Matradic Operations on Loop Cohomology

Joint work with Samson Saneblidze

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Millersville Univ of Pennsylvania

U. Penn Deformation Theory Seminar

August 18, 2011

Goal of the talk

- To identify a space X such that $H = H^* (\Omega X; \mathbb{Z}_2)$ comes equipped with an induced operation

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$$\omega_2^2 : H \otimes H \longrightarrow H \otimes H$$

- We assume existence of biassociahedra $KK = \{KK_{n,m}\}$ and bimultipliheda $JJ = \{JJ_{n,m}\}$

- The operation ω_2^2 will be defined in terms of $JJ_{2,1} = JJ_{1,2}$ and $JJ_{2,2}$

The Free Matrad

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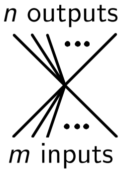
$\theta^n_m \quad \leftrightarrow$



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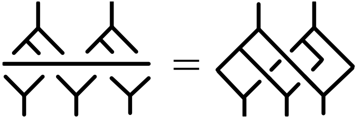 $\leftrightarrow$ a vertex of $KK_{2,3}$

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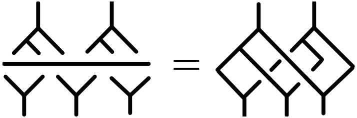

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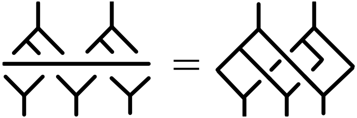 $\leftrightarrow$ a vertex of $KK_{2,3}$

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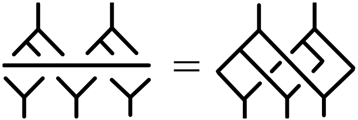

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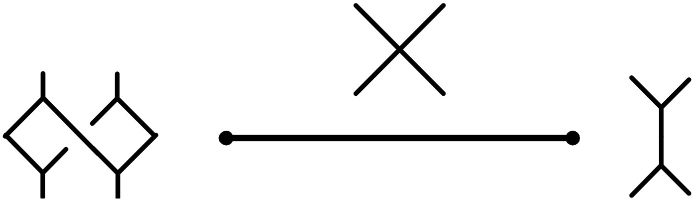
- $KK_{n,m}$ is contractible

The Biassociahedron $KK(2,2)$:



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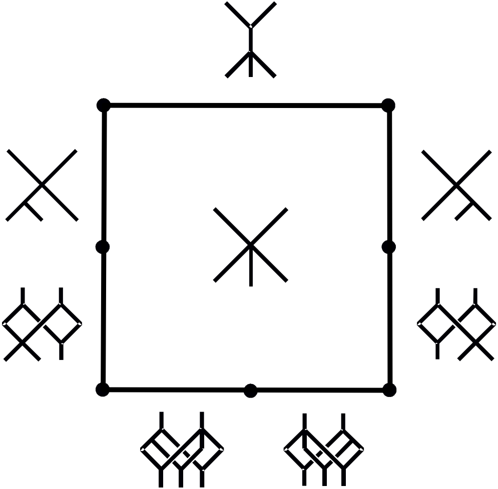

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- An A_∞-bialgebra (A, d, ω) is an algebra over $\mathcal{H}_\infty$

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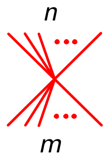
$\mathfrak{f}_m^n \quad \leftrightarrow$ 

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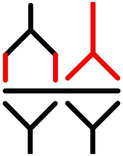 $\leftrightarrow$ an edge of $JJ_{2,2}$

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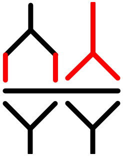 $\leftrightarrow$ an edge of $JJ_{2,2}$

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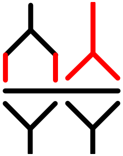 $\leftrightarrow$ an edge of $JJ_{2,2}$

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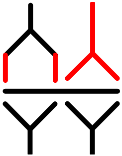 $\leftrightarrow$ an edge of $JJ_{2,2}$

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- $JJ_{n,m}$ is a subdivision of $KK_{n,m} \times I$

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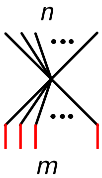
- As $KK_{n,m} \times 0 \quad \leftrightarrow$  $\leftrightarrow \quad \theta^n_m \left(\mathfrak{f}^1_1\right)^{\otimes m}$

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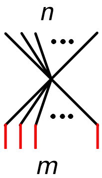
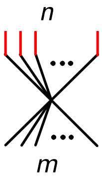
- As $KK_{n,m} \times 0 \quad \leftrightarrow$  $\leftrightarrow \quad \theta_m^n \left(\mathfrak{f}_1^1\right)^{\otimes m}$

