - \mu_B \otimes 1\right)$
- Then $\nabla \ominus_3 = f \nabla_B \phi = \nabla f \phi$
- Implies $\nabla\left(\ominus_3 - f\phi\right) = 0$

Transfer of A-infinity Structure (diagram chase)

$$\begin{array}{ccc}
\mu_B\,(\mu_B \otimes 1 - 1 \otimes \mu_B) & \xleftarrow{\nabla_B} & \phi \\
\bar{f} \downarrow & & \downarrow \bar{f} \\
f\mu_B\,(\mu_B \otimes 1 - 1 \otimes \mu_B) & \xleftarrow[\nabla]{} & \left\{ \begin{array}{l} f\phi \\ \ominus_3 \end{array} \right.
\end{array}$$

Transfer of A-infinity Structure (continued)

- Using surjectivity of $\bar{f}_*$, choose z such that $\bar{f}_* [z] = [\ominus_3 - f\phi]$

Transfer of A-infinity Structure (continued)

- Using surjectivity of $\bar{f}_*$, choose z such that $\bar{f}_*[z] = [\ominus_3 - f\phi]$
- Then $\ominus_3 - f\phi$ and fz are cohomologous

Transfer of A-infinity Structure (continued)

- Using surjectivity of $\bar{f}_*$, choose z such that $\bar{f}_*[z] = [\ominus_3 - f\phi]$
- Then $\ominus_3 - f\phi$ and fz are cohomologous
- Choose f_3 such that

$$\nabla f_3 = \ominus_3 - f\phi - fz = \ominus_3 - f(\phi + z)$$

Transfer of A-infinity Structure (continued)

- Using surjectivity of $\bar{f}_*$, choose z such that $\bar{f}_* [z] = [\ominus_3 - f\phi]$
- Then $\ominus_3 - f\phi$ and fz are cohomologous
- Choose f_3 such that

$$\nabla f_3 = \ominus_3 - f\phi - fz = \ominus_3 - f(\phi + z)$$

- Define $\mu_B^3 = \phi + z$

Transfer of A-infinity Structure (continued)

- Using surjectivity of $\bar{f}_*$, choose z such that $\bar{f}_* [z] = [\ominus_3 - f\phi]$
- Then $\ominus_3 - f\phi$ and fz are cohomologous
- Choose f_3 such that

$$\nabla f_3 = \ominus_3 - f\phi - fz = \ominus_3 - f(\phi + z)$$

- Define $\mu_B^3 = \phi + z$
- Then $\nabla f_3 = \ominus_3 - f\mu_B^3$

Transfer of A-infinity Structure (diagram chase)

$$\begin{array}{ccc}
\mu_B\,(\mu_B \otimes 1 - 1 \otimes \mu_B) & \xleftarrow{\nabla_B} & \phi \;\; ; \;\; z \in (\bar{f}_*)^{-1}\,[\ominus_3 - f\phi] \\
\bar{f} \downarrow & & \downarrow \bar{f} \\
f\mu_B\,(\mu_B \otimes 1 - 1 \otimes \mu_B) & \xleftarrow[\nabla]{} & \left\{ \begin{array}{l} f\phi \\ \ominus_3 \end{array} \right. \; ; \;\; fz \\
 & & \ominus_3 - f\phi - fz \;\; \xleftarrow[\nabla]{} \;\; f_3
\end{array}$$

- $\mu_B^3 = \phi + z$ and $\nabla f_3 = \ominus_3 - f\mu_B^3$

Transfer of A-infinity Structure (diagram chase)

$$
\begin{array}{ccc}
\mu_B(\mu_B \otimes 1 - 1 \otimes \mu_B) & \xleftarrow{\nabla_B} & \phi \;\; ; \;\; z \in (\bar{f}_*)^{-1}[\ominus_3 - f\phi] \\
\bar{f} \downarrow & & \downarrow \bar{f} \\
f\mu_B(\mu_B \otimes 1 - 1 \otimes \mu_B) & \underset{\nabla}{\longleftarrow} & \left\{ \begin{array}{l} f\phi \\ \ominus_3 \end{array} \right. \;\; ; \;\; fz \\
 & & \ominus_3 - f\phi - fz \;\; \underset{\nabla}{\longleftarrow} \;\; f_3
\end{array}
$$

