On the A_∞-bialgebra structure of $H_*(\Omega X; F)$

Joint work with S. Saneblidze, A. Berciano, and H. Abril

Presented by Ron Umble
Millersville Univ of Pennsylvania

U. Penn Deformation Theory Seminar

March 10, 2010

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- To consider some topological applications

Outline of the Talk

I. Guiding Principles

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II. Mophisms and Topological Applications

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The Biderivative

As a sum of tensor algebras:

$$\Omega H \oplus \Omega(H^{\otimes 2}) \oplus \Omega(H^{\otimes 3}) \oplus \cdots$$

$H^{\otimes 4}$	$(H^{\otimes 2})^{\otimes 4}$	$(H^{\otimes 3})^{\otimes 4}$
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$H^{\otimes 2}$	$(H^{\otimes 2})^{\otimes 2}$	$(H^{\otimes 3})^{\otimes 2}$
H	$H^{\otimes 2}$	$H^{\otimes 3}$

As a sum of tensor coalgebras:

$$
\begin{array}{cccc}
 & & & \vdots \\
 & & & \oplus \\
H^{\otimes 4} & (H^{\otimes 4})^{\otimes 2} & (H^{\otimes 4})^{\otimes 3} & B(H^{\otimes 4}) \\
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\end{array}
$$

The Biderivative

- These points of view are related by a permutation of factors

$$\sigma_{p,q} : \left(H^{\otimes p}\right)^{\otimes q} \xrightarrow{\approx} \left(H^{\otimes q}\right)^{\otimes p}$$

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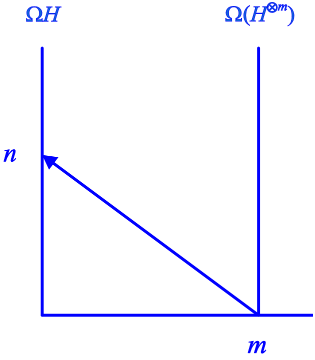
$$\left\{\omega_m^n \in Hom^{m+n-3}\left(H^{\otimes m}, H^{\otimes n}\right)\right\}$$

- ω_m^n acts on $\mathbb{N}^2$ via $(m, 1) \mapsto (1, n)$

The Biderivative

Think of the initial map $\omega_m^n : H^{\otimes m} \longrightarrow H^{\otimes n}$ two ways...



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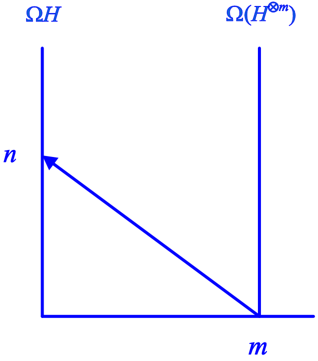
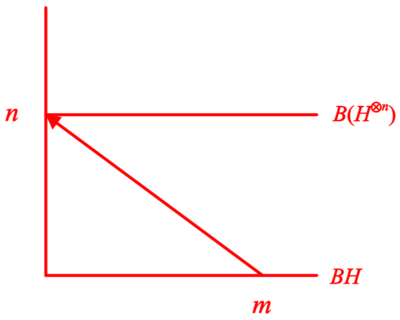
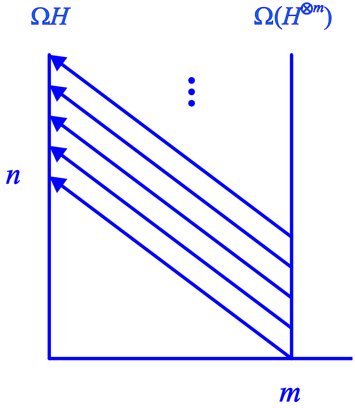




As a map of tensor algebras... and as a map of tensor coalgebras

Guiding principle: Extend ω_m^n as some "higher derivation"



