Computing the Cohomology Algebra of a Polyhedral Complex

Joint work with R. Gonzalez-Diaz & J. Lamar

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Polyhedral Complexes

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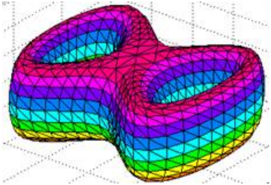

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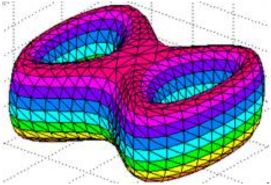
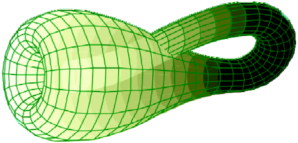


- X is *cubical* if its k-cells are k-cubes



Diagonal Approximations

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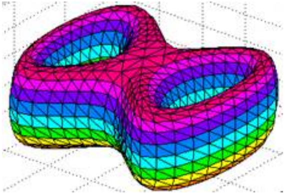
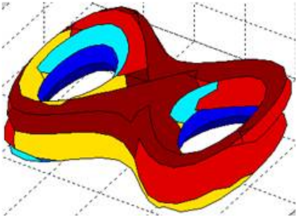
  - Commutes with the boundary operator

$$\Delta_X \partial = (\partial \otimes Id + Id \otimes \partial)\, \Delta_X$$

Goals of the Talk

1. *Transform a simplicial or cubical complex X into a polyhedral complex P*





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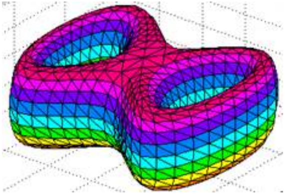
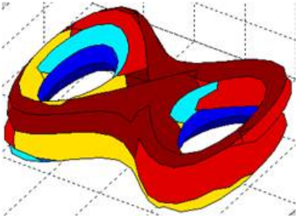
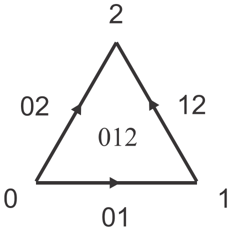




2. *Given a diagonal on $C_*(X)$, induce a diagonal on $C_*(P)$*

Alexander-Whitney Diagonal on the Simplex

$$\Delta_s(012\cdots n) = \sum_{i=0}^{n} 012\cdots i \otimes i\cdots n$$

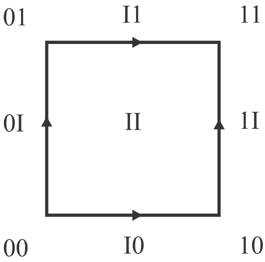


$$\Delta_s(012) = 0 \otimes 012 + 01 \otimes 12 + 012 \otimes 2$$

Serre Diagonal on the Cube

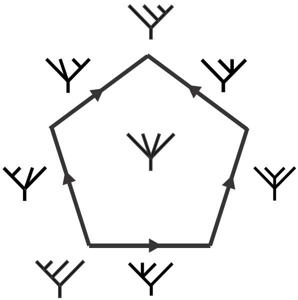
$$\Delta_{\mathtt{I}}\left(\mathtt{I}^n\right) = \sum_{(u_1,\ldots,u_n)\in\{0,\mathtt{I}\}^{\times n}} u_1 \cdots u_n \otimes u_1' \cdots u_n'$$

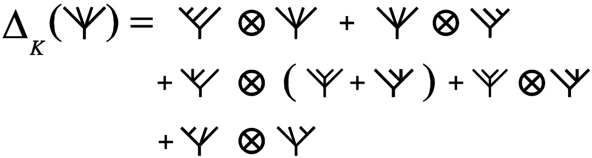
$$(0' = \mathtt{I} \text{ and } \mathtt{I}' = 1)$$