- $\mu_B^3 = \phi + z$ and $\nabla f_3 = \ominus_3 - f\mu_B^3$
- (B, d_B, μ_B, μ_B^3) is an A_3-algebra and $F_3 = f + f_2 + f_3$ is an A_3-map

Transfer of A-infinity Structure (diagram chase)

$$\begin{array}{ccc}
\mu_B\left(\mu_B \otimes 1 - 1 \otimes \mu_B\right) & \xleftarrow{\nabla_B} & \phi \;\; ; \;\; z \in \left(\bar{f}_*\right)^{-1}\left[\ominus_3 - f\phi\right] \\
\bar{f} \downarrow & & \downarrow \bar{f} \\
f\mu_B\left(\mu_B \otimes 1 - 1 \otimes \mu_B\right) & \xleftarrow[\nabla]{} & \left\{ \begin{array}{l} f\phi \\ \ominus_3 \end{array} \right. \; ; \; fz \\
 & & \ominus_3 - f\phi - fz \;\; \xleftarrow[\nabla]{} \;\; f_3
\end{array}$$

- $\mu_B^3 = \phi + z$ and $\nabla f_3 = \ominus_3 - f\mu_B^3$
- $\left(B, d_B, \mu_B, \mu_B^3\right)$ is an A_3-algebra and $F_3 = f + f_2 + f_3$ is an A_3-map
- Continue inductively

Example 1

$$
\begin{array}{ccccccccccc}
M^0 & \longrightarrow & 0 & \longrightarrow & M^2 & \longrightarrow & M^3 & \longrightarrow & M^4 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\mathbb{Z} & & & & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & & \mathbb{Z}_4 & & \mathbb{Z}_2 & & & & \\
 & & & & (a_2, b_2) & \longmapsto & (0, 2a_3) & & & & & & \\
 & & & & & & a_3 & \longmapsto & a_4 & & & &
\end{array}
$$

- $A = T^a M \,/\, \big(a_2^2 + a_4,\ a_3^2,\ a_4 a_3 + a_3 a_4,\ (a_2 a_3 + a_3 a_2)^2,\ a_2 b_2,$
  $b_2 a_2,\ a_3 b_2,\ b_2 a_3,\ b_2^2\big)$

Example 1

$$\begin{array}{ccccccccccccc}
M^0 & \to & 0 & \to & M^2 & \to & M^3 & \to & M^4 & \to & 0 & \to & \cdots \\
\mathbb{Z} & & & & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & & \mathbb{Z}_4 & & \mathbb{Z}_2 & & & & \\
 & & & & (a_2, b_2) & \mapsto & (0, 2a_3) & & & & & & \\
 & & & & & & a_3 & \mapsto & a_4 & & & &
\end{array}$$

- $A = T^a M \big/ \big(a_2^2 + a_4,\ a_3^2,\ a_4 a_3 + a_3 a_4,\ (a_2 a_3 + a_3 a_2)^2,\ a_2 b_2,$
  $b_2 a_2,\ a_3 b_2,\ b_2 a_3,\ b_2^2\big)$
- A is an A_∞-algebra with trivial higher order structure

The Cup Product

- $H^n(A) = \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z}_2 & n = 2, 5, 7 \\ 0 & \text{otherwise} \end{cases}$

The Cup Product

- $H^n(A) = \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z}_2 & n = 2, 5, 7 \\ 0 & \text{otherwise} \end{cases}$

- $$\begin{aligned} u &= [a_2] \in H^2 \\ v &= [a_2 a_3 + a_3 a_2] \in H^5 \\ w &= [a_2(a_2 a_3 + a_3 a_2)] = [(a_2 a_3 + a_3 a_2)a_2] \in H^7 \end{aligned}$$

The Cup Product

- $H^n(A) = \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z}_2 & n = 2, 5, 7 \\ 0 & \text{otherwise} \end{cases}$

- $$\begin{aligned} u = &\quad [a_2] \in H^2 \\ v = &\quad [a_2 a_3 + a_3 a_2] \in H^5 \\ w = &\quad [a_2(a_2 a_3 + a_3 a_2)] = [(a_2 a_3 + a_3 a_2) a_2] \in H^7 \end{aligned}$$

