Computing the Cohomology Ring of a Polyhedral Complex

Joint work with D. Kravatz, R. Gonzalez-Diaz & J. Lamar

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TGTS

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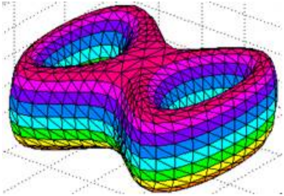
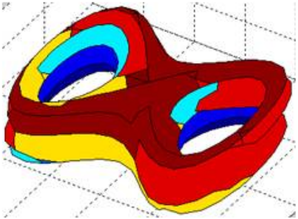
  - Homotopic to the geometric diagonal $\Delta : x \mapsto (x, x)$
  - Induces a *diagonal on cellular chains* $C_* (X)$

$$\text{i.e., a chain map } \Delta_X : C_* (X) \to C_* (X) \otimes C_* (X)$$

Goals of the Talk

1. *Transform a simplicial or cubical complex X into a polyhedral complex P*





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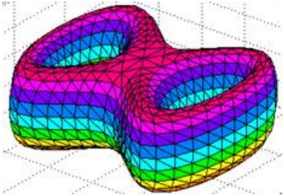
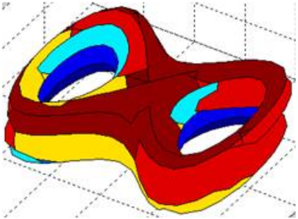
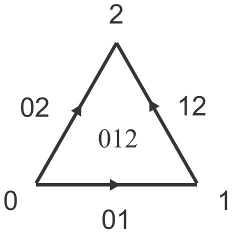




2. *Given a diagonal on $C_*(X)$, induce a diagonal on $C_*(P)$*

Alexander-Whitney Diagonal on the Simplex

$$\Delta_s(012\cdots n) = \sum_{i=0}^{n} 012\cdots i \otimes i\cdots n$$

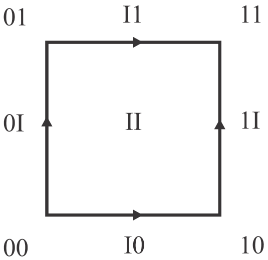


$$\Delta_s(012) = 0 \otimes 012 + 01 \otimes 12 + 012 \otimes 2$$

Serre Diagonal on the Cube

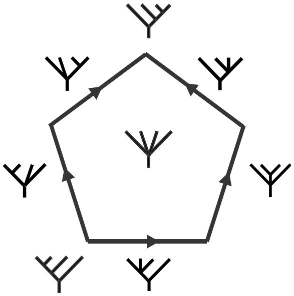
$$\Delta_{\mathtt{I}}\left(\mathtt{I}^n\right) = \sum_{(u_1,\ldots,u_n)\in\{0,\mathtt{I}\}^{\times n}} \pm u_1 \cdots u_n \otimes u_1' \cdots u_n'$$

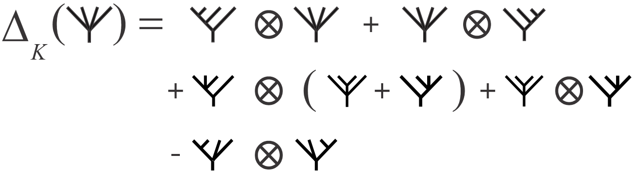
$$(0' = \mathtt{I} \text{ and } \mathtt{I}' = 1)$$



$$\Delta_{\mathtt{I}}\left(\mathtt{I}^2\right) = 00 \otimes \mathtt{II} - 0\mathtt{I} \otimes \mathtt{I}1 + \mathtt{I}0 \otimes 1\mathtt{I} + \mathtt{II} \otimes 11$$

S-U Diagonal on the Associahedron





Dan Kravatz's Diagonal on an n-gon (2008 Thesis)

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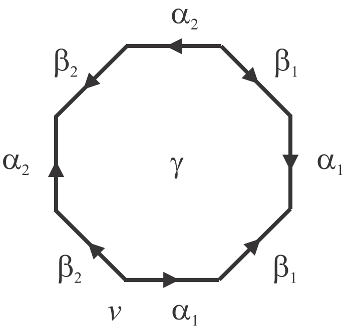
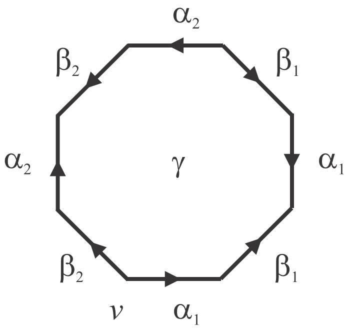
  - one 0-cell v
  - $4n$ 1-cells $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n$
  - one 2-cell γ