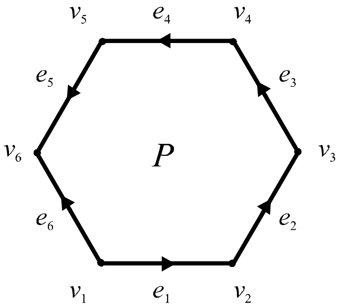
$$\Delta_{\mathtt{I}}\left(\mathtt{I}^2\right) = 00 \otimes \mathtt{II} + 0\mathtt{I} \otimes \mathtt{I}1 + \mathtt{I}0 \otimes 1\mathtt{I} + \mathtt{II} \otimes 11$$

S-U Diagonal on the Associahedron





A Diagonal on an n-gon P



- Vertices labeled $v_1, v_2, \ldots, v_n$

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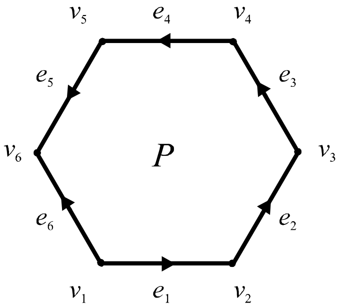

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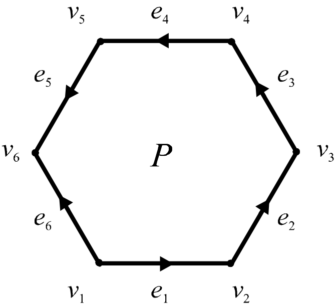

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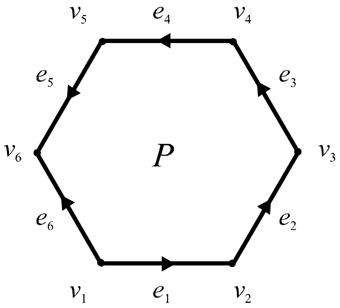

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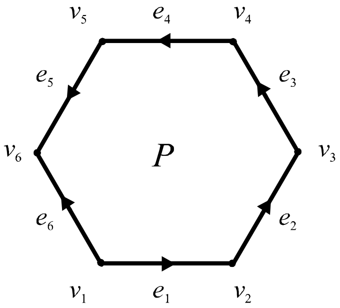

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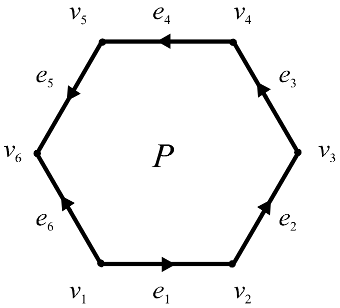

- **Theorem** (D. Kravatz, 2008 thesis) *There is a diagonal approximation on $C_*(P)$ defined by*

$$\Delta_P(v_i) = v_i \otimes v_i$$
$$\Delta_P(e_i) = v_i \otimes e_i + e_i \otimes v_{i+1} \text{ if } i < n$$
$$\Delta_P(e_n) = v_1 \otimes e_n + e_n \otimes v_n$$
$$\Delta_P(P) = v_1 \otimes P + P \otimes v_n + \sum_{0<i_1<i_2<n} e_{i_1} \otimes e_{i_2}$$

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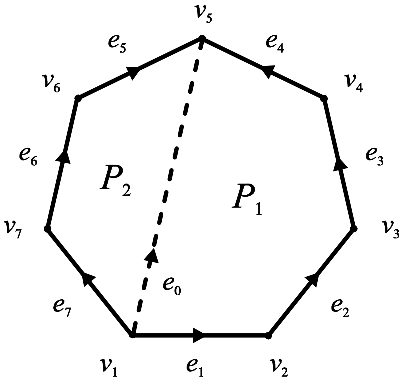

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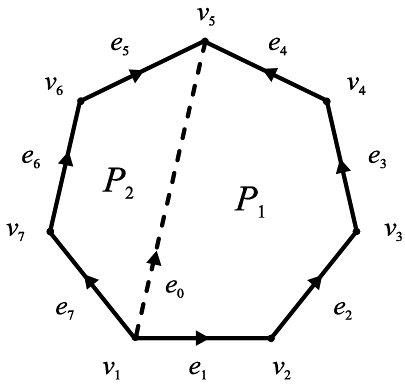

