Matrads, Matrahedra and A_∞-Bialgebras

Joint work with Samson Saneblidze

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- Identify cellular chains $C_*(K_n)$ with the A_∞-operad $\mathcal{A}_\infty$
- An A_∞-*algebra* is a graded module A together with a family of maps $\{\mu_n \in Hom(A^{\otimes n}, A)\}_{n\geq 1}$ and a chain map

$$\varphi : \mathcal{A}_\infty \to \left\{Hom\left(A^{\otimes n}, A\right)\right\}_{n\geq 1}$$

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- An A_∞-*bialgebra* is defined in the same way we defined an A_∞-algebra

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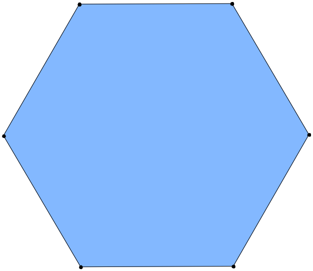
- P_3 is a plane hexagonal region



Permutahedra

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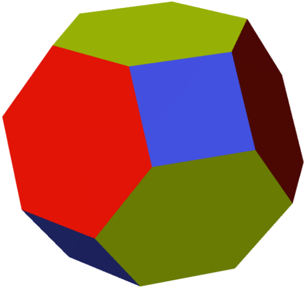

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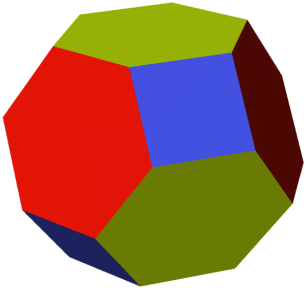

- P_n is an $(n-1)$-dim'l convex polyhedron

Combinatorics of Permutahedra

Let $\underline{n} = \{1, 2, \ldots n\}$

- $\{\text{Faces in codim } p\} \leftrightarrow \{\text{Partitions } U_1 | \cdots | U_{p+1} \text{ of } \underline{n}\}$

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- $P_1 : \; * \leftrightarrow \{1\}$
- $P_2 : \;$ edge $\leftrightarrow \{12\}$; vertices $\leftrightarrow \{1|2, 2|1\}$

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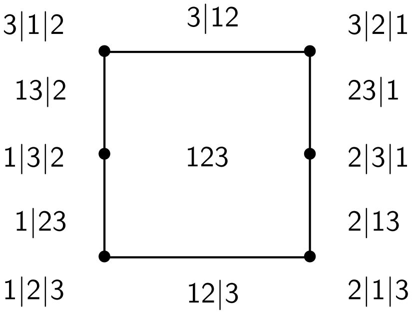
- P_3 as a subdivision of $P_2 \times I$:



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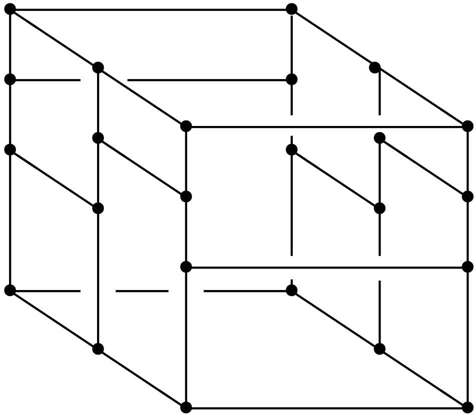
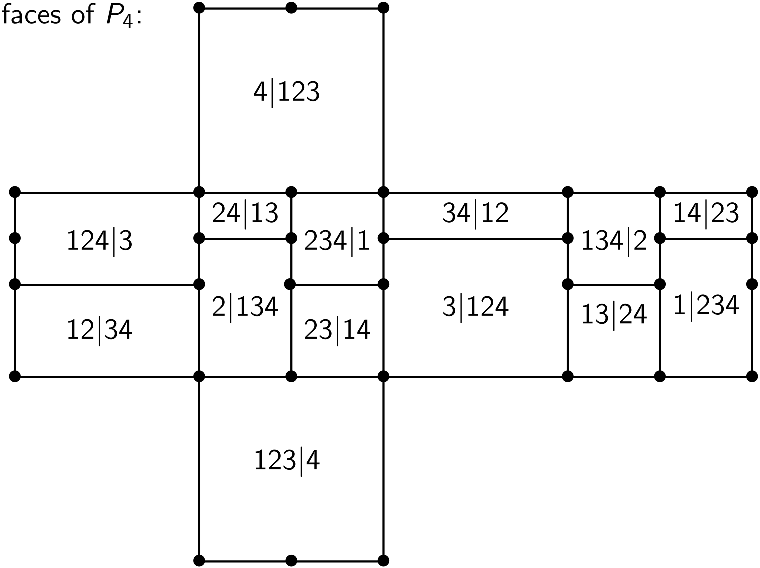


