A Non-operadic Operation on Loop Cohomology

Joint work with Samson Saneblidze

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The space X

- Denote $\bar{a}_i \in H^i\left(S^i;\mathbb{Z}_2\right)$, $i = 2, 3$, and $\bar{b} \in H^3\left(\Sigma\mathbb{C}P^2;\mathbb{Z}_2\right)$

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- Consider the following pullback of the path fibration:

$$\begin{array}{ccccc} K\left(\mathbb{Z}_2, 4\right) & \longrightarrow & X & \longrightarrow & \mathcal{L}K\left(\mathbb{Z}_2, 5\right) \\ & & p \downarrow & & \downarrow \\ & & \left(S^2 \times S^3\right) \vee \Sigma\mathbb{C}P^2 & \xrightarrow{f} & K\left(\mathbb{Z}_2, 5\right) \\ & & \bar{a}_2\bar{a}_3 + Sq^2\bar{b} & \underset{f^*}{\longleftarrow} & \iota_5 \end{array}$$

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- $H^*\left(X;\mathbb{Z}_2\right) = \left\{1, a_2, a_3, b, a_2a_3 = Sq^2b, \ldots\right\}$

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- The induced map $\phi : BA \otimes BA \longrightarrow A$ is given by
$$\phi\left([x] \otimes e\right) = \phi\left(e \otimes [x]\right) = x$$
$$\phi\left([b] \otimes [b]\right) = b \smile_1 b = a_2 a_3$$
and zero otherwise

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- Then $\mu\,([b] \otimes [b]) = [a_2 a_3]$

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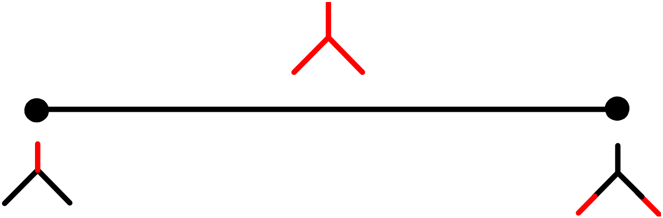
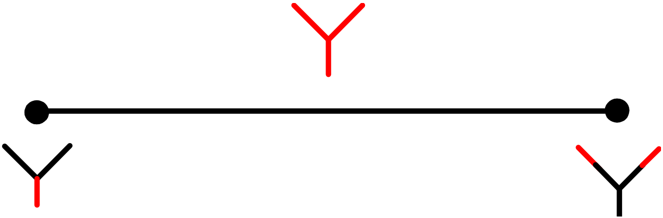
- The transfer process is controlled by *bimultiplihedra*

The Bimultiplihedron $JJ(1,2)$:



$$\nabla g_2^1 = g\mu_H + \mu\,(g \otimes g)$$

The Bimultiplihedron $JJ(2,1)$:



$$\nabla g_1^2 = \Delta g + (g \otimes g)\,\Delta_H$$

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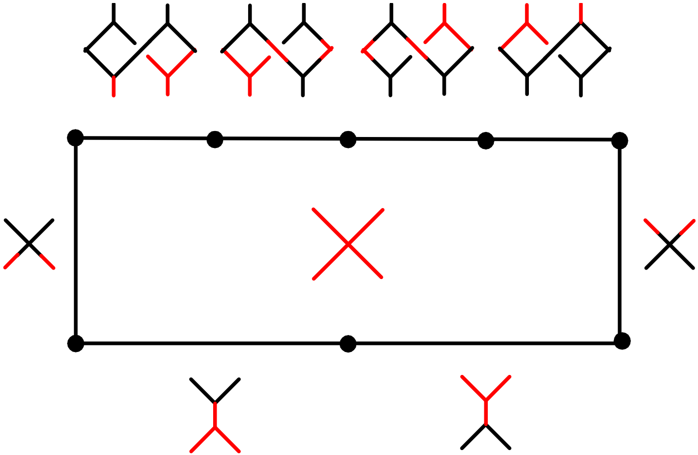

$$\begin{aligned}\nabla g_2^2 \quad = \quad & (\mu \otimes \mu)\,\sigma_{2,2}\left(\Delta g \otimes g_1^2 + g_1^2 \otimes (g \otimes g)\,\Delta_H\right) \\ & + \left(\mu\,(g \otimes g) \otimes g_2^1 + g_2^1 \otimes g\mu_H\right)\sigma_{2,2}\left(\Delta_H \otimes \Delta_H\right) \\ & + \omega_{BA}^{2,2}\,(g \otimes g) + (g \otimes g)\,\omega_H^{2,2} + \Delta g_2^1 + g_1^2 \mu_H\end{aligned}$$