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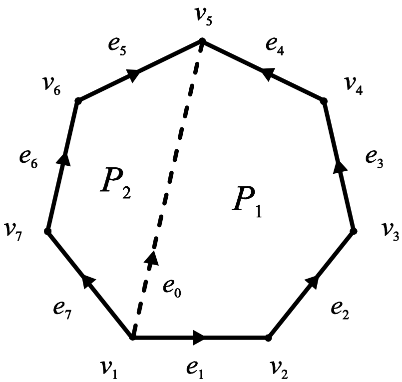

- Let P_1 be the subpolygon with vertices $v_1, v_2, \ldots, v_t$
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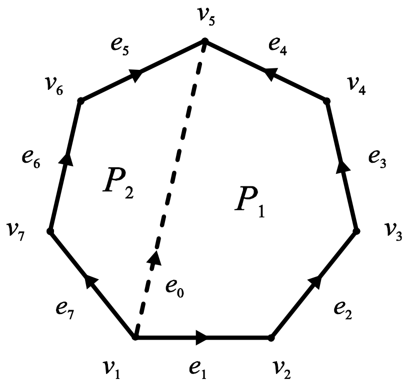

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A General Diagonal on an n-gon P

Corollary *Let P be an n-gon with initial vertex v_1 and terminal vertex v_t. Then*

$$\Delta'_P(P) = v_1 \otimes P + P \otimes v_t + \sum_{0<i_1<i_2<t} e_{i_1} \otimes e_{i_2} + \sum_{n \geq i_1 > i_2 \geq t} e_{i_1} \otimes e_{i_2}$$

is a diagonal approximation on $C_*(P)$

Application to Closed Compact Surfaces

Let X_g be a closed compact surface of genus g. The celebrated Classification of Closed Compact Surfaces states that X_g *is homeomorphic to a*

- *Sphere with* $g \geq 0$ *handles when orientable*

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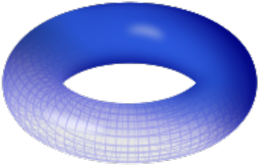
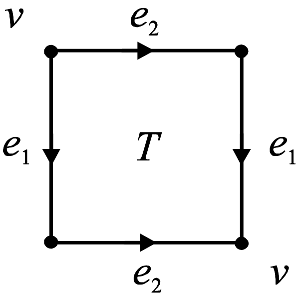
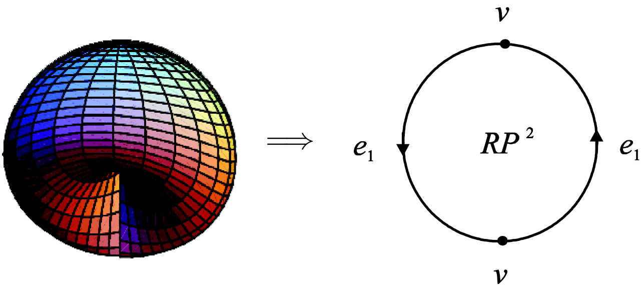
Polygonal Decomposition of a Torus