- $\mu_H(u \otimes v) = \mu_H(v \otimes u) = w$

The Transfer Map

- Define $f : H(A) \to A$ by

$$\begin{aligned} f(u) &= a_2 \\ f(v) &= a_2 a_3 + a_3 a_2 \\ f(w) &= a_2 (a_2 a_3 + a_3 a_2) \end{aligned}$$

The Transfer Map

- Define $f : H(A) \to A$ by

$$\begin{aligned} f(u) &= a_2 \\ f(v) &= a_2 a_3 + a_3 a_2 \\ f(w) &= a_2 (a_2 a_3 + a_3 a_2) \end{aligned}$$

- f has no right-homotopy inverse g

The Transfer Map

- Define $f : H(A) \longrightarrow A$ by

$$\begin{aligned} f(u) &= a_2 \\ f(v) &= a_2 a_3 + a_3 a_2 \\ f(w) &= a_2 (a_2 a_3 + a_3 a_2) \end{aligned}$$

- f has no right-homotopy inverse g

- Otherwise, $fg(b_2) = \lambda a_2$ so that $b_2 - \lambda a_2 = (1 - fg)(b_2)$

$$= (sd + ds)(b_2) = sd(b_2) = s(2a_3) = 2s(a_3) = 0$$

The Transfer Map

- Define $f : H(A) \to A$ by

$$\begin{aligned} f(u) &= a_2 \\ f(v) &= a_2 a_3 + a_3 a_2 \\ f(w) &= a_2 (a_2 a_3 + a_3 a_2) \end{aligned}$$

- f has no right-homotopy inverse g

- Otherwise, $fg(b_2) = \lambda a_2$ so that $b_2 - \lambda a_2 = (1 - fg)(b_2)$

$$= (sd + ds)(b_2) = sd(b_2) = s(2a_3) = 2s(a_3) = 0$$

- $\bar{f}$ is a quasi-isomorphism

Transfer of Structure

- Define $F = f + f_2 + f_3 + \cdots$ by

  $f_2(u \otimes u) = a_3$ and 0 otherwise

  $f_n = 0$ for $n \geq 3$

Transfer of Structure

- Define $F = f + f_2 + f_3 + \cdots$ by

  $f_2 (u \otimes u) = a_3$ and 0 otherwise

  $f_n = 0$ for $n \geq 3$

- Let $f^{-1} : A \to H$ be projection; define $\mu_H^3 = f^{-1} \mu_A (f_2 \otimes f - f \otimes f_2)$

Transfer of Structure

- Define $F = f + f_2 + f_3 + \cdots$ by

  $f_2(u \otimes u) = a_3$ and 0 otherwise

  $f_n = 0$ for $n \geq 3$

- Let $f^{-1} : A \to H$ be projection; define $\mu_H^3 = f^{-1}\mu_A(f_2 \otimes f - f \otimes f_2)$

- Then $\mu_H^3(u \otimes u \otimes u) = v$

Transfer of Structure

- Define $F = f + f_2 + f_3 + \cdots$ by

  $f_2(u \otimes u) = a_3$ and 0 otherwise

  $f_n = 0$ for $n \geq 3$

- Let $f^{-1} : A \to H$ be projection; define $\mu_H^3 = f^{-1}\mu_A(f_2 \otimes f - f \otimes f_2)$

- Then $\mu_H^3(u \otimes u \otimes u) = v$

- Define $\mu_H^n = 0$ for $n \geq 4$

Transfer of Structure

- Define $F = f + f_2 + f_3 + \cdots$ by

  $f_2 (u \otimes u) = a_3$ and 0 otherwise

  $f_n = 0$ for $n \geq 3$

- Let $f^{-1} : A \to H$ be projection; define $\mu_H^3 = f^{-1} \mu_A (f_2 \otimes f - f \otimes f_2)$
- Then $\mu_H^3 (u \otimes u \otimes u) = v$
- Define $\mu_H^n = 0$ for $n \geq 4$
- (H, μ_H, μ_H^3) is an A_∞-algebra and F is an A_∞-map