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- Choose a cocycle-selecting map $g : H \to BA$ such that

$$g\left(\mathrm{cls}\left[x_1|\cdots|x_n\right]\right) = \left[x_1|\cdots|x_n\right]$$

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- Let $\alpha_i = \mathrm{cls}\left[a_i\right]$ and $\beta = \mathrm{cls}\left[b\right]$
- $\mu\left(\left[b\right]\otimes\left[b\right]\right) = \left[a_2 a_3\right] = d\left[a_2\middle|a_3\right]$ implies

$$\mu_H\left(\beta\otimes\beta\right) = 0$$

Cocycle-selecting map g is homotopy multiplicative

- The Transfer Theorem implies that the $JJ_{1,2}$ relation holds:

$$\nabla g_2^1 = g\mu_H + \mu(g \otimes g)$$

for some cochain homotopy g_2^1

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- *The cochain homotopy* $g_2^1 \neq 0$ *and satisfies*

  $$g_2^1(\beta \otimes \beta) \in \{[a_2|a_3], [a_3|a_2]\}.$$

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- *The cochain homotopy g_1^2 satisfies*

  $$g_1^2\,(\beta) = \lambda \otimes [a_2] + [a_2] \otimes \rho$$

  for some $\lambda, \rho \in \mathbb{Z}_2$

Non-triviality of the (2,2)-operation

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- Let us evaluate the $JJ_{2,2}$ relation on $\beta\otimes\beta$

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- $\left(\mu (g \otimes g) \otimes g_2^1 + g_2^1 \otimes g \mu_H \right) \sigma_{2,2} (\Delta_H \otimes \Delta_H) (\beta \otimes \beta)$

  $$= \left(\mu (g \otimes g) \otimes g_2^1 + g_2^1 \otimes g \mu_H \right) \sigma_{2,2} (\Delta_H (\beta) \otimes \Delta_H (\beta))$$

  $$= e \otimes g_2^1 (\beta \otimes \beta) + g_2^1 (\beta \otimes \beta) \otimes e$$

  $$= \left(\Delta + \overline{\Delta} \right) g_2^1 (\beta \otimes \beta)$$

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- Adding $\Delta g_2^1(\beta \otimes \beta)$ reduces the $JJ_{2,2}$ relation to

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- $\overline{\Delta}g_2^1\left(\beta\otimes\beta\right)$ and $\left(g\otimes g\right)\omega_H^{2,2}\left(\beta\otimes\beta\right)$ are cohomologous

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- Adding $\Delta g_2^1\left(\beta \otimes \beta\right)$ reduces the $JJ_{2,2}$ relation to

$$\nabla g_2^2\left(\beta \otimes \beta\right) = \left(\overline{\Delta} g_2^1 + \left(g \otimes g\right) \omega_H^{2,2}\right)\left(\beta \otimes \beta\right)$$

- But $\nabla g_2^2\left(\beta \otimes \beta\right) = \left(d \otimes \mathbf{1} + \mathbf{1} \otimes d\right) g_2^2\left(\beta \otimes \beta\right)$
- $\overline{\Delta} g_2^1\left(\beta \otimes \beta\right)$ and $\left(g \otimes g\right) \omega_H^{2,2}\left(\beta \otimes \beta\right)$ are cohomologous
- $\mathrm{cls}\left\{\left(g \otimes g\right) \omega_H^{2,2}\left(\beta \otimes \beta\right)\right\} = \mathrm{cls}\left\{\overline{\Delta} g_2^1\left(\beta \otimes \beta\right)\right\}$

Non-triviality of the (2,2)-operation

- $\overline{\Delta} g_2^1 (\beta \otimes \beta) \in \{[a_2] \otimes [a_3], [a_3] \otimes [a_2]\}$ *does not cobound*

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- Since g is a cochain map, we conclude that

$$\omega_H^{2,2} (\beta \otimes \beta) \neq 0 \text{ in } H \otimes H$$

Summarizing...

- Let $K(\mathbb{Z}_2, 4) \to X \xrightarrow{p} (S^2 \times S^3) \vee \Sigma\mathbb{C}P^2$ be the fibration

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- **Conclusion:** *There is an induced non-operadic operation*

$$\omega_H^{2,2} : H \otimes H \to H \otimes H$$

of degree -1, *which is non-vanishing on* $\mathrm{cls}[b] \otimes \mathrm{cls}[b]$

Thank you!