Combinatorics of Permutahedra

- 2-faces of P_4:



The Poset of Vertices

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- Set $\mathcal{PP}_{n,0} = \mathcal{PP}_{0,n} = S_n$; then

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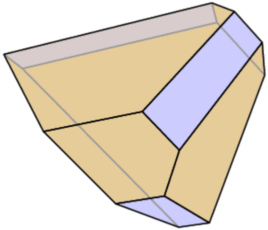

S. Devadoss rendering of K_5

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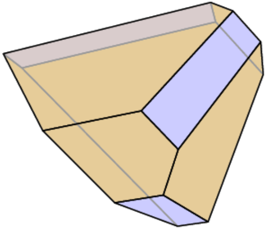

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- Markl's construction makes arbitrary choices; our construction uses the S-U diagonal on associahedra to eliminate all such choices

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- $\Delta_P (\curlywedge) = \curlywedge \times \curlywedge$ and $\Delta_P (\curlyvee) = \curlyvee \times \curlyvee$ (the index 2 is suppressed)

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- $X_m^{n+1} \times Y_n^{m+1}$ is a subposet of $S_m^{\times n+1} \times S_n^{\times m+1}$
- X_1^2, Y_1^2 and $X_1^2 \times Y_1^2 \cong P_1^{\times 4}$ are singleton sets

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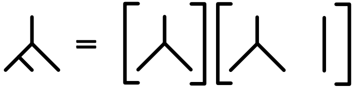
- $\wedge_1^{\times 2} = X_1^2 = \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix}$
- A leveled tree factors uniquely as the composition of its levels



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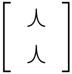
- $\wedge_1^{\times 2} = X_1^2 =$ 
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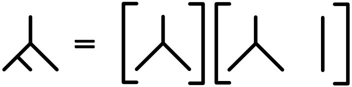

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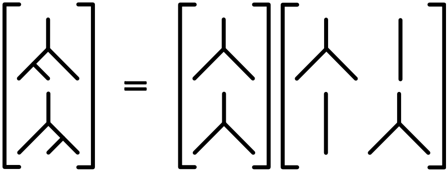

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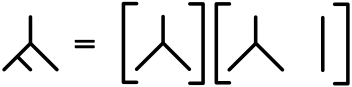

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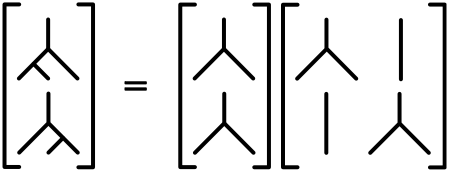

- Rows of x_i are the i^{th} levels of trees in x; each row of x_i contains ⅄ exactly once

Vertices as Matrix Products

- Vertices of $\Delta_P(\curlywedge_3) \subset P_2 \times P_2$ (the square) lie in

$$X_2^2 = \left\{ \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \curlywedge & \mathbf{1} \\ \curlywedge & \mathbf{1} \end{bmatrix} < \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \curlywedge & \mathbf{1} \\ \mathbf{1} & \curlywedge \end{bmatrix} < \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \mathbf{1} & \curlywedge \\ \mathbf{1} & \curlywedge \end{bmatrix} \right\}$$