Consistency with Markl's theory

- **Theorem (**Markl**).** *A left-homotopy inverse g of f extends to an A_∞-map $G : A \longrightarrow H$*

Consistency with Markl's theory

- **Theorem (**Markl**).** *A left-homotopy inverse g of f extends to an A_∞-map $G : A \to H$*

- **Example 1** (continued). Define $g : A \to H$ by

$$\begin{aligned} g\left(a_2\right) &= u \\ g\left(a_2 a_3\right) &= v \\ g\left(a_2 a_3 a_2\right) &= w \end{aligned}$$

and 0 otherwise

Consistency with Markl's theory

- **Theorem (**Markl**).** *A left-homotopy inverse g of f extends to an A_∞-map $G : A \to H$*

- **Example 1** (continued). Define $g : A \to H$ by

$$\begin{aligned} g\left(a_2\right) &= u \\ g\left(a_2 a_3\right) &= v \\ g\left(a_2 a_3 a_2\right) &= w \\ &\text{and 0 otherwise} \end{aligned}$$

- Then $gf = \mathbf{1}_H$

Consistency with Markl's theory

- **Theorem (**Markl**).** *A left-homotopy inverse g of f extends to an A_∞-map $G : A \to H$*

- **Example 1** (continued). Define $g : A \to H$ by

$$g(a_2) = u$$
$$g(a_2 a_3) = v$$
$$g(a_2 a_3 a_2) = w$$

and 0 otherwise

- Then $gf = \mathbf{1}_H$

- g fails to be an algebra map since $g(a_2)\, g(a_3) = 0 \neq v = g(a_2 a_3)$

Consistency with Markl's theory

- **Theorem (**Markl**).** *A left-homotopy inverse g of f extends to an A_∞-map $G : A \to H$*

- **Example 1** (continued). Define $g : A \to H$ by

$$\begin{aligned} g(a_2) &= u \\ g(a_2 a_3) &= v \\ g(a_2 a_3 a_2) &= w \\ &\text{and 0 otherwise} \end{aligned}$$

- Then $gf = \mathbf{1}_H$

- g fails to be an algebra map since $g(a_2)\, g(a_3) = 0 \neq v = g(a_2 a_3)$

- But as expected, g extends to an A_∞-map $G = g + g_2 + g_3 + \cdots$

The A-infinity map G

- Define $g_2 : A \longrightarrow H$ by

$$g_2\left(a_2 \otimes a_2^2\right) = v$$
$$g_2\left(a_2 \otimes a_2^3\right) = w$$

and 0 otherwise

The A-infinity map G

- Define $g_2 : A \longrightarrow H$ by

$$g_2\left(a_2 \otimes a_2^2\right) = v$$
$$g_2\left(a_2 \otimes a_2^3\right) = w$$
and 0 otherwise

- Since $\mu^3 = 0$, the "hexagon" relation reduces to

$$\nabla(g_3) = \mu_H(g \otimes g_2 + g_2 \otimes g) + g_2(\mu \otimes 1 + 1 \otimes \mu) + \mu_H^3(g \otimes g \otimes g)$$

The A-infinity map G

- Define $g_2 : A \longrightarrow H$ by

$$g_2\left(a_2 \otimes a_2^2\right) = v$$
$$g_2\left(a_2 \otimes a_2^3\right) = w$$
and 0 otherwise

- Since $\mu^3 = 0$, the "hexagon" relation reduces to

$$\nabla(g_3) = \mu_H(g \otimes g_2 + g_2 \otimes g) + g_2(\mu \otimes 1 + 1 \otimes \mu) + \mu_H^3(g \otimes g \otimes g)$$

- The right-hand side is identically 0

The A-infinity map G

- Define $g_2 : A \longrightarrow H$ by

$$\begin{aligned} g_2\left(a_2 \otimes a_2^2\right) &= v \\ g_2\left(a_2 \otimes a_2^3\right) &= w \\ &\text{and 0 otherwise} \end{aligned}$$

- Since $\mu^3 = 0$, the "hexagon" relation reduces to

$$\nabla\left(g_3\right) = \mu_H\left(g \otimes g_2 + g_2 \otimes g\right) + g_2\left(\mu \otimes 1 + 1 \otimes \mu\right) + \mu_H^3\left(g \otimes g \otimes g\right)$$

