

A Non-operadic Operation on Loop Cohomology

Joint work with Samson Saneblidze

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The space X

- Denote $\bar{a}_i \in H^i(S^i; \mathbb{Z}_2)$, $i = 2, 3$, and $\bar{b} \in H^3(\Sigma\mathbb{C}P^2; \mathbb{Z}_2)$

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- Consider the following pullback of the path fibration:

$$\begin{array}{ccccc}
 K(\mathbb{Z}_2, 4) & \longrightarrow & X & \longrightarrow & \mathcal{L}K(\mathbb{Z}_2, 5) \\
 & & p \downarrow & & \downarrow \\
 (S^2 \times S^3) \vee \Sigma\mathbb{C}P^2 & \xrightarrow{f} & K(\mathbb{Z}_2, 5) \\
 \bar{a}_2\bar{a}_3 + Sq^2\bar{b} & \xleftarrow{f^*} & l_5
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- ▶ $H^*(X; \mathbb{Z}_2) = \{1, a_2, a_3, b, a_2 a_3 = Sq^2 b, \dots\}$

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- ▶ The induced map $\phi: BA \otimes BA \rightarrow A$ is given by

$$\phi([x] \otimes e) = \phi(e \otimes [x]) = x$$

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and zero otherwise

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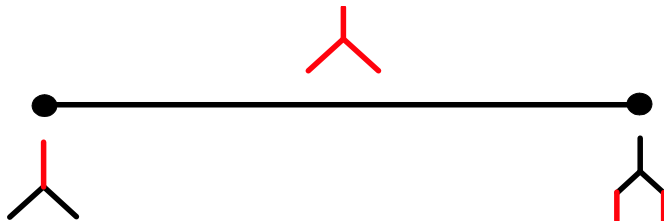
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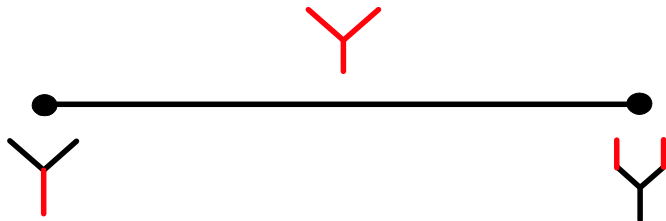
- ▶ The transfer process is controlled by *bimultiplihedra*

The Bimultiplihedron $JJ(1,2)$:



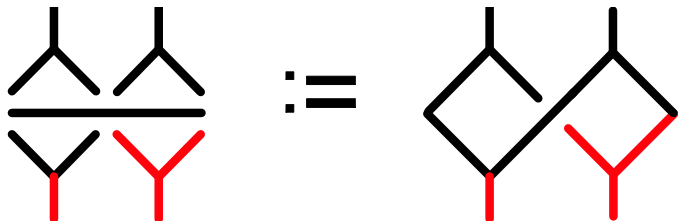
$$dg_2^1 = g\mu_H + \mu(g \otimes g)$$

The Bimultiplihedron $JJ(2,1)$:



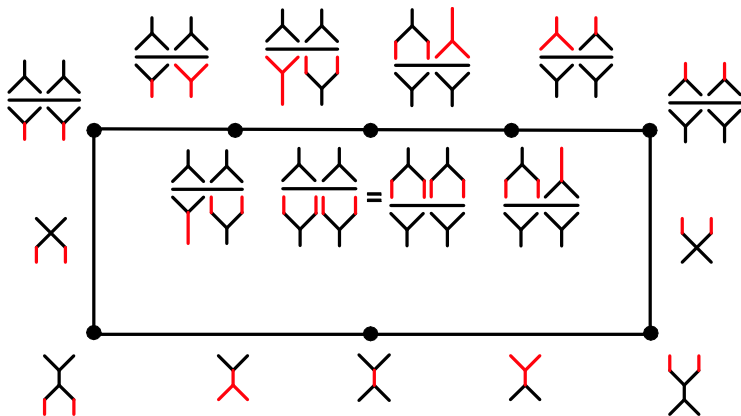
$$(d \otimes \mathbf{1} + \mathbf{1} \otimes d) g_1^2 = \Delta g + (g \otimes g) \Delta_H$$