- As $KK_{n,m} \times 1 \quad \leftrightarrow$  $\leftrightarrow \quad \left(\mathfrak{f}_1^1\right)^{\otimes n} \theta_m^n$

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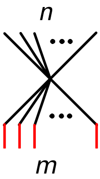
- As $KK_{n,m} \times 0 \quad \leftrightarrow$ 

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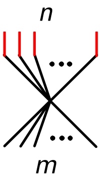
- As $KK_{n,m} \times 1 \quad \leftrightarrow$ 

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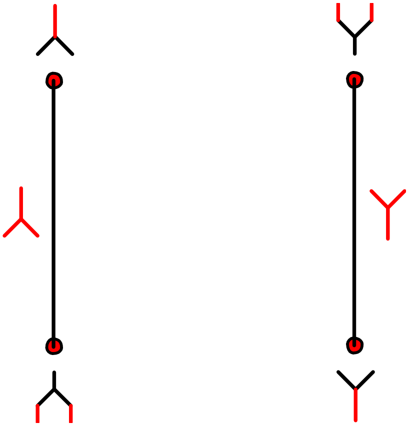
- This latter cell plays an important role in our transfer algorithm

The Bimultiplihedra $JJ(1,2)$ and $JJ(2,1)$:



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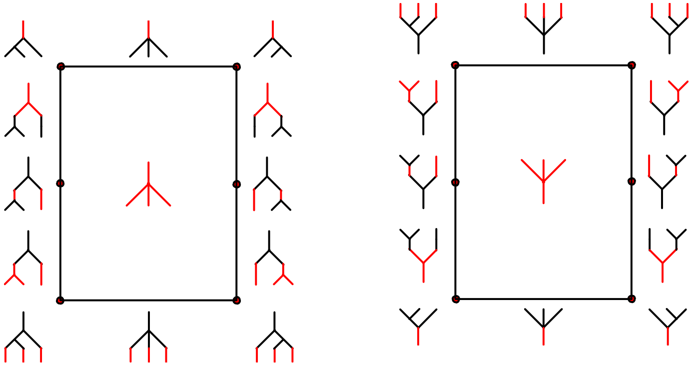

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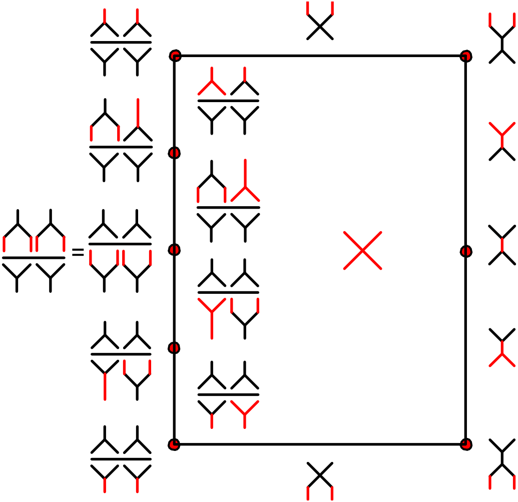

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- A morphism $G : A \Rightarrow B$ is an $\mathcal{H}_\infty$-bimodule

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- A chain map $g : A \to B$ of DGMs induces a cochain map

$$\tilde{g} : \mathcal{E}nd_{TA} \to U_{A,B}$$

defined on $u \in Hom(A^{\otimes m}, A^{\otimes n})$ by $\tilde{g}(u) = g^{\otimes n} u$

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  - g *extends to an* A_∞*-bialgebra map* $G = \{g_m^n \mid g_1^1 = g\} : A \Rightarrow B$

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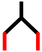
- Define β on the vertex  of $JJ_{1,2}$ by

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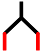
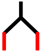
- Define β on the vertex  of $JJ_{1,2}$ by

$$\theta_2^1\left(\mathfrak{f}_1^1 \otimes \mathfrak{f}_1^1\right) =$$  $$\mapsto \omega_B^{1,2}\left(g \otimes g\right)$$

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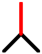
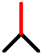
- Define $\alpha_A : C_0(\partial KK_{1,3}) \to Hom(A^{\otimes 3}, A)$ by

 $\mapsto \omega_A^{1,2}\left(\omega_A^{1,2} \otimes \mathbf{1}\right)$ and  $\mapsto \omega_A^{1,2}\left(\mathbf{1} \otimes \omega_A^{1,2}\right)$