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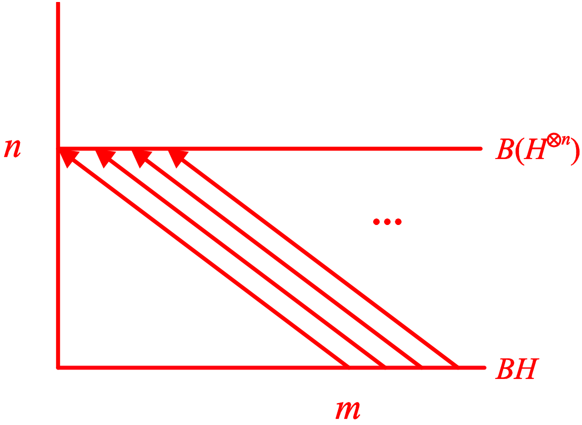

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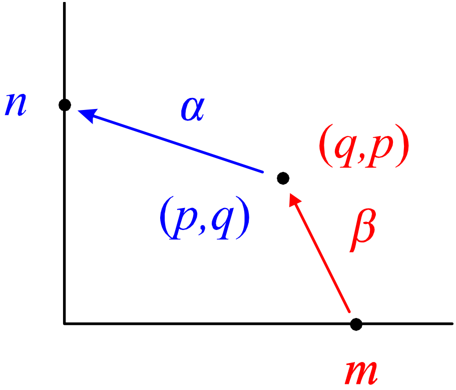
- The component of d_ω of interest here is some projection of $\bar{d}_\omega$
- In the examples presented here, the projection is the identity

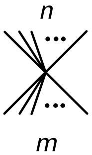
Markl's Fraction Product

- For $\beta : H^{\otimes m} \to (H^{\otimes q})^{\otimes p}$ and $\alpha : (H^{\otimes p})^{\otimes q} \to H^{\otimes n}$ define

$$\alpha / \beta = \alpha \circ \sigma_{q,p} \circ \beta$$



Markl's Fraction Product

- Picture $\omega_m^n =$ 

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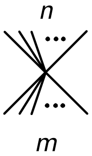
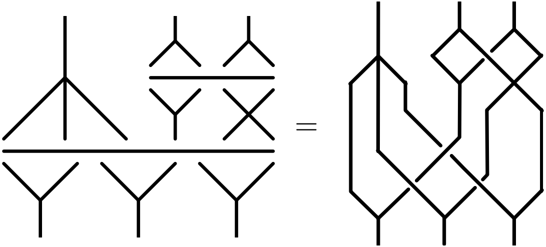
- Picture $\omega_m^n =$ 

- **Example:**



The Double Circle-Product

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- The $\circledcirc$-product is a restriction of the fraction product
- $A \circledcirc B = \begin{cases} A \diagup B, & \textit{if } A \textit{ and } B \textit{ are components of } d_\omega \\ 0, & \textit{otherwise} \end{cases}$

Structure Relations

- When $d_\omega \circledcirc d_\omega = 0$, the structure relation of bidegree (m, n) is

$$\sum \left\{ \textit{2-step paths } H^{\otimes m} \to * \to H^{\otimes n} \right\} = 0$$

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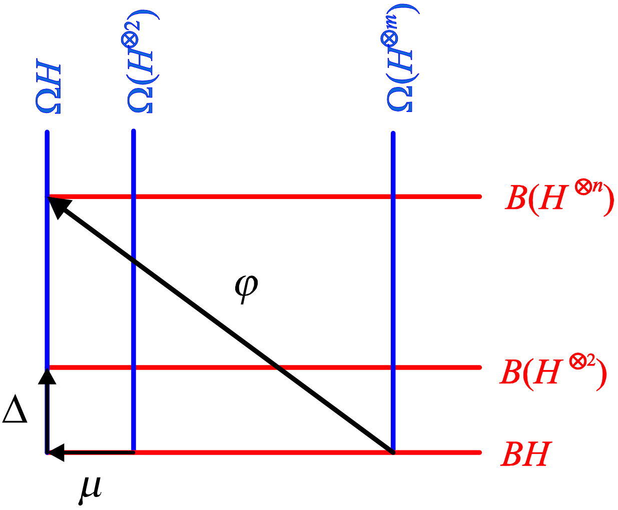
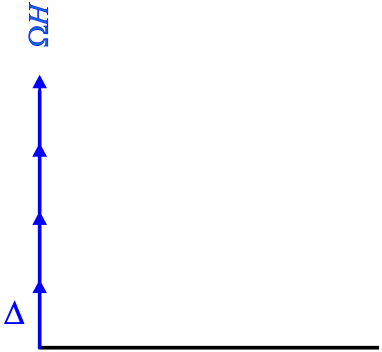
- Let $\varphi \in Hom^{m+n-3}\left(H^{\otimes m}, H^{\otimes n}\right)$