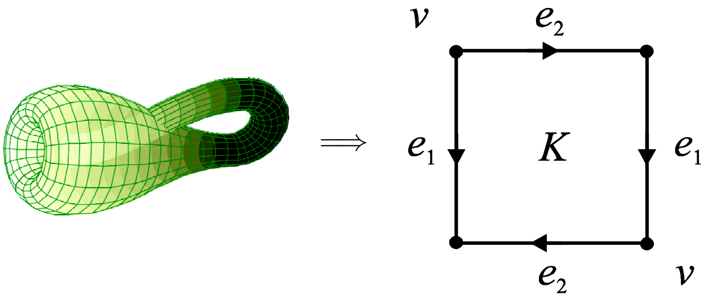
$\Longrightarrow$



Polygonal Decomposition of Real Projective Plane

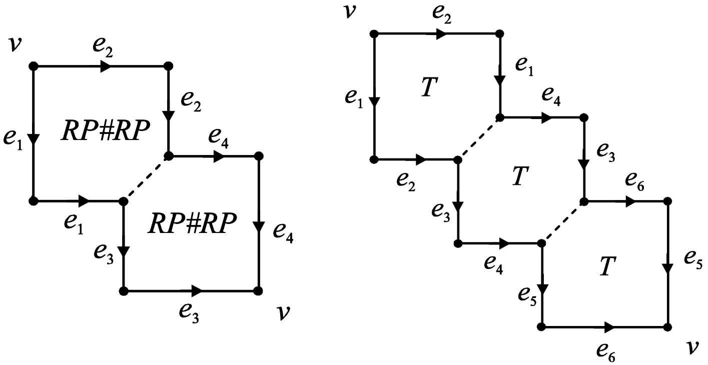


Polygonal Decomposition of a Klein Bottle



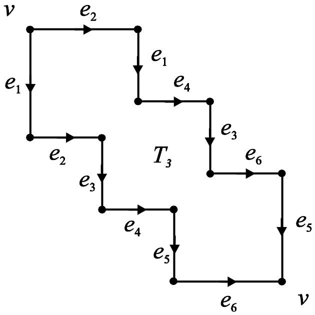
Connected Sums

To obtain the connected sum $X\#Y$ of two surfaces, remove the interior of a disk from X and from Y then glue the two surfaces together along their boundaries



Connected sums of four real projective planes and three tori

The g-fold Torus as a Quotient of a 4g-gon



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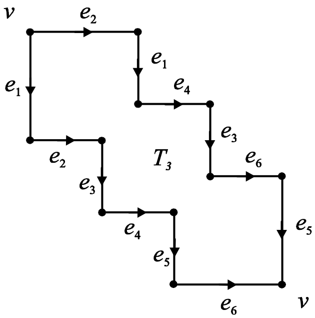

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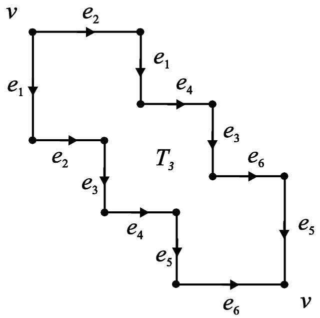

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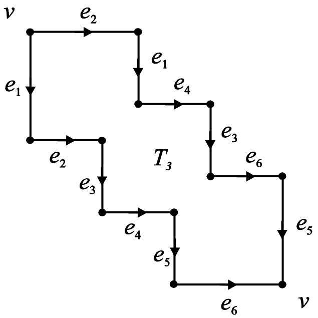

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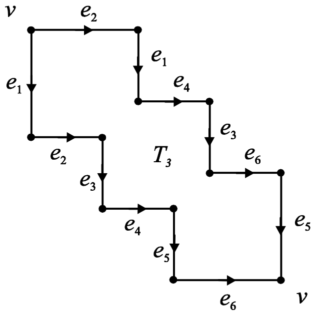
  - one 2-cell T_g

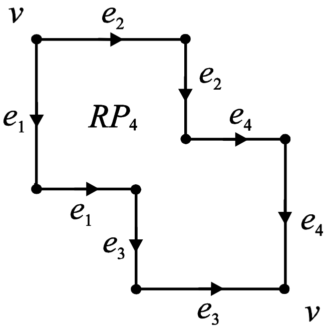
A Diagonal on the g-fold Torus



- *A diagonal on T_g is defined by*

$$\Delta_{T_g}(T_g) = v \otimes T_g + T_g \otimes v + \sum_{i=1}^{g} e_{2i-1} \otimes e_{2i} + e_{2i} \otimes e_{2i-1}$$