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- Factor y^T as a matrix product $y_1^T \cdots y_n^T$; then $y = y_n \cdots y_1$

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- Factor y^T as a matrix product $y_1^T \cdots y_n^T$; then $y = y_n \cdots y_1$
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- Columns of y_j are the j^{th} levels of trees in y; thus each column of y_j contains $\curlyvee$ exactly once
- Vertices of $\Delta_P(\curlyvee^3) \subset P_2 \times P_2$ lie in

$$Y_2^2 = \left\{ \begin{bmatrix} \curlyvee & \curlyvee \\ \mathbf{1} & \mathbf{1} \end{bmatrix} [\curlyvee \curlyvee] < \begin{bmatrix} \curlyvee & \mathbf{1} \\ \mathbf{1} & \curlyvee \end{bmatrix} [\curlyvee \curlyvee] < \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \curlyvee & \curlyvee \end{bmatrix} [\curlyvee \curlyvee] \right\}$$

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- $\Delta_P^{(2)}(\curlyvee) = \curlyvee \times \curlyvee \times \curlyvee$ implies $Y_1^3 = [\curlyvee\,\curlyvee\,\curlyvee]$; thus

$$X_2^2 \times Y_1^3 = \left\{ \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \curlywedge & \mathbf{1} \\ \curlywedge & \mathbf{1} \end{bmatrix} [\curlyvee\,\curlyvee\,\curlyvee] < \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \curlywedge & \mathbf{1} \\ \mathbf{1} & \curlywedge \end{bmatrix} [\curlyvee\,\curlyvee\,\curlyvee] \right.$$

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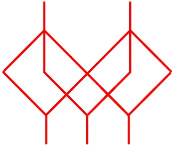
- Pictured as a non-planar upward-directed graph

 $\in M_{2,3}$

The Matrix Submodule of TTM

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- A pair of matrices $X = (x_{ij})$, $Y = (y_{ij}) \in \mathbb{N}^{q\times p}$ uniquely determines a submodule

$$\begin{array}{rcccccc} M_{Y,X} = & & (M_{y_{11},x_{11}} & \otimes\cdots\otimes & M_{y_{1p},x_{1p}}) & \otimes\cdots \\ & \otimes & (M_{y_{21},x_{21}} & \otimes\cdots\otimes & M_{y_{2p},x_{2p}}) & \otimes\cdots \\ & & \vdots & & \vdots & \\ & \otimes & (M_{y_{q1},x_{q1}} & \otimes\cdots\otimes & M_{y_{qp},x_{qp}}) & \subset \overline{\mathbf{M}} \end{array}$$

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- Each of the q rows is a submodule of $M^{\otimes p}$

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- A *monomial in TTM* is an element $B \in \left(G^{\otimes p}\right)^{\otimes q}$
- B is represented by the $q \times p$ matrix

$$[B] = \begin{bmatrix} g_{x_{1,1}}^{y_{1,1}} & \cdots & g_{x_{1,p}}^{y_{1,p}} \\ \vdots & & \vdots \\ g_{x_{q,1}}^{y_{q,1}} & \cdots & g_{x_{q,p}}^{y_{q,p}} \end{bmatrix}$$

with entries in G and rows identified with elements of $G^{\otimes p}$

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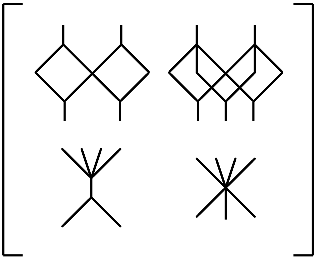
- A monomial $A \in \mathbf{M}$ is represented by a *bisequence matrix*

$$A = \begin{bmatrix} \alpha_{x_1}^{y_1} & \cdots & \alpha_{x_p}^{y_1} \\ \vdots & & \vdots \\ \alpha_{x_1}^{y_q} & \cdots & \alpha_{x_p}^{y_q} \end{bmatrix}$$