The Initial Maps



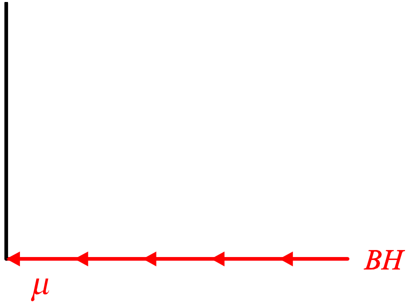
Constructing the Biderivative (Step 1a)

Extend Δ as a derivation: $\bar{\Delta} = \sum_{i=1}^{k} \mathbf{1}^{\otimes i-1} \otimes \Delta \otimes \mathbf{1}^{\otimes k-i}$



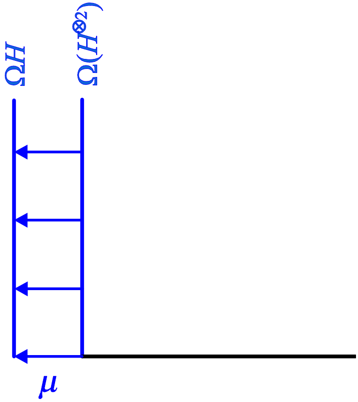
Constructing the Biderivative (Step 1b)

Extend μ as a coderivation: $\bar{\mu} = \sum_{i=1}^{k} \mathbf{1}^{\otimes i-1} \otimes \mu \otimes \mathbf{1}^{\otimes k-i}$



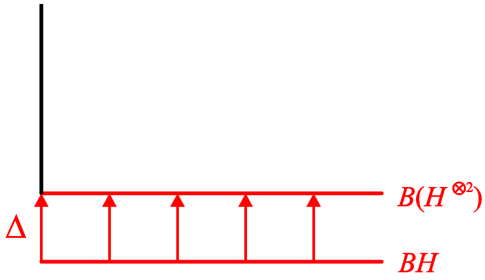
Constructing the Biderivative (Step 2a)

Extend μ as an algebra map: $\sum_{k \geq 1} \mu^{\otimes k}$



Constructing the Biderivative (Step 2b)

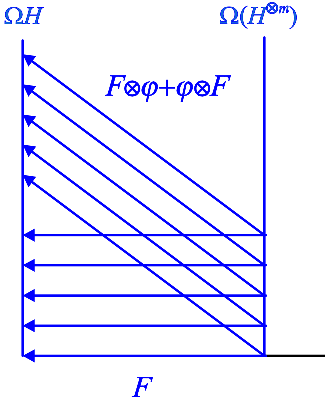
Extend Δ as a coalgebra map: $\sum_{k\geq 1} \Delta^{\otimes k}$



Constructing the Biderivative when $m > 2$ (Step 3a)

Extend $f = \mu(\mu \otimes \mathbf{1}) \cdots (\mu \otimes \mathbf{1}^{\otimes m-2})$ as an algebra map $F = \sum_{k \geq 1} f^{\otimes k}$

Then extend φ as an (F, F)-derivation: $F \otimes \varphi + \varphi \otimes F$



Constructing the Biderivative when $n>2$ (Step 3b)

Extend $g = \left(\Delta \otimes \mathbf{1}^{\otimes n-2}\right) \cdots \left(\Delta \otimes \mathbf{1}\right) \Delta$ as a coalgebra map $G = \sum_{k \geq 1} g^{\otimes k}$