Cohomology Ring of T#T



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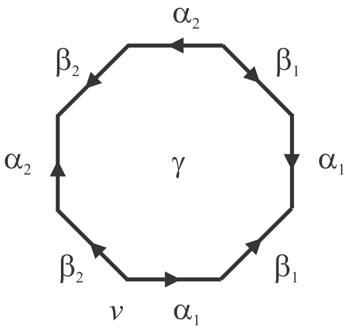

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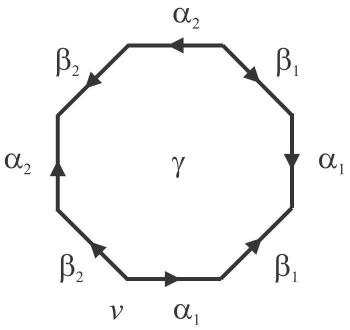

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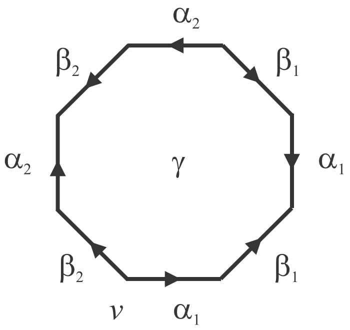

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- $H^*\left(T\#T\right)$ is a graded commutative ring with identity

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- Compute cohomology

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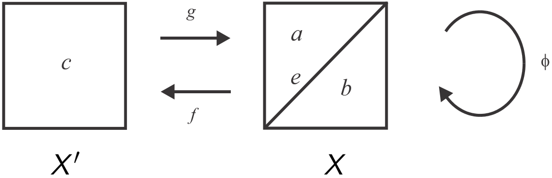
- Remove $int\,(a \cup b)$ and attach a $(k+1)$-cell c along $\partial\,(a \cup b)$
- Obtain the cell complex (X', ∂') with fewer cells

The Chain Contraction

$\exists$ chain maps $f : C_* (X) \to C_* (X')$, $g : C_* (X') \to C_* (X)$, and $\phi : C_* (X) \to C_{*+1} (X)$ defined on generators by

$$\begin{array}{ll} f(e) = \partial a - e & g(c) = a + b \\ f(a) = 0 & g(\sigma) = \sigma,\ \sigma \neq c \\ f(b) = c & \phi(e) = a \\ f(\sigma) = \sigma,\ \sigma \neq e, a, b & \phi(\sigma) = 0,\ \sigma \neq e \end{array}$$



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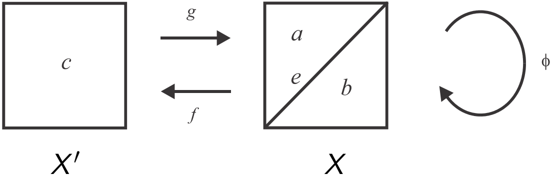

- $fg = Id_{C_*(X')}$ and ϕ is a chain homotopy from gf to $Id_{C_*(X)}$

$$\partial\phi + \phi\partial = Id_{C_*(X)} - gf$$

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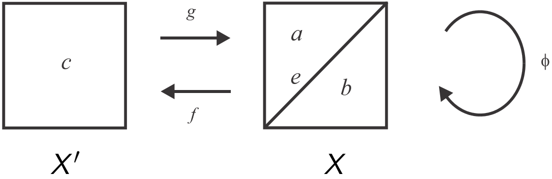

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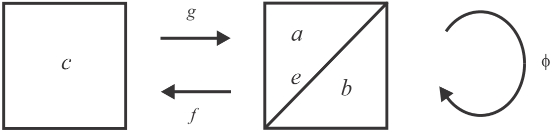

X' $\qquad\qquad$ X

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- (f, g, ϕ) is called a *chain contraction of* $C_*(X)$ *onto* $C_*(X')$ (Introduced by Henri Cartan 1904-2008)

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- **Theorem** *A chain contraction* $\left(f, g, \phi, C_*\left(X\right), C_*\left(X'\right)\right)$ *preserves the algebraic topology of* X