Bisequence Matrices

- We refer to A as a $q \times p$ *monomial and* denote $M_{\mathbf{x}}^{\mathbf{y}} = M_{\mathbf{y},\mathbf{x}}$

$$A \in \mathbf{M}^{\substack{2\\4}}_{23} \quad \leftrightarrow$$



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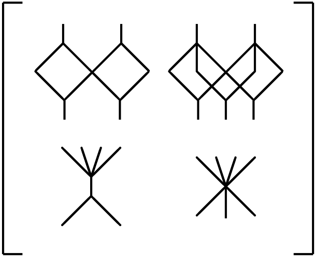

- A *Transverse Pair* (TP) of bisequence matrices has form

$$\begin{bmatrix} \alpha_p^{y_1} \\ \vdots \\ \alpha_p^{y_q} \end{bmatrix} \otimes \begin{bmatrix} \beta_{x_1}^q & \cdots & \beta_{x_p}^q \end{bmatrix} \in \mathbf{M}_p^{\mathbf{y}} \otimes \mathbf{M}_{\mathbf{x}}^q$$

- $A \otimes B \in \overline{\mathbf{M}} \otimes \overline{\mathbf{M}}$ is a *B̲lock T̲ransverse P̲air* (BTP) if there exist block decompositions $A = [A_{ij}]$ and $B = [B_{ij}]$ such that $A_{ij} \otimes B_{ij}$ is a TP for each (i, j).

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- A pair $A \otimes B \in \mathbf{M}^{\substack{1\\5\\4\\3}}_{21} \otimes \mathbf{M}^{\substack{3\\1}}_{123}$ is a BTP via

$$A = \begin{bmatrix} \alpha_2^1 & \alpha_1^1 \\ \alpha_2^5 & \alpha_1^5 \\ \alpha_2^4 & \alpha_1^4 \\ \alpha_2^3 & \alpha_1^3 \end{bmatrix} \qquad B = \begin{bmatrix} \beta_1^3 & \beta_2^3 & \beta_3^3 \\ \beta_1^1 & \beta_2^1 & \beta_3^1 \end{bmatrix}$$

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- BTP decomposition is unique when it exists

Upsilon Products

- Given a map $\overline{\gamma} : TM \otimes TM \to M$, extend the component

$$\gamma = \left\{ \gamma_{\mathbf{x}}^{\mathbf{y}} : \mathbf{M}_{p}^{\mathbf{y}} \otimes \mathbf{M}_{\mathbf{x}}^{q} \to \mathbf{M}_{|\mathbf{x}|}^{|\mathbf{y}|} \right\}$$

to a global product $\Upsilon : \overline{\mathbf{M}} \otimes \overline{\mathbf{M}} \to \overline{\mathbf{M}}$ by setting $\Upsilon \left(A \otimes B \right) = 0$ unless $A \otimes B$ is a BTP, in which case

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- There is unique Υ-factorization in $\overline{\mathbf{M}}$

Upsilon Products

$$\Upsilon : \mathbf{M}_{21}^{\substack{1\\5\\4\\3}} \otimes \mathbf{M}_{123}^{\substack{3\\1}} \rightarrow \mathbf{M}_{33}^{\substack{10\\3}}$$

$$\begin{bmatrix} \alpha_2^1 & \alpha_1^1 \\ \alpha_2^5 & \alpha_1^5 \\ \alpha_2^4 & \alpha_1^4 \\ \alpha_2^3 & \alpha_1^3 \end{bmatrix} \begin{bmatrix} \beta_1^3 \ \ \beta_2^3 & \beta_3^3 \\ \beta_1^1 \ \ \beta_2^1 & \beta_3^1 \end{bmatrix} = \begin{bmatrix} \begin{matrix} \alpha_2^1 \\ \alpha_2^5 \\ \alpha_2^4 \end{matrix} & \beta_1^3 \ \ \beta_2^3 & \begin{matrix} \alpha_1^1 \\ \alpha_1^5 \\ \alpha_1^4 \end{matrix} & \beta_3^3 \\ \alpha_2^3 & \beta_1^1 \ \ \beta_2^1 & \alpha_1^3 & \beta_3^1 \end{bmatrix}$$