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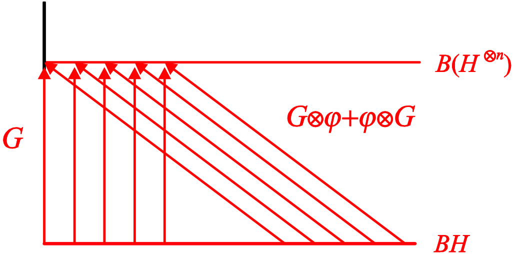
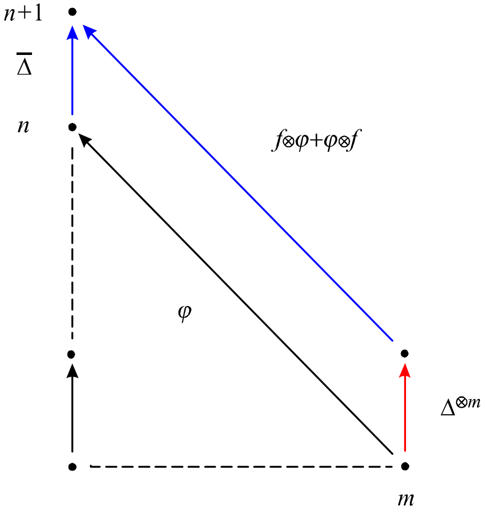
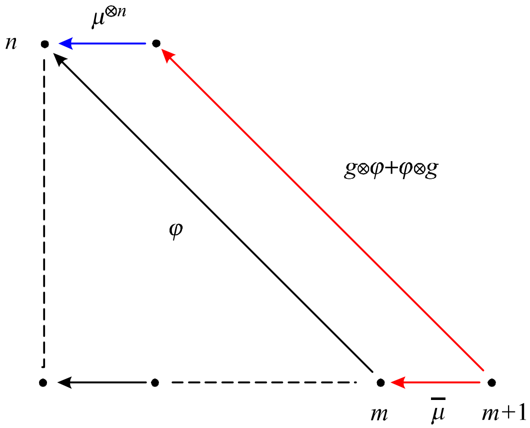


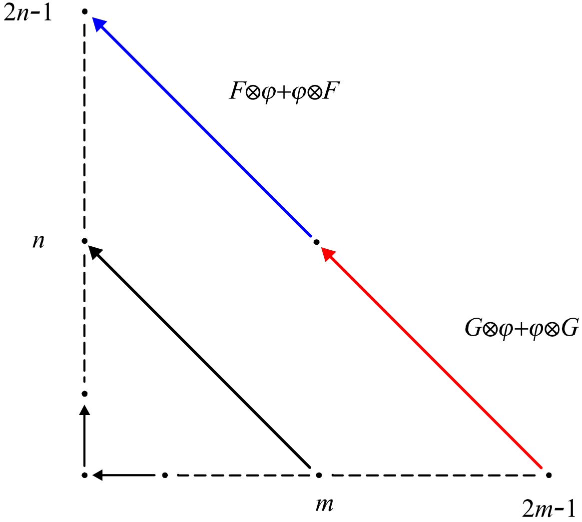
Structure Relation 1



Structure Relation 2



Structure Relation 3



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The Biderivative when $m>2$, $n=1$

- Relation 1 expresses the compatibility of $\varphi = \mu_m$ and Δ :

$$\Delta\varphi = (f \otimes \varphi + \varphi \otimes f)\,\sigma_{2,m}\Delta^{\otimes m},$$

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- We call such structures "Hopf A_∞-algebras"

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The Biderivative when $m{=}1$, $n{>}2$

- Relation 2 expresses the compatibility of $\varphi = \Delta_n$ and μ :

$$\varphi\mu = \mu^{\otimes n}\sigma_{n,2}\left(g \otimes \varphi + \varphi \otimes g\right),$$

$$g = \left(\Delta \otimes \mathbf{1}^{\otimes n-2}\right)\cdots\left(\Delta \otimes \mathbf{1}\right)\Delta$$

- Relations 1 and 3 are classical A_∞-coalgebra relations:

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- We call such structures "Hopf A_∞-coalgebras"

Definition

Let Λ be a commutative ring with unity; let $m, n \neq 2$.