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$$\Delta_X : C_*(X) \to C_*(X) \otimes C_*(X)$$

the composition

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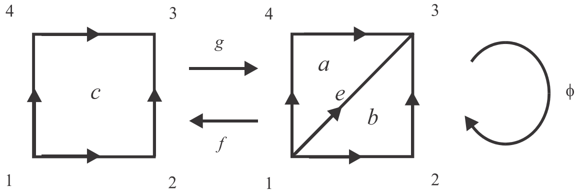
Example: Merging Adjacent 2-Simplices



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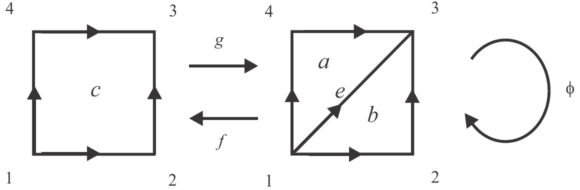

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- $= (f \otimes f)[1 \otimes a - 14 \otimes 43 + a \otimes 3$
  $+ 1 \otimes b + 12 \otimes 23 + b \otimes 3]$

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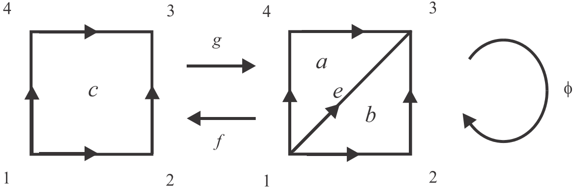

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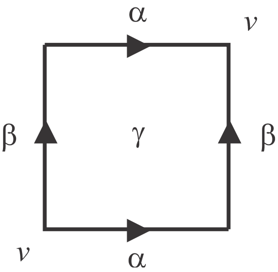
- $= (f \otimes f)[1 \otimes a - 14 \otimes 43 + a \otimes 3$
  $+ 1 \otimes b + 12 \otimes 23 + b \otimes 3]$
- $= 1 \otimes c + 12 \otimes 23 - 14 \otimes 43 + c \otimes 3$

The Cohomology Ring of a Torus

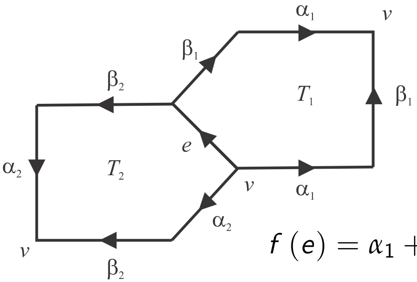


$$\Delta_T(\gamma) = v \otimes \gamma + \alpha \otimes \beta - \beta \otimes \alpha + \gamma \otimes v$$

$$\begin{aligned}
(\alpha^* \smile \beta^*)(\gamma) &= m(\alpha^* \otimes \beta^*)\Delta_T(\gamma) \\
&= m(\alpha^* \otimes \beta^*)(\alpha \otimes \beta) \\
&= 1 = \gamma^*(\gamma)
\end{aligned}$$

$$\alpha^* \smile \beta^* = -\beta^* \smile \alpha^* = \gamma^*$$

The Cohomology Ring of T#T via Chain Contraction

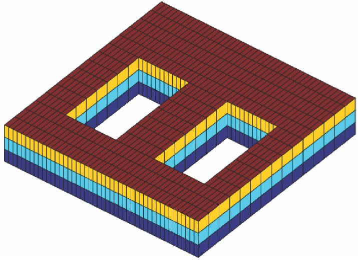


$$f(e) = \alpha_1 + \beta_1 - \alpha_1 - \beta_1 = 0$$

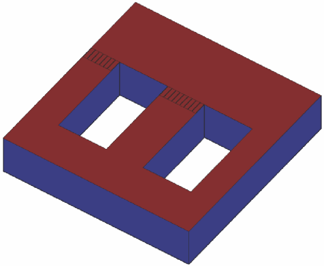
$$\begin{aligned}\overline{\Delta}_{T\#T}(\gamma) &= (f \otimes f)\left[\alpha_1 \otimes \beta_1 - e \otimes (\beta_1 + \alpha_1) - \beta_1 \otimes \alpha_1 \right. \\ &\quad \left. -\alpha_2 \otimes \beta_2 + e \otimes (\beta_2 + \alpha_2) + \beta_2 \otimes \alpha_2\right] \\ &= \alpha_1 \otimes \beta_1 - \beta_1 \otimes \alpha_1 - \alpha_2 \otimes \beta_2 + \beta_2 \otimes \alpha_2\end{aligned}$$