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- *AB* is an *(i,j)-edge pair* if

  - $A \otimes B$ is a BTP

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- $A_m B_n$ is the only possible edge pair in

$$A_1 \cdots A_m B_n \cdots B_1 \in X_m^{n+1} \times Y_n^{m+1}$$

Matrix Edge Pairs

- The pair $x_2 y_2$ in

$$x_1 x_2 y_2 y_1 = \begin{bmatrix} \curlywedge \\ \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \curlywedge & \mathbf{1} \\ \curlywedge & \mathbf{1} \\ \curlywedge & \mathbf{1} \end{bmatrix} \begin{bmatrix} \curlyvee & \curlyvee & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \curlyvee \end{bmatrix} [\curlyvee \ \curlyvee \ \curlyvee]$$

is a $(1,1)$-edge pair

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- But

$$\begin{bmatrix} \curlywedge \\ \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \curlywedge & \mathbf{1} \\ \curlywedge & \mathbf{1} \\ \curlywedge & \mathbf{1} \end{bmatrix} \begin{bmatrix} \curlyvee & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \curlyvee & \curlyvee \end{bmatrix} \begin{bmatrix} \curlyvee & \curlyvee & \curlyvee \end{bmatrix}$$

  has no edge pairs

Edges of a Poset

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- 1-dim'l elements of $\mathcal{H}_\infty$ generated by indecomposables $\{\mathbf{1}, \theta_2^1, \theta_1^2, \theta_2^2\}$ are represented by edges of a poset $Z_{n,m}$ related to but disjoint from $X_m^{n+1} \times Y_n^{m+1}$

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- The (i,j)-*transposition* of an (i,j)-edge pair AB is the pair $B^{*j}A^{i*}$
- If $c = C_1 \cdots C_r$ and $C_k C_{k+1}$ is an (i,j)-edge pair,

$$\mathcal{T}_{ij}^{k}(c) = C_1 \cdots C_{k+1}^{*j} C_k^{i*} \cdots C_r$$

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- Iterate $\mathcal{T}$ on each $u \in X_m^{n+1} \times Y_n^{m+1}$ and obtain

$$Z_{n,m} = \left\{ \mathcal{T}_{i_t j_t}^{k_t} \cdots \mathcal{T}_{i_1 j_1}^{k_1}(u) \;\middle|\; u \in X_m^{n+1} \times Y_n^{m+1} \right\}$$

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- The action of $\mathcal{T}_{i_t j_t}^{k_t} \cdots \mathcal{T}_{i_1 j_1}^{k_1}$ on $u \in X_m^{n+1} \times Y_n^{m+1}$ uniquely determines an (m,n)-shuffle σ; thus we denote

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- Define $\mathcal{T}_{\mathrm{Id}} = \mathrm{Id}$
- If $\sigma \neq \mathrm{Id}$ cannot be realized as a composition of (i, j)-transpositions on u, then $\mathcal{T}_\sigma(u)$ is undefined

Sigma-Partitions

- Given $u_1 \leq u_2 \in X_m^{n+1} \times Y_n^{m+1}$, define $\mathcal{T}_\sigma(u_1) \leq \mathcal{T}_\sigma(u_2)$ if either (u_1, u_2) is an edge of $X_m^{n+1} \times Y_n^{m+1}$ or u_2 is "σ -compatible" with u_1Let $a = a_1|\cdots|a_m \in S_m$ and $b = b_1|\cdots|b_n \in S_n$