- The right-hand side is identically 0

- Define $g_i = 0$ for $i \geq 3$

Second Transfer Theorem

The proof above extends immediately to A_∞-bialgebras

- In this case, f induces a cochain map

$$\tilde{f} : \left(Hom\left(B^{\otimes m}, B^{\otimes n} \right); \nabla_B \right) \to Hom\left(B^{\otimes m}, A^{\otimes n}; \nabla \right)$$

given by $\tilde{f}(u) = f^{\otimes n} u$

Second Transfer Theorem

The proof above extends immediately to A_∞-bialgebras

- In this case, f induces a cochain map

$$\tilde{f} : \left(Hom\left(B^{\otimes m}, B^{\otimes n}\right); \nabla_B\right) \to Hom\left(B^{\otimes m}, A^{\otimes n}; \nabla\right)$$

given by $\tilde{f}(u) = f^{\otimes n} u$

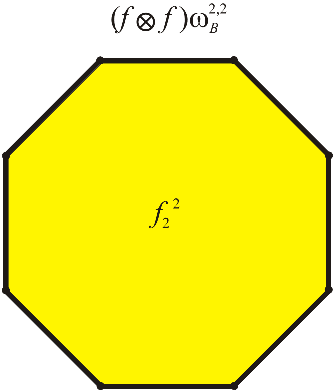
- **Theorem 2.** *Let A be an A_∞-bialgebra and let $f : B \to A$ be a chain map. If $\tilde{f}$ is a quasi-isomorphism, then*

  (i) f transfers the A_∞-bialgebra structure from A to B; the induced structure on B is unique up to automorphism

  *(ii) There is a map $F : B \Rightarrow A$ of A_∞-*bialgebras *extending f*

Generalized Multiplihedra

- Transfer is controlled by generalized multiplihedra $\{JJ_{m,n}\}_{m,n\geq 1}$ of which $JJ_{n,1} = JJ_{1,n} = J_n$



The Multiplihedron $JJ_{2,2}$

The General Transfer Procedure

- The procedure is exactly the same as before

The General Transfer Procedure

- The procedure is exactly the same as before
- $Q^{m+n-3} = \partial JJ_{n,m} - \{\text{interior of the cell identified with } f^{\otimes n}\omega_B^{n,m}\}$

The General Transfer Procedure

- The procedure is exactly the same as before
- $Q^{m+n-3} = \partial JJ_{n,m} - \{$interior of the cell identified with $f^{\otimes n}\omega_B^{n,m}\}$
- $\ominus_m^n \in Hom^{m+n-3}\left(B^{\otimes m}, A^{\otimes n}\right)$ is identified with Q^{m+n-3}

The General Transfer Procedure

- The procedure is exactly the same as before
- $Q^{m+n-3} = \partial JJ_{n,m} - \{\text{interior of the cell identified with } f^{\otimes n}\omega_B^{n,m}\}$
- $\ominus_m^n \in Hom^{m+n-3}\left(B^{\otimes m}, A^{\otimes n}\right)$ is identified with Q^{m+n-3}
- *Given* $\left\{\omega_B^{j,i},\ f_i^j\right\}_{1\leq i+j<k}$ define f_m^n and $\omega_B^{n,m}$ for each (m,n) with $m+n=k$ *so that*

$$\nabla f_m^n = \ominus_m^n - f^{\otimes n}\omega_B^{n,m}$$

The General Transfer Procedure

- The procedure is exactly the same as before
- $Q^{m+n-3} = \partial JJ_{n,m} - \{\text{interior of the cell identified with } f^{\otimes n}\omega_B^{n,m}\}$
- $\ominus_m^n \in Hom^{m+n-3}\left(B^{\otimes m}, A^{\otimes n}\right)$ is identified with Q^{m+n-3}
- *Given* $\left\{\omega_B^{j,i},\ f_i^j\right\}_{1\leq i+j<k}$ define f_m^n and $\omega_B^{n,m}$ for each (m,n) with $m+n=k$ *so that*

$$\nabla f_m^n = \ominus_m^n - f^{\otimes n}\omega_B^{n,m}$$

- Continue inductively

Example 2: Transfer of A-infinity Bialgebra Structure

- Let (M, d) be as in Example 1

Example 2: Transfer of A-infinity Bialgebra Structure

- Let (M, d) be as in Example 1
- Extend primitive Δ on M to $T^a M$ as an algebra map