- Extend β to the vertex  $\subset JJ_{1,2}$ via  $\mapsto g\omega_A^{1,2}$

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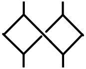

$$\mapsto \left(\omega_A^{1,2} \otimes \omega_A^{1,2}\right) \sigma_{2,2} \left(\omega_A^{2,1} \otimes \omega_A^{2,1}\right)$$

and

$$\mapsto \omega_A^{2,1} \omega_A^{1,2}$$

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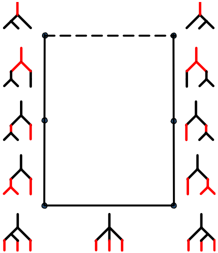
- Define β on $\partial JJ_{1,3} \smallsetminus \operatorname{int} KK_{1,3} \times 1$:



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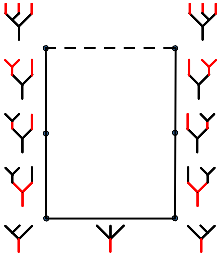
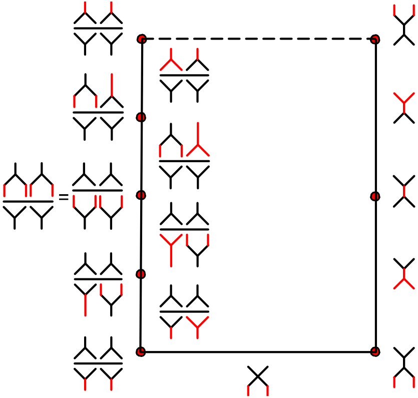


The Transfer Algorithm: Initialization

Having defined β on  and , define β on $\partial JJ_{2,2} \smallsetminus \operatorname{int} KK_{2,2} \times 1$:



The Transfer Algorithm: Induction Hypothesis

Given $m+n \geq 4$, assume for $i+j < m+n$, $ij \neq 1$, there exist maps

- $\alpha_A : C_*(KK_{j,i}) \to Hom\left(A^{\otimes i}, A^{\otimes j}\right)$ of matrads sending $\theta_i^j \mapsto \omega_A^{j,i}$

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The Transfer Algorithm: Induction

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- Define α_A on a monomial in $C_* \left(\partial KK_{n,m} \right)$ to be its corresponding fraction product of operations in $\{\omega_A^{j,i}\}$; let

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- This completes the induction

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- $A = H^*\left(X; \mathbb{Z}_2\right) = \left\{1, a_2, a_3, b, a_2a_3 = Sq^2 b, \ldots\right\}$

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- Induced map $\phi = E'_{1,0} + E'_{0,1} + E'_{1,1} : BA \otimes BA \to A$ acts nontrivially:

$$\phi([b] \otimes [b]) = a_2 a_3$$

Induced multiplication on BA

- ϕ lifts to a chain map $\mu : BA \otimes BA \to BA$ of DG coalgebras given by

$$\mu = \sum_{k \geq 0} \downarrow^{\otimes k+1} \phi^{\otimes k+1} \bar{\psi}^{(k)}$$

where $\bar{\psi}^{(0)} = \mathrm{Id}_{BA \otimes BA}$ and $\bar{\psi}^{(k)}$ is the left-iterated reduced coproduct

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- Nontrivial product $\mu\left([b] \otimes [b]\right) = [a_2 a_3]$

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- $H = H^*(BA) \approx H^*(\Omega X; \mathbb{Z}_2)$ as graded coalgebras
- Denote: $\alpha_i = \text{cls}[a_i] \qquad \beta = \text{cls}[b]$
- Choose a cycle-selecting map $g : H \to BA$ such that

$$g(\text{cls}[x_1|\cdots|x_n]) = [x_1|\cdots|x_n]$$

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- *There is a homotopy g_2 satisfying the relation above such that*

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Non-operadic operation defined

- The Transfer Theorem implies the relation on $JJ_{2,2}$ holds:

$$\begin{aligned}\nabla g_2^2 &= (\mu \otimes \mu)\,\sigma_{2,2}\left(\Delta g \otimes g^2 + g^2 \otimes (g \otimes g)\,\Delta_H\right) + g^2 \varphi \\ &\quad + (\mu\,(g \otimes g) \otimes g_2 + g_2 \otimes g\varphi)\,\sigma_{2,2}\,(\Delta_H \otimes \Delta_H) + \Delta g_2 \\ &\quad + (g \otimes g)\,\omega_2^2\end{aligned}$$

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- $[a_2]\otimes[a_3]$ represents a non-zero class in $BA\otimes BA$

- Thus no choice of g_2^2 kills the obstruction $[a_2]\otimes[a_3]$

Conclusion:

- *There is an operation ω_2^2 and a cochain homotopy g_2^2 satisfying the relation on $JJ_{2,2}$ such that*

$$\omega_2^2\,(\beta \otimes \beta) = \alpha_2 \otimes \alpha_3$$

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- We have identified the first non-operadic cohomology operation

The End

Thank you!