Fraction notation

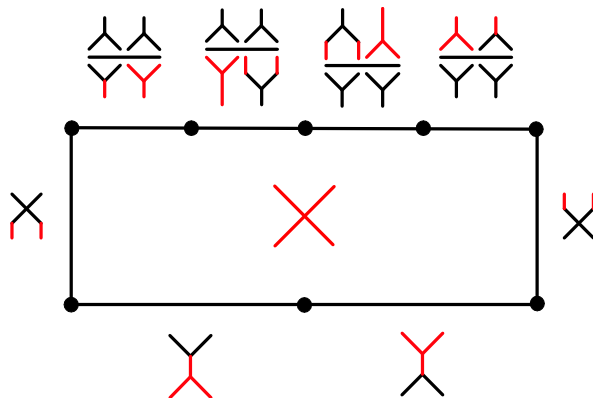


$$(\mu \otimes \mu) \sigma_{2,2} (\Delta g \otimes g_1^2)$$

The boundary of $JJ(2,2)$:



The $JJ(2,2)$ relation :



$$\begin{aligned}
 (d \otimes \mathbf{1} + \mathbf{1} \otimes d) g_2^2 &= (\mu \otimes \mu) \sigma_{2,2} (\Delta g \otimes g_1^2 + g_1^2 \otimes (g \otimes g) \Delta_H) \\
 &\quad + (\mu (g \otimes g) \otimes g_2^1 + g_2^1 \otimes g \mu_H) \sigma_{2,2} (\Delta_H \otimes \Delta_H) \\
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- $\mu([b] \otimes [b]) = [a_2 a_3] = d[a_2 | a_3]$ implies

$$\mu_H(\beta \otimes \beta) = 0$$

g is homotopy multiplicative:

- ▶ The Transfer Theorem implies that the $JJ_{1,2}$ relation holds:

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$$g_2^1(\beta \otimes \beta) = [a_i | a_{5-i}] \text{ for some } i \in \{2, 3\}$$

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$$g_1^2 (\beta) = \lambda \otimes [a_2] + [a_2] \otimes \rho$$

for some $\lambda, \rho \in \mathbb{Z}_2$

Non-triviality of the (2,2)-operation

- The Transfer Theorem implies that the $JJ_{2,2}$ relation holds:

$$\begin{aligned}(d \otimes \mathbf{1} + \mathbf{1} \otimes d) g_2^2 &= (\mu \otimes \mu) \sigma_{2,2} (\Delta g \otimes g_1^2 + g_1^2 \otimes (g \otimes g) \Delta_H) \\ &\quad + (\mu (g \otimes g) \otimes g_2^1 + g_2^1 \otimes g \mu_H) \sigma_{2,2} (\Delta_H \otimes \Delta_H) \\ &\quad + \omega_{BA}^{2,2} (g \otimes g) + (g \otimes g) \omega_H^{2,2} + \Delta g_2^1 + g_1^2 \mu_H\end{aligned}$$

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- ▶ Let's evaluate the $JJ_{2,2}$ relation on $\beta \otimes \beta$

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 $= (\mu \otimes \mu) \sigma_{2,2} (\Delta [b] \otimes g_1^2 (\beta) + g_1^2 (\beta) \otimes \Delta [b])$
 $= 0$ for every choice of $g_1^2 (\beta)$

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- ▶
$$\begin{aligned}(\mu (g \otimes g) \otimes g_2^1 + g_2^1 \otimes g \mu_H) \sigma_{2,2} (\Delta_H \otimes \Delta_H) (\beta \otimes \beta) \\&= (\mu (g \otimes g) \otimes g_2^1 + g_2^1 \otimes g \mu_H) \sigma_{2,2} (\Delta_H (\beta) \otimes \Delta_H (\beta)) \\&= 1 \otimes g_2^1 (\beta \otimes \beta) + g_2^1 (\beta \otimes \beta) \otimes 1 \\&= (\Delta + \overline{\Delta}) g_2^1 (\beta \otimes \beta)\end{aligned}$$

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- ▶ Thus the right-hand side of the $JJ_{2,2}$ relation reduces to

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- ▶ Thus $(d \otimes \mathbf{1} + \mathbf{1} \otimes d) g_2^2 (\beta \otimes \beta)$

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Non-triviality of the (2,2)-operation

- ▶ Thus the right-hand side of the $JJ_{2,2}$ relation reduces to

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- ▶ Since g is a cochain map, we conclude

$$\omega_H^{2,2} (\beta \otimes \beta) = \alpha_i \otimes \alpha_{5-i}$$

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- ▶ **Conclusion:** *There is an induced non-operadic operation*

$$\omega_H^{2,2} : H \otimes H \rightarrow H \otimes H$$

of degree -1 , which is non-vanishing on $\text{cls}[b] \otimes \text{cls}[b]$

Thank you!