An A_∞*-***bialgebra of type** (m, n) *is a* Λ*-Hopf algebra* (H, Δ, μ) *of finite type together with an operation* $\varphi \in Hom^{m+n-3}(H^{\otimes m}, H^{\otimes n})$ *such that*

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Open question

- Since odd dimensional spheres are rational H-spaces, $H = H_*\left(S^1; H_*\left(S^{2m-3}; \mathbb{Q}\right)\right)$ admits a non-trivial A_∞-bialgebra structure of type (m, n)

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- **Question 1.** Is there an H-space X, appropriate coefficients Λ, and an induced operation φ such that $\left(H_* \left(X; \Lambda\right), \Delta, \varphi, \mu\right)$ is an A_∞-bialgebra?

- **Example 2.** (Berciano-U) *For* $i \geq 0$, $n \geq 3$, and p *an odd prime, the submodule*

$$E(v_i, 2np^i + 1) \otimes \Gamma(w_i, 2np^{i+1} + 2) \subset H_*(\mathbb{Z}, n; \mathbb{Z}_p)$$

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- **Question 2.** What is the global A_∞-bialg structure of $H_*(\mathbb{Z}, n; \mathbb{Z}_p)$?

Group Cohomology

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$$H^*(C_p \times \cdots \times C_p; \mathbb{Z}_p)\,?$$

- **Question 3b.** Is there an isomorphism of A_∞-bialgebras
$$H^*(C_p \times \cdots \times C_p; \mathbb{Z}_p) \approx H^*(C_p; \mathbb{Z}_p)^{\otimes p}\,?$$

II. Morphisms and Topological Applications

Morphisms of A–infinity Algebras

- A morphism $G : H \Rightarrow C$ of A_∞-algebras is defined in terms of cochain homotopies

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- The parameter space of g_k is the *multiplihedron* J_k

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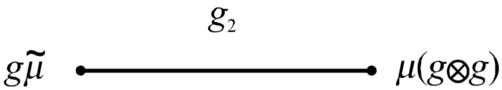
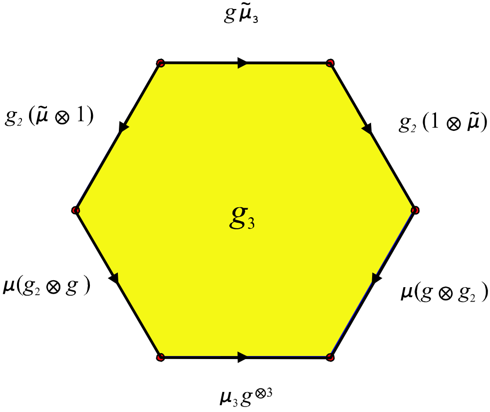
- The endpoints of J_2 are identified with components of the coboundary

$$\nabla g_2 = \mu\left(g \otimes g\right) - g\tilde{\mu}$$



Multiplihedra

- J_3 is an hexagonal plane region identified with $g_3 : H^{\otimes 3} \longrightarrow C$



$$\nabla g_3 = \mu_3 g^{\otimes 3} + \mu\left(g_2 \otimes g - g \otimes g_2\right) + g_2\left(\tilde{\mu} \otimes 1 - 1 \otimes \tilde{\mu}\right) - g\mu_3$$

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First Transfer Theorem

- **Transfer Problem 1:** *Let H* be a DGM, let *C be an A_∞-algebra, and let $g = g_1 : H \to C$ be a cochain map. Given $\{\tilde{\mu}_k, g_k\}_{k<n}$ construct $\tilde{\mu}_n$ and g_n so that*
$$\nabla g_n = \Theta_n - g\tilde{\mu}_n$$

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- There is the dual statement for A_∞-coalgebras