Example 2: Transfer of A-infinity Bialgebra Structure

- Let (M, d) be as in Example 1
- Extend primitive Δ on M to $T^a M$ as an algebra map
- d is a coderivation of Δ since $T^a M$ is primitively generated

Example 2: Transfer of A-infinity Bialgebra Structure

- Let (M, d) be as in Example 1
- Extend primitive Δ on M to $T^a M$ as an algebra map
- d is a coderivation of Δ since $T^a M$ is primitively generated
- $(T^a M, d, \Delta)$ is a DG bialgebra

Example 2: Transfer of A-infinity Bialgebra Structure

- Let (M, d) be as in Example 1
- Extend primitive Δ on M to $T^a M$ as an algebra map
- d is a coderivation of Δ since $T^a M$ is primitively generated
- $(T^a M, d, \Delta)$ is a DG bialgebra
- $A = T^a M \,/\, \big(a_2^2 + a_4,\ a_3^2,\ a_4 a_3 + a_3 a_4,\ (a_2 a_3 + a_3 a_2)^2,\ a_2 b_2,$
  $b_2 a_2,\ a_3 b_2,\ b_2 a_3,\ b_2^2\big)$

Example 2: Transfer of A-infinity Bialgebra Structure

- Let (M, d) be as in Example 1
- Extend primitive Δ on M to $T^a M$ as an algebra map
- d is a coderivation of Δ since $T^a M$ is primitively generated
- $(T^a M, d, \Delta)$ is a DG bialgebra
- $A = T^a M \,/\, \big(a_2^2 + a_4,\ a_3^2,\ a_4 a_3 + a_3 a_4,\ (a_2 a_3 + a_3 a_2)^2,\ a_2 b_2,$
  $b_2 a_2,\ a_3 b_2,\ b_2 a_3,\ b_2^2\big)$
- $H = H^*(A)$

Example 2 (continued)

- All A_∞-bialgebra relations hold; one holds non-trivially:

$$\Delta\mu_H^3 = \left[\mu_H\left(\mu_H \otimes 1\right) \otimes \mu_H^3 + \mu_H^3 \otimes \mu_H\left(1 \otimes \mu_H\right)\right]\sigma_{2,3}\Delta^{\otimes 3}$$

Example 2 (continued)

- All A_∞-bialgebra relations hold; one holds non-trivially:

$$\Delta\mu_H^3 = \left[\mu_H\left(\mu_H \otimes 1\right) \otimes \mu_H^3 + \mu_H^3 \otimes \mu_H\left(1 \otimes \mu_H\right)\right]\sigma_{2,3}\Delta^{\otimes 3}$$

- $\left(H, \mu_H, \mu_H^3, \Delta\right)$ is an A_∞-bialgebra and F is an A_∞-bialgebra map