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- The action of an (m, n)-shuffle σ on $(a; b)$ produces contiguous blocks $\mathbf{m}_i \subseteq a$ and $\mathbf{n}_j \subseteq b$:

$$\sigma(a;b) = \begin{cases} \mathbf{m}_1, \mathbf{n}_1, \mathbf{m}_2, \mathbf{n}_2, \ldots, \mathbf{n}_{k-1}, \mathbf{m}_k, & \sigma(a_1) = a_1, \ \sigma(b_n) \neq b_n \\ \mathbf{m}_1, \mathbf{n}_1, \mathbf{m}_2, \mathbf{n}_2, \ldots, \mathbf{m}_k, \mathbf{n}_k, & \sigma(a_1) = a_1, \ \sigma(b_n) = b_n \\ \mathbf{n}_1, \mathbf{m}_1, \mathbf{n}_2, \mathbf{m}_2, \ldots, \mathbf{n}_k, \mathbf{m}_k, & \sigma(a_1) \neq a_1, \ \sigma(b_n) \neq b_n \\ \mathbf{n}_1, \mathbf{m}_1, \mathbf{n}_2, \mathbf{m}_2, \ldots, \mathbf{m}_k, \mathbf{n}_{k+1}, & \sigma(a_1) \neq a_1, \ \sigma(b_n) = b_n. \end{cases}$$

Sigma-Partitions

- Given $u_1 \leq u_2 \in X_m^{n+1} \times Y_n^{m+1}$, define $\mathcal{T}_\sigma(u_1) \leq \mathcal{T}_\sigma(u_2)$ if either (u_1, u_2) is an edge of $X_m^{n+1} \times Y_n^{m+1}$ or u_2 is "σ -compatible" with u_1Let $a = a_1|\cdots|a_m \in S_m$ and $b = b_1|\cdots|b_n \in S_n$

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- The *σ-partition* of $(a;b)$ is the cell $\mathbf{m}_1|\cdots|\mathbf{m}_k \times \mathbf{n}_\ell|\cdots|\mathbf{n}_1 \subset P_m \times P_n$ obtained by unshuffling and removing block delimiters from $\mathbf{m}_i$ and $\mathbf{n}_j$

Sigma-Compatibility

- $u_1 = A_1 \cdots A_m B_n \cdots B_1,\ u_2 = A_1' \cdots A_m' B_n' \cdots B_1' \in X_m^{n+1} \times Y_n^{m+1}$

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- For each j, obtain a pair of down-rooted trees and corresponding partitions $b_j = j_1 | \cdots | j_n$ and $b'_j = j'_1 | \cdots | j'_n$ of $\underline{n}$
- u_2 is *σ-compatible* with u_1 if for each (i,j), the σ-partition of $(a_i; b_j)$ contains an oriented path of edges from $a_i \times b_j$ to $a'_i \times b'_j$

The Poset PP(1,2)

- The action of $\mathcal{T}$ on

$$u_1 = \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \curlywedge & \mathbf{1} \\ \curlywedge & \mathbf{1} \end{bmatrix} [\curlyvee\ \curlyvee\ \curlyvee],\ u_3 = \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \mathbf{1} & \curlywedge \\ \mathbf{1} & \curlywedge \end{bmatrix} [\curlyvee\ \curlyvee\ \curlyvee]$$

produces four elements of $Z_{1,2}$:

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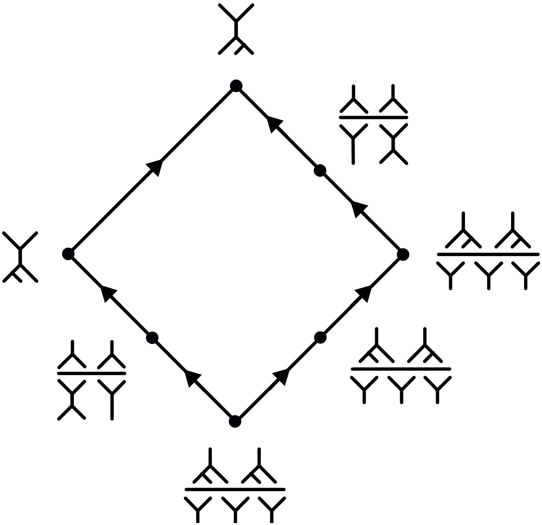
The Digraph of PP(1,2)



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- In $\mathcal{V}_4$:

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$$\updownarrow$$

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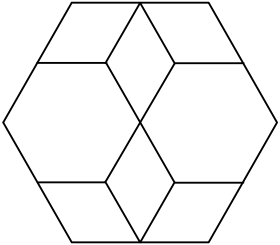
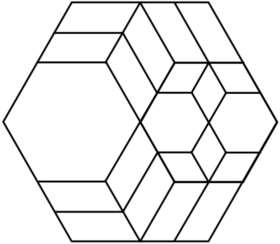
   a. Replacing the codim 1 cell $\underline{m}|\,(\underline{n}+m) \subset P_{m+n}$ with

$$\left|X_m^{n+1} \times Y_n^{m+1}\right| = \Delta_P^{(n)}(\curlywedge_{m+1}) \times \Delta_P^{(m)}(\curlywedge_{n+1})$$

(the (m,n)*-subdivision of* P_{m+n})



$|X_3^2| = \Delta_P^{(1)}(\curlywedge_4)$



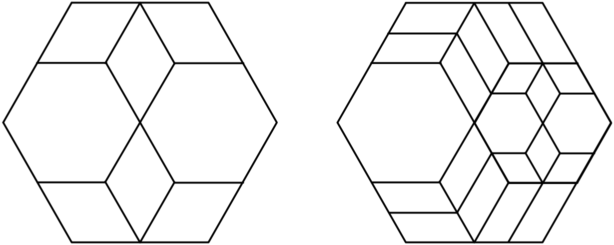
$|X_3^3| = \Delta_P^{(2)}(\curlywedge_4)$

Constructing Matrahedra

1. Construct $PP_{n,m} = |\mathcal{PP}_{n,m}|$ as a subdivision of P_{m+n} by
   a. Replacing the codim 1 cell $\underline{m} | (\underline{n} + m) \subset P_{m+n}$ with

$$\left| X_m^{n+1} \times Y_n^{m+1} \right| = \Delta_P^{(n)} \left(\curlywedge_{m+1} \right) \times \Delta_P^{(m)} \left(\curlywedge_{n+1} \right)$$

(the (m,n)*-subdivision of* P_{m+n})



$\left| X_3^2 \right| = \Delta_P^{(1)} \left(\curlywedge_4 \right)$ $\qquad$ $\left| X_3^3 \right| = \Delta_P^{(2)} \left(\curlywedge_4 \right)$

   b. Using the action of $\mathcal{T}_\sigma$ to propagate the (m,n)-subdivision and subdivide remaining cells

Constructing Matrahedra

2. $\mathcal{P}_{n,m} = \{A_1 \cdots A_m B_n \cdots B_1 \in \mathcal{PP}_{m+n} \mid A_i$ has constant columns and B_j has constant rows$\}$

Constructing Matrahedra

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    a. Elements of $\mathcal{P}_{n,m}$ index the vertices of P_{m+n}

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Constructing Matrahedra

3. Construct $KK_{n+1,m+1}$ as a subdivision of $K_{n+1,m+1}$ that commutes the diagram

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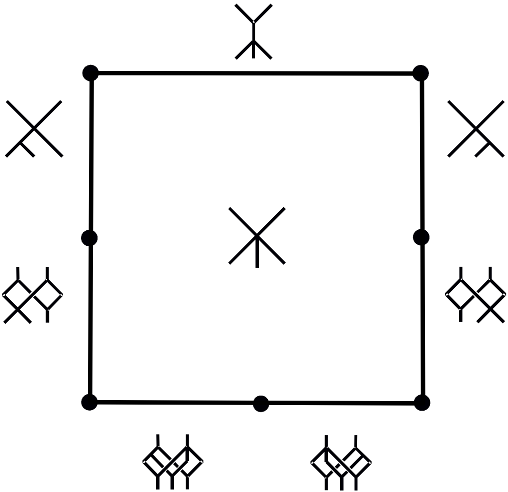
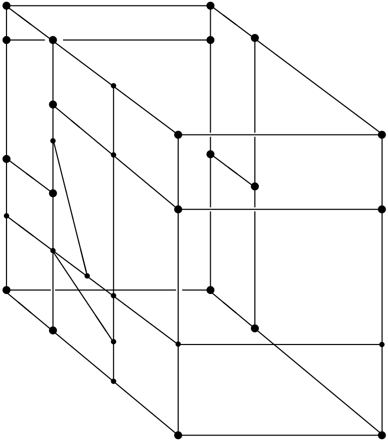
4. Matrix products associated with boundary components of a single cell $e \subset KK$ uniquely determine the matrix product associated with e