The Induced A-infinity Structure on Homology

- **Corollary 1.** *If C is a DGA over a field F, there is a cycle-selecting homomorphism $g : H(C) \to C$ and an induced A_∞-algebra structure on $H(C)$*

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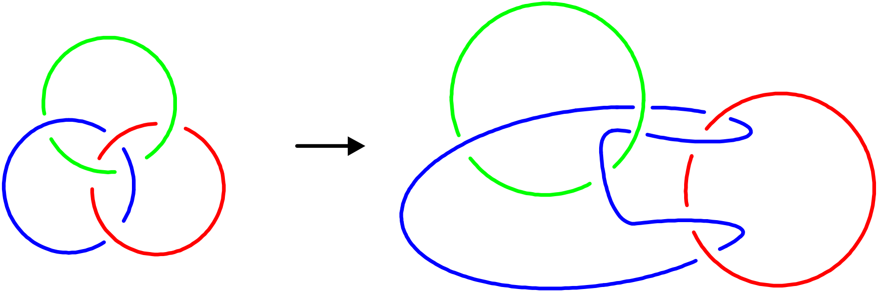
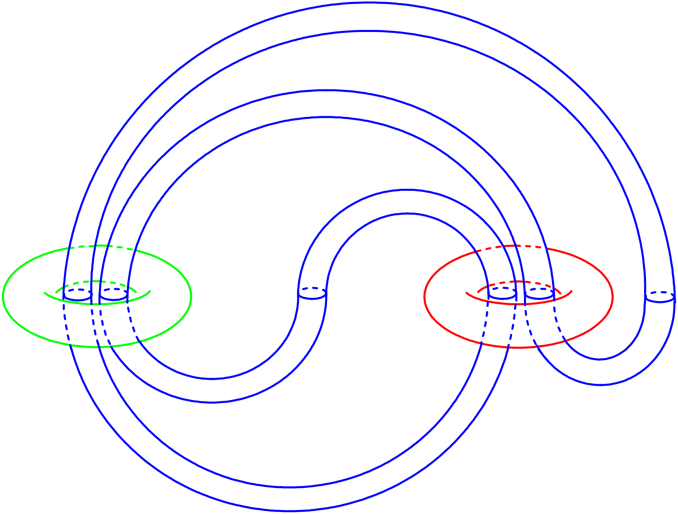
- **Example 4.** (Abril-U) *Let BR be a tubular nbh of the Borromean Rings. There is a non-trivial induced A_∞-coalgebra structure on $H_*\left(S^3 - BR; \mathbb{Z}_2\right)$*

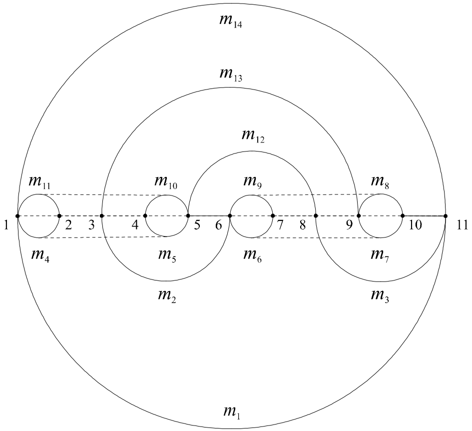
The Borromean Rings



A Tubular Neighborhood of the Borromean Rings



A Cellular Decomposition of the Link Complement



Edges and vertices (side view)

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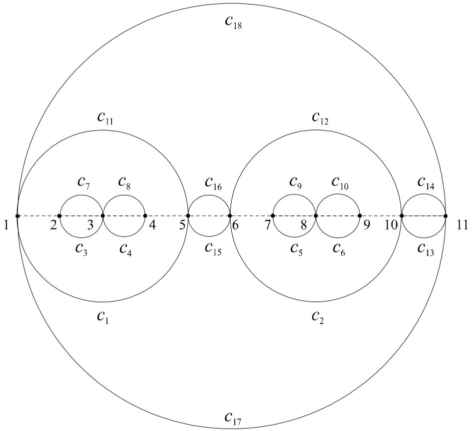

Edges and vertices (top view)