Homology of Loop Spaces

- Let $\mathbf{k}$ be a field and let X be a space

  Theorem 3. *$H_*(\Omega X; \mathbf{k})$ is an A_∞-bialgebra*

- Let $\mathbf{k}$ be a field and let X be a space

Theorem 3. *$H_*(\Omega X; \mathbf{k})$ is an A_∞-bialgebra*

- **Proof**. Choose a cycle-selection homomorphism

$$f : H_*(\Omega X; \mathbf{k}) \to C_*(\Omega X; \mathbf{k})$$

Homology of Loop Spaces

- Let $\mathbf{k}$ be a field and let X be a space

  Theorem 3. $H_*(\Omega X; \mathbf{k})$ *is an* A_∞*-bialgebra*

- **Proof**. Choose a cycle-selection homomorphism

$$f : H_*(\Omega X; \mathbf{k}) \longrightarrow C_*(\Omega X; \mathbf{k})$$

- $\tilde{f}$ is a quasi-isomorphism since $H_*(\Omega X; \mathbf{k})$ is free

- Let $\mathbf{k}$ be a field and let X be a space

  Theorem 3. *$H_*(\Omega X;\mathbf{k})$ is an A_∞-bialgebra*

- **Proof**. Choose a cycle-selection homomorphism

$$f : H_*(\Omega X;\mathbf{k}) \longrightarrow C_*(\Omega X;\mathbf{k})$$

- $\tilde{f}$ is a quasi-isomorphism since $H_*(\Omega X;\mathbf{k})$ is free

- Apply Theorem 2

Rational Cohomology of Loop Spaces

- Let X be a simply connected space

Rational Cohomology of Loop Spaces

- Let X be a simply connected space
- $H = H^*(\Omega X; \mathbb{Q})$ is an A_∞-coalgebra with operations $\Delta^n : H \to H^{\otimes n}$

Rational Cohomology of Loop Spaces

- Let X be a simply connected space
- $H = H^*(\Omega X; \mathbb{Q})$ is an A_∞-coalgebra with operations $\Delta^n : H \to H^{\otimes n}$
- The cup product μ is compatible with Δ^n in the following sense:

Rational Cohomology of Loop Spaces

- Let X be a simply connected space
- $H = H^*(\Omega X; \mathbb{Q})$ is an A_∞-coalgebra with operations $\Delta^n : H \to H^{\otimes n}$
- The cup product μ is compatible with Δ^n in the following sense:
- Compatibility with $\Delta = \Delta^2$ is expressed by the classical Hopf relation

$$\Delta\mu = (\mu \otimes \mu)\, \sigma_{2,2}\, (\Delta \otimes \Delta)$$

Rational Cohomology of Loop Spaces

- Let X be a simply connected space
- $H = H^*(\Omega X; \mathbb{Q})$ is an A_∞-coalgebra with operations $\Delta^n : H \to H^{\otimes n}$
- The cup product μ is compatible with Δ^n in the following sense:
- Compatibility with $\Delta = \Delta^2$ is expressed by the classical Hopf relation

$$\Delta\mu = (\mu \otimes \mu)\,\sigma_{2,2}\,(\Delta \otimes \Delta)$$

- Compatibility with Δ^3 is expressed by the relation

$$\Delta^3\mu = \mu^{\otimes 3}\sigma_{3,2}\left[(\Delta \otimes 1)\,\Delta \otimes \Delta^3 + \Delta^3 \otimes (1 \otimes \Delta)\,\Delta\right]$$

Rational Cohomology of Loop Spaces

- Compatibility with Δ^n is expressed in terms of the S-U diagonal on cellular chains of associahedra:

$$\Delta_K : C_*(K) \to C_*(K) \otimes C_*(K)$$

Rational Cohomology of Loop Spaces

- Compatibility with Δ^n is expressed in terms of the S-U diagonal on cellular chains of associahedra:

$$\Delta_K : C_*(K) \to C_*(K) \otimes C_*(K)$$

- Let e^{n-2} denote the top dimensional cell of K_n

Rational Cohomology of Loop Spaces

- Compatibility with Δ^n is expressed in terms of the S-U diagonal on cellular chains of associahedra:

$$\Delta_K : C_*(K) \to C_*(K) \otimes C_*(K)$$

- Let e^{n-2} denote the top dimensional cell of K_n

- Represent the A_∞-coalgebra structure by

$$\xi : C_*(K) \to Hom\left(H, H^{\otimes n}\right)$$

Rational Cohomology of Loop Spaces

- Compatibility with Δ^n is expressed in terms of the S-U diagonal on cellular chains of associahedra:

$$\Delta_K : C_*(K) \to C_*(K) \otimes C_*(K)$$

- Let e^{n-2} denote the top dimensional cell of K_n

- Represent the A_∞-coalgebra structure by

$$\xi : C_*(K) \to Hom\left(H, H^{\otimes n}\right)$$

- Then $\xi\left(e^{n-2}\right) = \Delta^n$ and

$$\Delta^n \mu = \mu^{\otimes n} \sigma_{n,2}\left[(\xi \otimes \xi)\,\Delta_K\left(e^{n-2}\right)\right]$$

The End

Thank you!