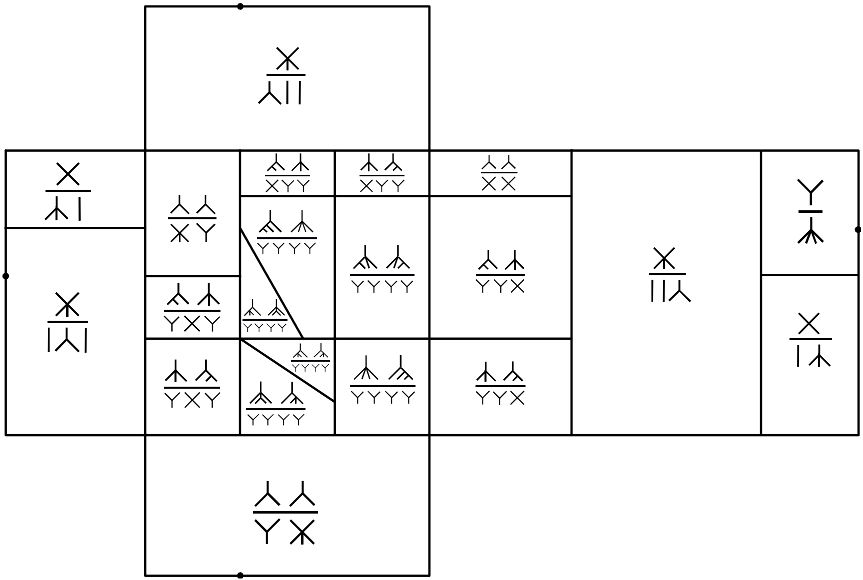
The Matrahedron KK(2,3)



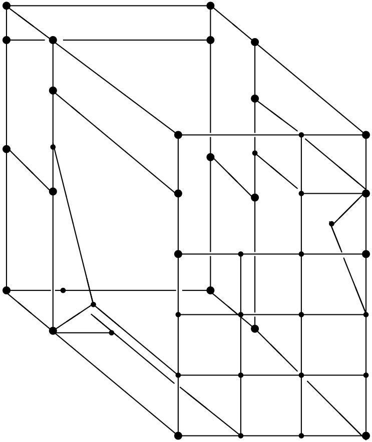
The Matrahedron KK(2,4)



The 2-Faces of KK(2,4)



The Matrahedron KK(3,3)



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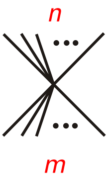
- $\theta^n_m =$  $\in (\mathcal{H}_\infty)_{n,m}$

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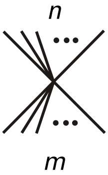
- $\theta_m^n =$ 

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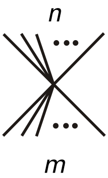
- $\theta_m^n =$  $\in (\mathcal{H}_\infty)_{n,m}$

- θ_m^n is associated with the top dimensional cell e^{m+n-3} of $KK_{n,m}$
- The relations in $\mathcal{H}_\infty$ are encoded by the cellular boundary in KK

- An A_∞-*bialgebra* is a bigraded module H together with a chain map

$$\varphi : \mathcal{H}_\infty \to \left\{ Hom\left(H^{\otimes m}, H^{\otimes n}\right)\right\}_{m,n\geq 1}$$

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A-infinity Bialgebras

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- Recently S. Saneblidze proved that if F is a field and X is a space, $H_*\left(\Omega X; F\right)$ admits a canonical A_∞-bialgebra structure (the main result in our forthcoming paper)

The End

Thank you!