$$\alpha_i^* \smile \beta_i^* = -\beta_i^* \smile \alpha_i^* = \gamma^*$$

Computational Considerations



X

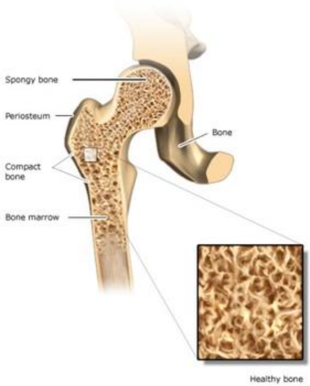


X'

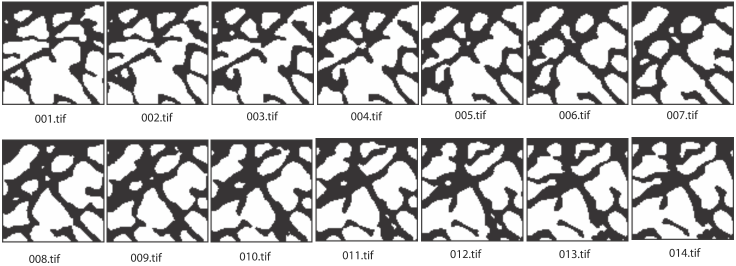
	Number of 2-cells	Cup product computed in
X	1,638	28.00 sec
X'	46	1.04 sec

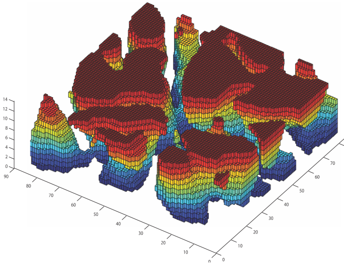
Trabecular Bone

- Makes up the inner layer of the bone and has a spongy, honeycomb-like structure.

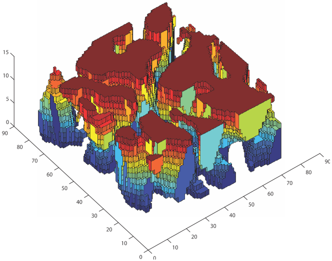


Micro-CT Images of a Trabecular Bone



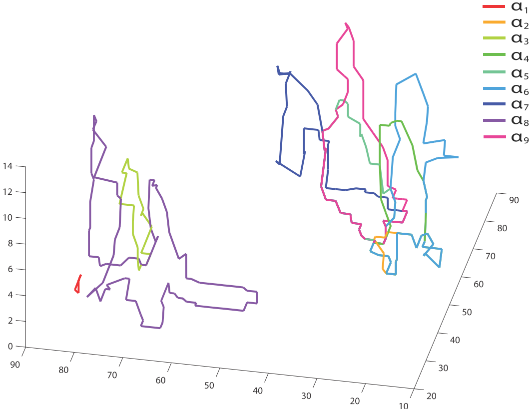


X



X'

Representative 1-cycles



Non-vanishing cup products: $\alpha_2^*\alpha_4^*$, $\alpha_2^*\alpha_5^*$, $\alpha_2^*\alpha_9^*$, $\alpha_3^*\alpha_8^*$, $\alpha_4^*\alpha_5^*$

Recent Work with Barbara Nimershiem and Merv Fansler

Barbara and I constructed a cellular decomposition of the link complement of a n-component Brunnian link and imposed Dan Kravatz's diagonal on each cell to obtain a global diagonal. In his 2016 honors thesis, Merv Fansler extended our diagonal to an A_∞-coalgebra structure on cellular chains in the case $n = 3$, then used a chain contraction to compute to compute the 3-ary operation on homology. This direct computational way of detecting the linkage in the Borromean Rings is elucidating and satisfying.

Current Project with Quinn Minnich

Using Dan Kravatz's diagonal, Quinn Minnich recently discovered a non-trivial A_∞-colagebra structure on cellular chains of an n-gon. When the initial and terminal vertices are adjacent, Kravatz's diagonal induces a non-trivial k-ary operation for each $k < n$. This answers the following question, which to my knowledge is open: Do there exist finite cell complexes $\{X_n\}$ such that $C_*(X_n)$ admits an A_∞-colagebra structure with a non-vanishing k-ary operation for all $k < n$? I'm pleased to report that the answer is YES!!

Thank you!