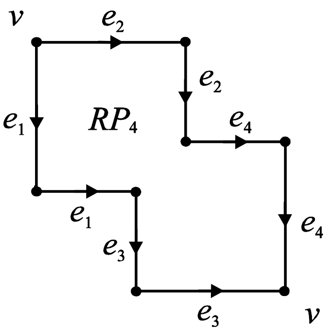
A Diagonal on the g-fold Projective Plane



- *A diagonal on RP_g is defined by*

$$\Delta_{RP_g}(RP_g) = v \otimes RP_g + RP_g \otimes v + \sum_{i=1}^{g} e_i \otimes e_i$$

A Diagonal on the g-fold Projective Plane



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- Δ_{T_g} *and* Δ_{RP_g} *are strikingly different and determine the homeomorphism type of the surface*

Cohomology of a Closed Compact Surface

- Choose a polygonal decomposition of X_g

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- If $x \in H_k (X_g)$, define $x^* (e) = \begin{cases} 1, & \text{if } e = x \\ 0, & \text{otherwise} \end{cases}$
- $H^k (X_g) = \{x^* : x \in H_k (X_g)\}$

Cohomology Algebra of a Closed Compact Surface

- Given $x^*, y^* \in H^*(X)$, define $x^* \smile y^* = m(x^* \otimes y^*)\Delta_X$

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- **Example**

$$\Delta_{T_g}(T_g) = v \otimes T_g + T_g \otimes v + \sum_{i=1}^{g} e_{2i-1} \otimes e_{2i} + e_{2i} \otimes e_{2i-1}$$

$$\begin{aligned}
(e_{2i-1}^* \smile e_{2i}^*)(T_g) &= m(e_{2i-1}^* \otimes e_{2i}^*)\Delta_{T_g}(T_g) \\
&= m(e_{2i-1}^* \otimes e_{2i}^*)(e_{2i-1} \otimes e_{2i}) \\
&= m(1 \otimes 1) = 1
\end{aligned}$$

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$$\begin{aligned} (e_{2i-1}^* \smile e_{2i}^*)(T_g) &= m(e_{2i-1}^* \otimes e_{2i}^*)\Delta_{T_g}(T_g) \\ &= m(e_{2i-1}^* \otimes e_{2i}^*)(e_{2i-1} \otimes e_{2i}) \\ &= m(1 \otimes 1) = 1 \end{aligned}$$

- $e_{2i-1}^* \smile e_{2i}^*$ acting non-trivially on T_g implies

$$e_{2i-1}^* \smile e_{2i}^* = T_g^*$$

Cohomology Algebra of a g-Fold Torus

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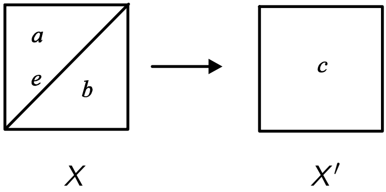
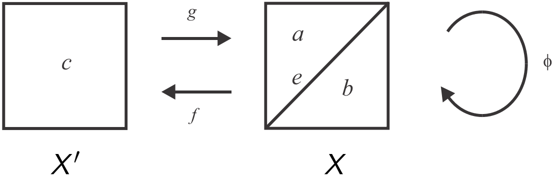
- Obtain the cell complex (X', ∂') with fewer cells



The Chain Contraction



There exist chain maps

- $f : C_*(X) \to C_*(X')$

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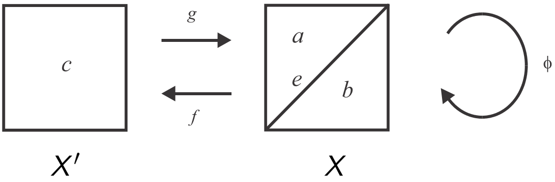