The A-infinity Colagebra Structure on Homology

- $H_i\left(S^3 - BR; \mathbb{Z}_2\right) = \begin{cases} \langle 1 \rangle, & i = 0 \\ \langle u, v, w \rangle, & i = 1 \\ \langle y, z \rangle, & i = 2 \\ 0, & \text{otherwise} \end{cases}$

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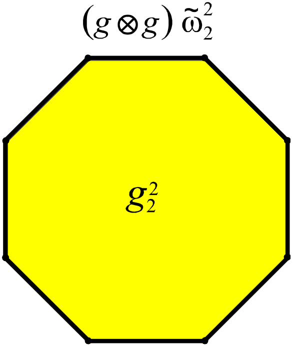
- $H_*\left(S^3 - BR; \mathbb{Z}_2\right)$ is primitively generated and

$$\Delta_3(y) = u|u|w + v|u|w + w|v|u + w|u|u$$

$$\Delta_3(z) = u|u|v + u|v|v + v|u|u + v|u|v + v|w|u + w|u|v$$

Morphisms of A-infinity Bialgebras

A morphism $G : H \Rightarrow C$ of A_∞-bialgebras is defined in terms of higher homotopies $g_m^n \in Hom^{m+n-2}\left(H^{\otimes m}, C^{\otimes n}\right)$ whose parameter spaces are *bimultiplihedra* $JJ_{n,m}$



The Bimultiplihedron $JJ_{2,2}$

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induces an A_∞-bialgebra structure on $H_* (\Omega X; F)$

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- Then $\xi \left(e^{n-2} \right) = \Delta_n$ and

$$\Delta_n \mu = \mu^{\otimes n} \circledcirc \left[\left(\xi \otimes \xi \right) \Delta_K \left(e^{n-2} \right) \right]$$

(Δ_n is some "higher" derivation of algebras)

- **Theorem 4.** *The A_∞-coalgebra structure on $H = H_*(X; F)$ extends to an A_∞-bialgebra structure on $T^a H$, which is trivial iff the A_∞-coalgebra structure on H is trivial*

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- **Theorem 4.** *The A_∞-coalgebra structure on $H = H_*(X;F)$ extends to an A_∞-bialgebra structure on $T^a H$, which is trivial iff the A_∞-coalgebra structure on H is trivial*

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- **Theorem 6.** *$H_*(\Omega\Sigma X;F)$ is an A_∞-bialgebra of the form $(H,\Delta_n,\mu)_{n\geq 2}$, where Δ_n is some "higher" derivation*

Open Question in Link Homotopy

- **Example 5.** $H_*\left(\Omega\Sigma(S^3 - BR);\mathbb{Z}_2\right)$ *is a non-trivial* A_∞*-bialgebra with more structure than the* A_∞-coalgebra $H_*\left(S^3 - BR;\mathbb{Z}_2\right)$

Open Question in Link Homotopy

- **Example 5.** $H_* \left(\Omega\Sigma(S^3 - BR); \mathbb{Z}_2\right)$ *is a non-trivial* A_∞*-bialgebra with more structure than the* A_∞-coalgebra $H_* \left(S^3 - BR; \mathbb{Z}_2\right)$

- Given a tubular nbh N of a link in S^3, there is the link homotopy invariant $H_* \left(\Omega\Sigma(S^3 - N); \mathbb{Z}_2\right)$

Open Question in Link Homotopy

- **Example 5.** $H_* \left(\Omega\Sigma(S^3 - BR); \mathbb{Z}_2\right)$ *is a non-trivial A_∞-bialgebra with more structure than the A_∞-*coalgebra $H_* \left(S^3 - BR; \mathbb{Z}_2\right)$
- Given a tubular nbh N of a link in S^3, there is the link homotopy invariant $H_* \left(\Omega\Sigma(S^3 - N); \mathbb{Z}_2\right)$
- **Question 4.** What new link homotopy information, if any, is given by $H_* \left(\Omega\Sigma(S^3 - N); \mathbb{Z}_2\right)$?

The End

Thank you!