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- $f : C_*(X) \to C_*(X')$
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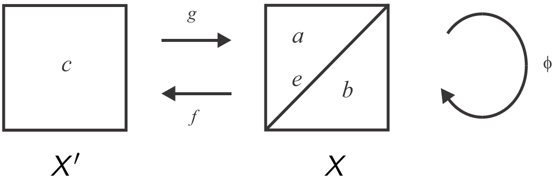

There exist chain maps

- $f : C_* (X) \to C_* (X')$
- $g : C_* (X') \to C_* (X)$
- $\phi : C_* (X) \to C_{*+1} (X)$ defined on generators by

$$
\begin{array}{ll}
f(e) = \partial a + e & g(c) = a + b \\
f(a) = 0 & g(\sigma) = \sigma, \ \sigma \neq c \\
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$$

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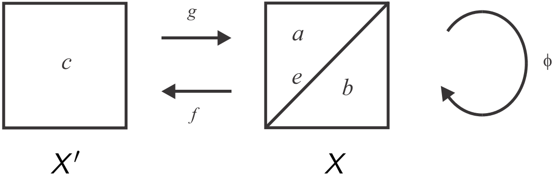

- $fg = Id_{C_*(X')}$ and ϕ is a chain homotopy from gf to $Id_{C_*(X)}$

$$\partial\phi + \phi\partial = Id_{C_*(X)} + gf$$

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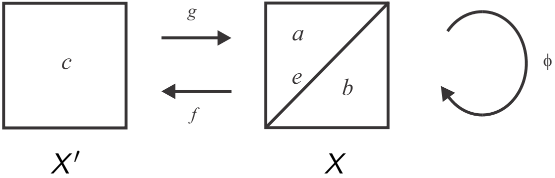

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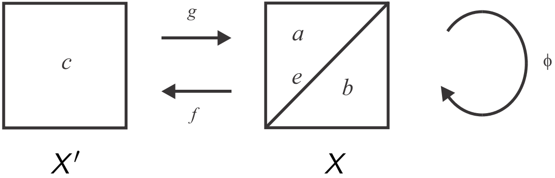

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- (f, g, ϕ) is called a *chain contraction of* $C_*(X)$ *onto* $C_*(X')$

(Introduced by Henri Cartan 1904-2008)

The Transfer Theorem

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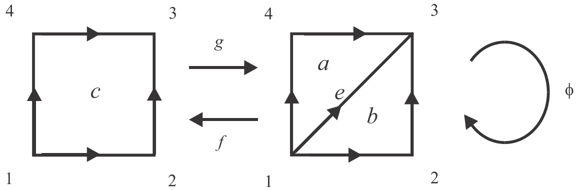
Example: Merging Adjacent 2-Simplices



X' $\qquad\qquad$ X

$$\begin{aligned} f(e) &= \partial a + e \\ f(a) &= 0 \\ f(b) &= c \\ f(\sigma) &= \sigma, \ \ \sigma \neq e, a, b \end{aligned} \qquad \begin{aligned} g(c) &= a + b \\ g(\sigma) &= \sigma, \ \ \sigma \neq c \\ \phi(e) &= a \\ \phi(\sigma) &= 0, \ \ \sigma \neq e \end{aligned}$$

- $\Delta_{X'}(c) = [(f \otimes f) \circ \Delta_X \circ g](c) = (f \otimes f)(\Delta_X(a+b))$

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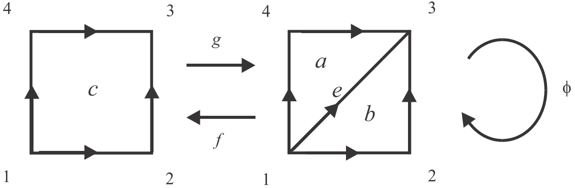

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- $\qquad = (f \otimes f)[1 \otimes a + 14 \otimes 43 + a \otimes 3 + 1 \otimes b + 12 \otimes 23 + b \otimes 3]$

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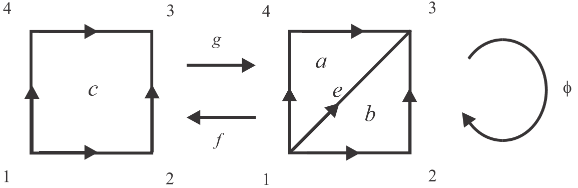

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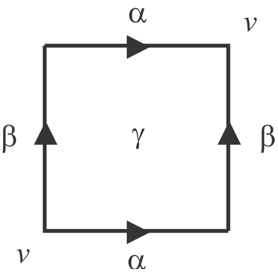
  $+ 1 \otimes b + 12 \otimes 23 + b \otimes 3]$
- $= 1 \otimes c + 12 \otimes 23 + 14 \otimes 43 + c \otimes 3$

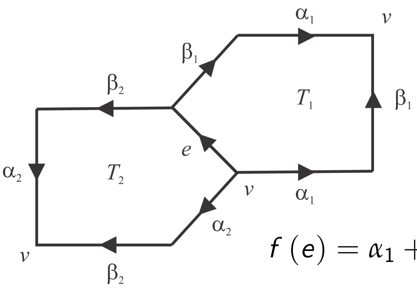
The Cohomology Algebra of a Torus



$$\Delta_T(\gamma) = v \otimes \gamma + \alpha \otimes \beta + \beta \otimes \alpha + \gamma \otimes v$$

$$\alpha^* \smile \beta^* = \beta^* \smile \alpha^* = \gamma^*$$

The Cohomology Algebra of T#T via Chain Contraction



$$f(e) = \alpha_1 + \beta_1 + \alpha_1 + \beta_1 = 0$$

$$\begin{aligned}
\bar{\Delta}'_{T\#T}(T\#T) &= [(f \otimes f) \circ \bar{\Delta}_X \circ g](T\#T) \\
&= [(f \otimes f) \circ \bar{\Delta}_X](T_1 + T_2) \\
&= (f \otimes f)[\alpha_1 \otimes \beta_1 + e \otimes (\beta_1 + \alpha_1) + \beta_1 \otimes \alpha_1 \\
&\qquad + \alpha_2 \otimes \beta_2 + e \otimes (\beta_2 + \alpha_2) + \beta_2 \otimes \alpha_2] \\
&= \alpha_1 \otimes \beta_1 + \beta_1 \otimes \alpha_1 + \alpha_2 \otimes \beta_2 + \beta_2 \otimes \alpha_2
\end{aligned}$$

$$\alpha_i^* \smile \beta_i^* = \beta_i^* \smile \alpha_i^* = T\#T^*$$

Generalization of Kravatz's Diagonal on an n-gon

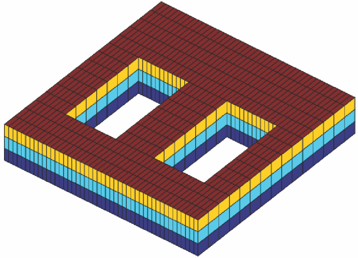
Theorem (G-L-U) *Let X be a* 3D *polyhedral complex with vertices numbered arbitrarily from* 1 *to n. Represent a k-gon P of X as an ordered k-tuple of vertices $\langle i_1, \ldots, i_k \rangle$, where $i_1 = \min \{i_1, \ldots, i_k\}$, i_1 is adjacent to i_k, and i_j is adjacent to i_{j+1} for $1 < j < k$. Then*

$$\begin{aligned}\Delta_P(P) &= i_1 \otimes P + P \otimes i_{m(k)} \\ &\quad + \sum_{j=2}^{m(k)-1} (u_2 + e_2 + \cdots + \lambda_j e_j) \otimes e_j \\ &\quad\quad + \sum_{j=m(k)}^{k-1} [(1+\lambda_j) e_j + e_{j+1} + \cdots + e_{k-1} + u_k] \otimes e_j,\end{aligned}$$

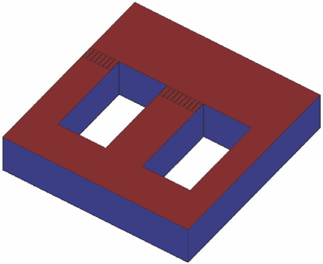
where $i_{m(k)} = \max \{i_2, \ldots, i_k\}$, $\lambda_j = 0$ *iff* $i_j < i_{j+1}$,

$\{u_j = \langle i_1, i_j \rangle\}_{2 \leq j \leq k}$ and $\{e_j = \langle i_j i_{j+1} \rangle\}_{2 \leq j \leq k-1}$

Computational Considerations



X

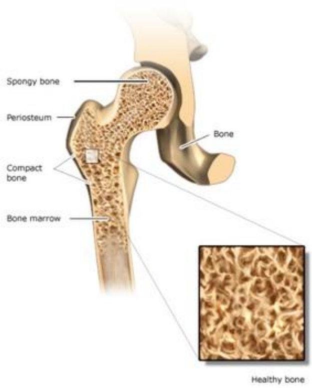


X'

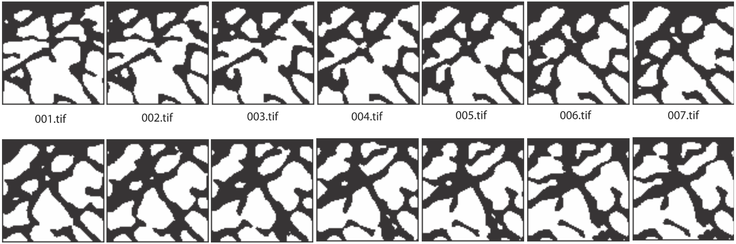
	Number of 2-cells	Cup product computed in
X	1,638	28.00 sec
X'	46	1.04 sec

Trabecular Bone

- Makes up the inner layer of the bone and has a spongy, honeycomb-like structure.

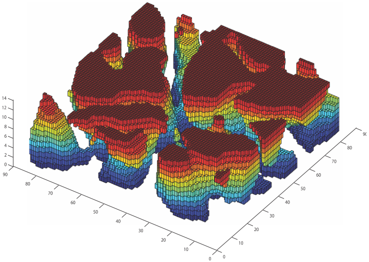


Micro-CT Images of a Trabecular Bone

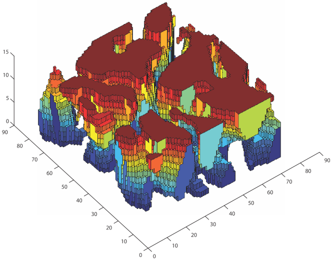


001.tif 002.tif 003.tif 004.tif 005.tif 006.tif 007.tif

008.tif 009.tif 010.tif 011.tif 012.tif 013.tif 014.tif

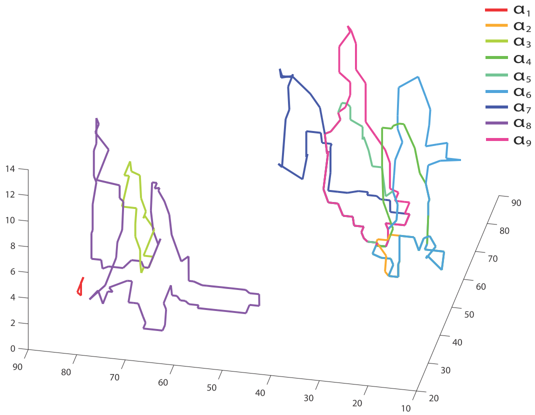


X



X'

Representative 1-cycles



Non-vanishing cup products: $\alpha_2^*\alpha_4^*$, $\alpha_2^*\alpha_5^*$, $\alpha_2^*\alpha_9^*$, $\alpha_3^*\alpha_8^*$, $\alpha_4^*\alpha_5^*$

- Computational methods such as these allow us to identify diseased tissue

Thank you!